Spanning weakly even trees of graphs

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Abstract

Let G be a graph (with multiple edges allowed) and let T be a tree in G. We say that T is *even* if every leaf of T belongs to the same part of the bipartition of T, and that T is *weakly even* if every leaf of T that has maximum degree in G belongs to the same part of the bipartition of T. We confirm two recent conjectures of Jackson and Yoshimoto by showing that every connected graph that is not a regular bipartite graph has a spanning weakly even tree.

Keywords: Even tree; Weakly even tree; 2-factor; Weak 2-factor.

1 Introduction

In this paper graphs are finite and may contain multiple edges but not loops. We use uv to denote an edge from u to v; if there is more than one such edge, which edge we mean will either not matter or be clear from context. Let T be a tree in a graph G. We say that T is *even* if all leaves of T belong to the same part of the bipartition of T. More generally, we say that T is *weakly even* if all leaves of T that have maximum degree in G belong to the same part of the bipartition of T.

For our proofs it is convenient to consider a specific ordered bipartition (X, Y) of a tree T in G and to insist that the leaves of T with maximum degree in G belong to X. We introduce some appropriate terminology. Given an ordered bipartition (X, Y) of a bipartite graph H, a vertex of H is *type*-0 or *type*-1 if it belongs to X or Y, respectively. If $w \in V(H)$ and $\lambda \in \{0, 1\}$, the (w, λ)-*bipartition* of H is the bipartition of H for which w has type λ. If H is equipped with this bipartition, we say H is a (w, λ)-*graph* (or (w, λ)-*tree*, (w, λ)-*cycle*, etc., as appropriate). A tree $T \subseteq G$ with ordered bipartition (X, Y) is *even* if all leaves of T are type-0 (belong to X), and *weakly even* if all leaves of T that have maximum degree in G are type-0.

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S. Saito asked which regular connected graphs have a spanning even tree. Jackson and Yoshimoto [5] obtained the following partial answer to the question.

Theorem 1 ([5])**.** *Suppose G is a regular nonbipartite connected graph that has a 2-factor, $w \in V(G)$ and $\lambda \in \{0, 1\}$. Then G has a spanning even (w, λ)-tree.*

They conjectured that connected regular bipartite graphs are the only connected graphs that do not have a spanning even tree.

Conjecture 2. *Every regular nonbipartite connected graph has a spanning even tree.*

As an extension of Conjecture 2, Jackson and Yoshimoto [5] also posed the following conjecture.

Conjecture 3. *Every connected graph that is not a regular bipartite graph has a spanning weakly even tree.*

Conjecture 3 implies Conjecture 2: in a regular connected graph, every vertex has maximum degree, and so a spanning tree is weakly even if and only if it is even. In this paper we confirm Conjecture 3 and hence Conjecture 2. Our result combines work by Ai, Gao, Liu, and Yue [1], and by Ellingham, Huang, Shan, and Špacapan [3].

Most of the work to confirm Conjecture 3 is in the proof of the following theorem, which we postpone to the next section.

Theorem 4. *Let G be a 2-edge-connected graph that is not regular bipartite, $w \in V(G)$ and $\lambda \in \{0, 1\}$. Then G has a spanning weakly even (w, λ)-tree.*

Using Theorem 4 we can prove Theorem 5, which verifies Conjecture 3.

Theorem 5. *Let G be a connected graph that is not regular bipartite, $w \in V(G)$ and $\lambda \in \{0, 1\}$. Then G has a spanning weakly even (w, λ)-tree.*

Proof. We proceed by induction on $|V(G)|$. If $|V(G)| \leq 2$ then G is regular bipartite, and the theorem holds vacuously. Therefore, we may assume that $|V(G)| \geq 3$, which implies $\Delta(G) \geq 2$, and that the theorem holds for graphs of smaller order than G. If G is 2-edge-connected, then the theorem holds by Theorem 4, so we may assume that G has a cutedge x_1x_2. Let G_1 and G_2 be the components of $G - x_1x_2$, with $x_1 \in V(G_1)$, $x_2 \in V(G_2)$.

Claim. *Let $i \in \{1, 2\}$, $w_i \in V(G_i)$, and $\lambda_i \in \{0, 1\}$. Then there is a spanning (w_i, λ_i)-tree T_i of G_i that is weakly even in G except possibly at x_i (i.e., no vertex of T_i except possibly x_i is a type-1 leaf of T_i with maximum degree in G).*

Proof of Claim. Note that all vertices of G_i have the same degree in G_i as in G, except x_i. If G_i is regular bipartite then $\Delta(G_i) < \Delta(G)$. Hence any spanning (w_i, λ_i)-tree T_i of G_i is weakly even in G except possibly at x_i. If G_i is not regular bipartite, then by the induction hypothesis there is a spanning (w_i, λ_i)-tree T_i of G_i that is weakly even in G_i, and hence weakly even in G except possibly at x_i. □

We may assume that $w \in V(G_1)$. By the Claim G_1 has a spanning (w, λ)-tree T_1 that is weakly even in G except possibly at x_1. Let λ_2 be the type opposite to the type of x_1 in T_1. By the Claim, G_2 has a spanning (x_2, λ_2)-tree T_2 that is weakly even in G except possibly at x_2. The bipartitions of T_1 and T_2 agree with the (w, λ)-bipartition of $T = T_1 \cup T_2 \cup \{x_1x_2\}$. If either x_1 or x_2 is a leaf of T then it has degree $1 < \Delta(G)$ in G, and all other leaves of T satisfy the weakly even condition in G, so T is a spanning weakly even tree in G. □

2 Proof of Theorem 4

We start with some preliminaries. We assume that every cycle in a graph G has a fixed orientation. When discussing a particular cycle C and $u, v \in V(C)$ we use u^- and u^+ to denote the immediate predecessor and successor, respectively, of u on C, and uCv to mean the subpath of C from u to v following the orientation of C (uCv is a single vertex if $u = v$). A spanning subgraph H of G is a *weak 2-factor* if each component of H is either a cycle or a path (possibly a single vertex), and the endvertices of the path components of H have degree less than $\Delta(G)$ in G. For any positive integer k, let $[k] = \{1, \dots, k\}$.

Theorem 6 ([2, 4, 6])**.** *If G is a connected r-regular graph, $r \geq 2$, and G is 2-edge-connected or has at most $r-1$ cutedges, then G has a 2-factor.*

The following generalizes [5, Lemma 6].

Lemma 7. *Let G be a connected graph that is not regular bipartite, and $Y \subseteq V(G)$ an independent set in G. Suppose that all vertices of Y have maximum degree in G. Then for any $X \subseteq V(G) \setminus Y$ with $|X| \leq |Y|$, we have $E(Y, V(G) \setminus (X \cup Y)) \neq \emptyset$.*

Proof. Let G, X and Y be as given in the lemma. If $E(Y, V(G) \setminus (X \cup Y)) = \emptyset$, then

$$|E(X,Y)| = \sum_{y \in Y} d(y) = \Delta(G)|Y| \geq \Delta(G)|X| \geq \sum_{x \in X} d(x)$$
$$= |E(X,Y)| + 2|E(X)| + |E(X, V(G) \setminus (X \cup Y))| \geq |E(X,Y)|.$$

Therefore, $|E(X)| = |E(X, V(G) \setminus (X \cup Y))| = 0$, $|X| = |Y|$, and $d(x) = \Delta(G)$ for all $x \in X$. This implies that G is a $\Delta(G)$-regular bipartite graph, a contradiction. □

Now we can prove Theorem 4. The proof uses a similar argument to the original proof of Theorem 1 by Jackson and Yoshimoto.

Proof of Theorem 4. Let G be a 2-edge-connected graph that is not regular bipartite, and let $w \in V(G)$ and $\lambda \in \{0, 1\}$ be given. If G is regular, then since G is 2-edge-connected it has a 2-factor by Theorem 6, and hence a spanning even tree with w of type λ by Theorem 1. Assume therefore that G is not a regular graph. Since G is 2-edge-connected, it follows that $\Delta(G) \geq 3$.

We will find a weak 2-factor in G and then construct a spanning weakly even tree by using most of the edges of the weak 2-factor and some other edges.

Claim 1. *The graph G has a weak 2-factor.*

Proof. Let G' be another copy of G. For each $v \in V(G)$ with $d_G(v) < \Delta(G)$, add $\Delta(G) - d_G(v)$ edges joining v and the copy of v in G'. Denote by G^* the resulting multigraph. Then G^* is $\Delta(G)$-regular and has at most one cutedge. By Theorem 6, G^* has a 2-factor F^*. Let F be the spanning subgraph of G such that $E(F) = E(F^*) \cap E(G)$. Since F^* is a 2-factor, each component of F is either a cycle or a path. Moreover, each endvertex of a path component of F is incident with an edge in $E(F^*) \setminus E(G)$ and thus has degree less than $\Delta(G)$. Therefore F is a weak 2-factor of G. □

Fix a weak 2-factor F of G. A tree in G is *good* (with respect to F) if its vertex set is the union of vertex sets of some components of F. Let T be a good weakly even (w, λ)-tree in G of maximum order; if no such tree exists, let T be the null graph. We claim that T is a spanning tree of G. We suppose that $V(T) \neq V(G)$ and show that this always leads to a contradiction.

Claim 2. *Suppose that T is null. Then the component C_0 of F containing w is an even cycle. If (X_0, Y_0) is the (w, λ)-bipartition of C_0, then all vertices of Y_0 have maximum degree in G, and Y_0 is an independent set in G.*

Proof. If C_0 is a path, then both ends of C_0 have degree less than $\Delta(G)$ in G. Therefore, C_0 is a good even (w, λ)-tree, which contradicts the choice of T. If C_0 is an odd cycle, then w^+C_0w is a good even $(w, 0)$-tree, and $w^{++}C_0w^+$ is a good even $(w, 1)$-tree. Therefore, there is a good even (w, λ)-tree, which contradicts the choice of T. Thus, C_0 must be an even cycle.

If there is $u \in Y_0$ that has degree less than $\Delta(G)$ in G, then $T = u^+C_0u$ is a good weakly even (w, λ)-tree (since $u^+ \in X_0$ is the only leaf of T that possibly has degree $\Delta(G)$ in G), contradicting the choice of T. Therefore, all vertices of Y_0 have degree $\Delta(G)$ in G.

Suppose that G has an edge uv joining $u, v \in Y_0$. Consider $T_1 = u^+C_0u^- \cup \{uv\}$ and $T_2 = v^+C_0v^- \cup \{uv\}$. If $w \neq u$ then T_1 is a good even (w, λ)-tree, and if $w = u$ then T_2 is a good even (w, λ)-tree. Either situation contradicts the choice of T. Thus, Y_0 is an independent set in G. □

If T is null, let C_0 be the component (even cycle) of F that contains w, and let (X_0, Y_0) be the (w, λ)-bipartition of C_0. If T is nonnull, let (X_0, Y_0) be the (w, λ)-bipartition of T.

Claim 3. *Suppose that T is nonnull.*

(a) *Then $E_G(X_0, V(G) \setminus V(T)) = \emptyset$ and $E_G(Y_0, V(G) \setminus V(T)) \neq \emptyset$.*

(b) *Suppose that $y_0z \in E(G)$ with $y_0 \in Y_0$ and $z \notin V(T)$. Then the component C of F containing z is an even cycle. If (X, Y) is the $(z, 0)$-bipartition of C (so that $z \in X$), then all vertices of Y have maximum degree in G, and Y is an independent set in G.*

Proof. (a) If $E_G(X_0, V(G) \setminus V(T)) \neq \emptyset$, then there exists $x_0z \in E(G)$ such that $x_0 \in X_0$ and $z \notin V(T)$. Let C be the component of F containing z. If C is a path, then $T' = T \cup \{x_0z\} \cup C$ is a good weakly even (w, λ)-tree. Indeed, each leaf of T' that is not a leaf of T has degree less than $\Delta(G)$ in G (since F is a weak 2-factor). If C is an odd cycle, then $T \cup \{x_0z\} \cup z^{++}Cz^+$ is a good weakly even (w, λ)-tree. If C is an even cycle, then $T \cup \{x_0z\} \cup z^+Cz$ is a good weakly even (w, λ)-tree. All three situations contradict the maximality of T, and therefore $E_G(X_0, V(G) \setminus V(T)) = \emptyset$.

If $E_G(Y_0, V(G) \setminus V(T)) = \emptyset$ then $E_G(V(T), V(G) \setminus V(T)) = \emptyset$, which (since $V(T) \neq \emptyset$ and $V(T) \neq V(G)$) contradicts the fact that G is connected. Hence, $E_G(Y_0, V(G) \setminus V(T)) \neq \emptyset$.

(b) If C is a path, then each leaf of $T' = T \cup \{y_0z\} \cup C$ that is not a leaf of T is an endvertex of C and so has degree less than $\Delta(G)$ in G. Thus, T' is a good weakly even (w, λ)-tree. If C is an odd cycle, then $T \cup \{y_0z\} \cup z^+Cz$ is a good weakly even (w, λ)-tree. In both cases we contradict the maximality of T. Hence, C is an even cycle.

If there is $u \in Y$ that has degree less than $\Delta(G)$ in G, then $T' = T \cup \{y_0z\} \cup u^+Cu$ is a good weakly even (w, λ)-tree (since u is the only vertex of Y that is a leaf of T' but not of T), contradicting the maximality of T. Therefore, all vertices of Y have degree $\Delta(G)$ in G.

Suppose that G has an edge uv joining $u, v \in Y$. Let $T' = T \cup \{y_0z, uv\} \cup u^+Cu^-$. Then T' is a good weakly even (w, λ)-tree because every vertex that is not a leaf of T but possibly a leaf of T' (namely u^-, u, u^+) is type-0. This contradicts the maximality of T. Thus, Y is an independent set in G. □

Since T is a good tree, $F' = F - V(T)$ is a union of components of F. Let $\mathcal{C}_{\text{even}}$ be the set of all even cycles of F', and $F'_{\text{even}} = \bigcup_{C \in \mathcal{C}_{\text{even}}} C$. If T is null then $C_0 \subseteq F'_{\text{even}}$, and we say a bipartition (X, Y) of F'_{even} is *consistent* if $X_0 \subseteq X$ and $Y_0 \subseteq Y$. If T is nonnull, every bipartition of F'_{even} is considered to be consistent. Let $C_1, C_2, \ldots, C_\ell$ be a sequence of $\ell \geq 1$ distinct cycles in $\mathcal{C}_{\text{even}}$. Given such a sequence, we define $X_i = X \cap V(C_i)$ and $Y_i = Y \cap V(C_i)$ for $i \in [\ell]$. The sequence is *admissible* with respect to a consistent bipartition (X, Y) of F'_{even} if (1) either T is null and $C_1 = C_0$, or T is nonnull and $E(Y_0, X_1) \neq \emptyset$, and (2) $E(Y_i, X_{i+1}) \neq \emptyset$ for $i \in [\ell - 1]$. A cycle $C \in \mathcal{C}_{\text{even}}$ is *admissible* if it is the final (equivalently, any) cycle of an admissible sequence.

Choose a consistent bipartition (X, Y) of F'_{even} such that the number of admissible cycles with respect to (X, Y) is as large as possible, and let $\mathcal{C}_{\text{adm}}$ be the set of admissible cycles with respect to (X, Y). If T is null then $C_0 \in \mathcal{C}_{\text{adm}}$, and if T is nonnull, then $\mathcal{C}_{\text{adm}}$ is nonempty by Claim 3. Let $F'_{\text{adm}} = \bigcup_{C \in \mathcal{C}_{\text{adm}}} C$, $X_{\text{adm}} = X \cap V(F'_{\text{adm}})$, and $Y_{\text{adm}} = Y \cap V(F'_{\text{adm}})$. By the maximality of $\mathcal{C}_{\text{adm}}$, if $E(V(F'_{\text{adm}}) \cap Y, V(C)) \neq \emptyset$ for a component C of F' that is vertex-disjoint from F'_{adm}, then C must be an odd cycle or a path.

Claim 4. *All vertices in Y_{adm} have maximum degree in G.*

Proof. Suppose that the claim is false and let $C_1, C_2, \ldots, C_\ell$ be a shortest admissible sequence with respect to (X, Y) for which some vertex z in $Y \cap C_\ell$ does not have maximum degree in G. Note that, by Claims 2 and 3, all vertices of Y_1 have maximum degree in G, so $\ell \geq 2$. There is $y_i x_{i+1} \in E(G)$ with $y_i \in Y_i$ and $x_{i+1} \in X_{i+1}$ for $i \in [\ell - 1]$, and for $i = 0$ if T is nonnull. Let T^* be null if T is null, and $T \cup \{y_0 x_1\}$ if T is nonnull. Define

$$T' = T^* \cup (\textstyle\bigcup_{i=1}^{\ell-1} y_i^+ C_i y_i x_{i+1}) \cup z^+ C_\ell z.$$

Then T' is a good weakly even (w, λ)-tree since z, which has degree less than $\Delta(G)$, is the only vertex in Y that is a leaf of T' and not a leaf of T. This contradicts the maximality of T. ☐

We now consider two cases.

Case 1. Suppose Y_{adm} is an independent set in G.

In this case, Lemma 7 implies that there is an edge uz of G joining $u \in Y_{\text{adm}}$ and $z \in V(G) \setminus V(F'_{\text{adm}})$. Then there is an admissible sequence $C_1, C_2, \ldots, C_\ell$ with $u \in V(C_\ell)$. There is $y_i x_{i+1} \in E(G)$ with $y_i \in Y_i$ and $x_{i+1} \in X_{i+1}$ for $i \in [\ell - 1]$, and for $i = 0$ if T is nonnull. Let T^* be null if T is null, and $T \cup \{y_0 x_1\}$ if T is nonnull.

If $z \notin V(T)$, let C be the component of F' that contains z. Then C is an odd cycle or a path, by the maximality of $\mathcal{C}_{\text{adm}}$. Let $Q = z^+ C z$ if C is an odd cycle and $Q = C$ if C is a path. Define

$$T' = T^* \cup (\textstyle\bigcup_{i=1}^{\ell-1} y_i^+ C_i y_i x_{i+1}) \cup u^+ C_\ell u z \cup Q.$$

Then T' is a good weakly even (w, λ)-tree. This contradicts the maximality of T.

If $z \in V(T)$, this implies that T is nonnull. Since u is a neighbor of z not in $V(T)$, by Claim 3, $z \in Y_0$. Define

$$T' = T \cup \{y_0 x_1, uz\} \cup (\textstyle\bigcup_{i=1}^{\ell-1} y_i^+ C_i y_i x_{i+1}) \cup u^+ C_\ell u^-.$$

Then T' is a good weakly even (w, λ)-tree because every vertex that is not a leaf of T but possibly a leaf of T' (including u) is type-0. This contradicts the maximality of T.

Case 2. Suppose Y_{adm} is not an independent set in G.

Then there is an edge yz where $y, z \in Y_{\text{adm}}$. Suppose that y is a vertex of $C \in \mathcal{C}_{\text{adm}}$ and z is a vertex of $\widetilde{C} \in \mathcal{C}_{\text{adm}}$, where possibly $C = \widetilde{C}$. Let $C_1, C_2, \ldots, C_\ell$ be an admissible sequence with $C = C_\ell$. There is $y_i x_{i+1} \in E(G)$ with $y_i \in Y_i$ and $x_{i+1} \in X_{i+1}$ for $i \in [\ell - 1]$, and for $i = 0$ if T is nonnull. Let $\widetilde{C}_1, \widetilde{C}_2, \ldots, \widetilde{C}_k$ be an admissible sequence with $\widetilde{C} = \widetilde{C}_k$, and let $\widetilde{X}_j = X \cap V(\widetilde{C}_j)$, $\widetilde{Y}_j = Y \cap V(\widetilde{C}_j)$ for $j \in [k]$. There is $\widetilde{y}_j \widetilde{x}_{j+1} \in E(G)$ with $\widetilde{y}_j \in \widetilde{Y}_j$ and $\widetilde{x}_{j+1} \in \widetilde{X}_{j+1}$ for $j \in [k - 1]$, and for $j = 0$ if T is nonnull. Let T^* be null if T is null, and $T \cup \{y_0 x_1\}$ if T is nonnull.

Suppose first that C and $\widetilde{C}$ belong to a common admissible sequence. Without loss of generality we may suppose that $\widetilde{C} \in \{C_1, C_2, \ldots, C_\ell\}$. Define

$$T' = T^* \cup \{yz\} \cup (\textstyle\bigcup_{i=1}^{\ell-1} y_i^+ C_i y_i x_{i+1}) \cup y^+ C_\ell y^-.$$

Then T' is a good weakly even (w, λ)-tree because every vertex that is not a leaf of T but possibly a leaf of T' (including y) is type-0. This contradicts the maximality of T.

Next, suppose that $\{C_1, C_2, \ldots, C_\ell\} \cap \{\widetilde{C}_1, \widetilde{C}_2, \ldots, \widetilde{C}_k\} = \emptyset$. Then T is nonnull. Let

$$T' = T \cup \{y_0x_1, \widetilde{y}_0\widetilde{x}_1, yz\} \cup (\bigcup_{i=1}^{\ell-1} y_i^+ C_i y_i x_{i+1}) \cup (\bigcup_{j=1}^{k-1} \widetilde{y}_j^+ \widetilde{C}_j \widetilde{y}_j \widetilde{x}_{j+1}) \cup y^+ C_\ell y^- \cup z^+ \widetilde{C}_k z.$$

Again T' is a good weakly even (w, λ)-tree because every vertex that is not a leaf of T but possibly a leaf of T' (including y) is type-0. This contradicts the maximality of T.

Finally, suppose that $\{C_1, C_2, \ldots, C_\ell\} \cap \{\widetilde{C}_1, \widetilde{C}_2, \ldots, \widetilde{C}_k\} \neq \emptyset$, but $C = C_\ell \notin \{\widetilde{C}_1, \widetilde{C}_2, \ldots, \widetilde{C}_k\}$ and $\widetilde{C} = \widetilde{C}_k \notin \{C_1, C_2, \ldots, C_\ell\}$. Let $b \in [k-1]$ be the largest integer such that $\widetilde{C}_b \in \{C_1, C_2, \ldots, C_\ell\}$. Then $\widetilde{C}_b = C_a$ for some $a \in [\ell - 1]$ and $\{C_1, C_2, \ldots, C_\ell\} \cap \{\widetilde{C}_{b+1}, \widetilde{C}_{b+2}, \ldots, \widetilde{C}_k\} = \emptyset$. Let

$$T' = T^* \cup \{\widetilde{y}_b\widetilde{x}_{b+1}, yz\} \cup (\bigcup_{i=1}^{\ell-1} y_i^+ C_i y_i x_{i+1}) \cup (\bigcup_{j=b+1}^{k-1} \widetilde{y}_j^+ \widetilde{C}_j \widetilde{y}_j \widetilde{x}_{j+1}) \cup y^+ C_\ell y^- \cup z^+ \widetilde{C}_k z.$$

Once more T' is a good weakly even (w, λ)-tree because every vertex that is not a leaf of T but possibly a leaf of T' (including y) is type-0. This contradicts the maximality of T.

In all situations we reach a contradiction, so we conclude that $V(T) = V(G)$, as required. □

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