Unavoidable induced subgraphs of large and infinite 2-edge-connected graphs

Sarah Allred * M. N. Ellingham †

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Abstract

In 1930, Ramsey proved that every large graph contains either a large clique or a large edgeless graph as an induced subgraph. It is well known that every large connected graph contains a long path, a large clique, or a large star as an induced subgraph. Recently Allred, Ding, and Oporowski presented the unavoidable large induced subgraphs for large 2-connected graphs and for infinite 2-connected graphs. In this paper we establish the 2-edge-connected analogues of these results. As consequences we obtain results on unavoidable large subgraphs, topological minors, minors, induced topological minors, induced minors, and Eulerian subgraphs in large and infinite 2-edge-connected graphs. When appropriate we extend our results to multigraphs.

1 Introduction

1.1 Background

In this paper graphs are simple, and multigraphs may have multiple edges but not loops. Graphs and multigraphs are assumed to be finite unless we explicitly indicate that they are infinite. Terms and symbols not defined here follow West [17]. For notational simplicity we use ∞ to mean countable infinity, $\aleph_0$, and we generalize notation for finite graphs such as $K_r, K_{k,r}, K_{1,1,r}$ to countably infinite graphs $K_\infty, K_{k,\infty}, K_{1,1,\infty}$, and so on, if there is no ambiguity. Sometimes there may be more than one infinite analogue of a class of finite graphs (for example, there are one-way and two-way infinite paths), and we give specific definitions.

This paper presents results on unavoidable large induced subgraphs in 2-edge-connected graphs. Many results in graph theory depend on the fact that sufficiently large (or infinite) graphs contain one of a family of unavoidable substructures. For example, Ding, Oporowski, Thomas, and Vertigan [11] use results on unavoidable topological minors in 3-connected graphs from [15] to obtain results on 2-crossing-critical graphs, and Ding and Marshall [8] use Theorem 1.6 below to obtain a characterization of graphs with no large

*Department of Mathematics & Statistics, University of South Alabama, Mobile, AL, USA (sarahallred@southalabama.edu)
†Department of Mathematics, Vanderbilt University, Nashville, TN, USA (mark.ellingham@vanderbilt.edu)

theta graph as a minor. Results guaranteeing the existence of unavoidable substructures are therefore of fundamental importance.

Two foundational results of this kind for infinite graphs are due to König in 1927 and Ramsey in 1930. Denote the *ray* or *one-way infinite path* $v_1v_2v_3\dots$ by P_∞. König's result includes the following as an important special case.

Theorem 1.1 (König [14]). *Every infinite connected graph contains either $K_{1,\infty}$ or P_∞ as a subgraph.*

Ramsey proved a result for complete uniform hypergraphs whose edges are given finitely many colors. The simplest case is for complete graphs whose edges are given two colors, which may be stated as follows.

Theorem 1.2 (Ramsey [16]). *Every infinite graph contains either K_∞ or its complement $\overline{K}_\infty$ as an induced subgraph.*

König's result and Ramsey's result differ in two ways. First, they guarantee different kinds of substructure: König's result guarantees a subgraph, while Ramsey's result guarantees an induced subgraph. In other words, they use different containment orderings for graphs. Second, König's result has a connectivity condition, while Ramsey's does not. However, they also have similarities. Both Theorems 1.1 and 1.2 apply to graphs whose number of vertices is any infinite cardinal, although they guarantee only countably infinite substructures. And both results have counterparts for sufficiently large finite graphs: Theorem 1.3 is an easy exercise, and Theorem 1.4 was also proved by Ramsey.

Theorem 1.3. *For every positive integer r, there is an integer $f_{1.3}(r)$ such that every graph on at least $f_{1.3}(r)$ vertices has either $K_{1,r}$ or P_r as a subgraph.*

Theorem 1.4 (Ramsey [16]). *For every positive integer r, there is an integer $f_{1.4}(r)$ such that every graph on at least $f_{1.4}(r)$ vertices contains either K_r or $\overline{K}_r$ as an induced subgraph.*

Besides the subgraph and induced subgraph orderings, there are results involving unavoidable substructures for the topological minor and minor orderings of graphs, based on various connectivity conditions. The four orderings can be ranked from strongest to weakest as induced subgraph, subgraph, topological minor, and minor. A result on unavoidable substructures in a class of graphs for a given ordering generally also yields results for all weaker orderings. Results for weaker orderings are usually easier to state because the complicated unavoidable structures for a stronger ordering have simpler unavoidable substructures under a weaker ordering. Oporowski, Oxley, and Thomas [15] found unavoidable topological minors for 2-connected, 3-connected, and internally 4-connected graphs, which imply results on minors.

The induced subgraph ordering is the strongest of the four related orderings (induced subgraph, subgraph, topological minor, minor). Induced subgraphs occur in characterizations of a number of graph classes. For example, the class of line graphs, and its superclass claw-free graphs are both characterized by finitely many unavoidable induced subgraphs, and chordal graphs and perfect graphs are characterized by simple infinite sets of unavoidable induced subgraphs. Since the induced subgraph ordering is the strongest of the four, it is the most difficult to obtain unavoidable substructure results for. The only known connectivity-based results for unavoidable induced subgraphs are for connected and 2-connected graphs. For connected graphs we have the following.

Theorem 1.5. *Every infinite connected graph contains K_∞, $K_{1,\infty}$, or P_∞ as an induced subgraph.*

Theorem 1.6 ([9, (5.3)] or [7, Proposition 9.4.1])**.** *For every positive integer r, there is an integer $f_{1.6}(r)$ such that every connected graph on at least $f_{1.6}(r)$ vertices contains K_r, $K_{1,r}$, or P_r as an induced subgraph.*

Theorem 1.5 follows from Theorems 1.1 and 1.2, and Theorem 1.6 can be proved using Theorem 1.4. The induced subgraph results Theorems 1.5 and 1.6 immediately imply the subgraph results Theorems 1.1 and 1.3, respectively.

Our results on unavoidable large induced subgraphs for 2-edge-connected graphs are obtained by using the unavoidable induced subgraphs for 2-connected graphs found by the first author, Ding, and Oporowski [1, 2]. Both the previous 2-connected results and our 2-edge-connected results are discussed in the next subsection. The results for 2-edge-connected graphs are not just simple consequences of what is known for 2-connected graphs; they involve completely new classes of graphs. Our theorems imply results on unavoidable subgraphs, topological minors, minors, induced topological minors, and induced minors, as well as a strengthening of a recent result of Goddard and LaVey [13] on large Eulerian subgraphs in 2-edge-connected graphs.

1.2 Induced subgraphs and higher connectivity

In this subsection we describe the known unavoidable induced subgraphs for 2-connected graphs, and state our new results for 2-edge-connected graphs. In both cases we need to make a number of definitions before describing the results.

In a particular class of graphs, we desire first that the unavoidable substructures are still in the class of graphs, and secondly, if possible the unavoidable substructures are minimal with respect to membership in the graph class. By Theorem 1.4, large complete graphs will be in the list of unavoidable induced subgraphs no matter the connectivity requirement, even though they are not minimal.

In [1], the first author, Ding, and Oporowski considered the induced subgraph ordering for 2-connected graphs. They defined two families of graphs that generalize stars and paths to 2-connected graphs. The first family Θ_r, where $r \geq 3$ is an integer or $r = \infty$, is the class of graphs that consist of r internally disjoint paths between two specified vertices, which can be considered as a generalization of the star $K_{1,r}$ for 2-connected graphs.

The description of the second family requires some preliminary definitions. For a path P, we denote the subpath of P with initial vertex u and final vertex v by $P[u,v]$, and we denote the subpath $P[u,v]-\{u,v\}$ by $P(u,v)$ (which is empty if $u = v$). The subpaths $P(u,v]$ and $P[u,v)$ are defined analogously. A *ladder* is a triple (L,P,Q) that consists of a graph L whose vertices all lie on two disjoint induced paths $P = p_1p_2\ldots p_\ell$ and $Q = q_1q_2\ldots q_m$ of L, called *rails*, and where $p_1q_1, p_\ell q_m \in E(L)$. The edges of L that belong to neither P nor Q are called *rungs*. Two rungs p_aq_b and p_cq_d *cross* if $a < c$ and $d < b$. We also say that $\{p_aq_b, p_cq_d\}$ is a *cross* whose *P-span* is $P[p_a,p_c]$, and whose *Q-span* is $Q[q_d,q_b]$; the *span* is the union of the P-span and the Q-span. A cross whose P-span and Q-span are both single edges is called *trivial*.

We define a *clean ladder* to be a ladder L where all crosses are trivial and $|V(L)| \geq 3$ (so clean ladders are 2-connected). Let Λ_r be the set of all clean ladders of order at least r, which can be considered as a generalization of the path P_r for 2-connected graphs. Every cycle of length at least r belongs to Λ_r. Note that graphs in Λ_r are in general not minimally 2-edge-connected (in the induced subgraph ordering).

Theorem 1.7 ([1, (1.4)])**.** *For every integer $r \geq 3$, there is an integer $f_{1.7}(r)$ such that every 2-connected graph of order at least $f_{1.7}(r)$ contains K_r or a member of the family $\Theta_r \cup \Lambda_r$ as an induced subgraph.*

In [2], the first author, Ding, and Oporowski extended this theorem to infinite 2-connected graphs. To do this, one of the families we need is Θ_∞. Figure 1.1 shows the structure of graphs in this family; the graphs in Figure 1.1(a) form the class $\mathcal{K}_{2,\infty}$ consisting of all subdivisions of $K_{2,\infty}$. In our figures thin line segments represent single edges, while thick line segments represent paths, which may be a single edge.

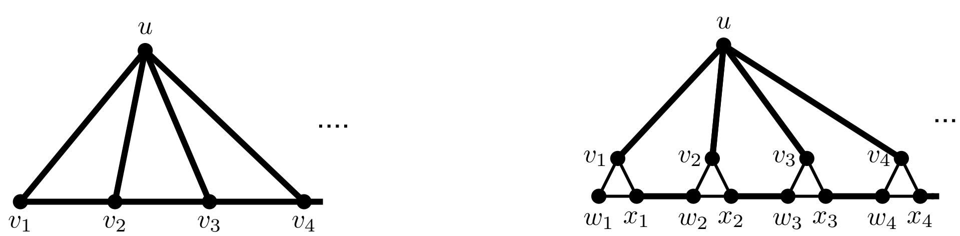
(a) $\mathcal{K}_{2,\infty}$, graphs with no edge u_1u_2

(b) Graphs with edge u_1u_2

Figure 1.1: Structure of Θ_∞

For several graph classes below we use the operation of *replacing a vertex by a triangle*: if v has degree $d \leq 3$ and neighbors u_i, $1 \leq i \leq d$, we delete v, add a triangle of new vertices $(v_1v_2v_3)$, and add edges u_iv_i for $1 \leq i \leq d$.

We use certain fan-like structures. Let F_∞ denote the graph that consists of a vertex u, a ray $v_1v_2v_3\dots$ (called the *rim*), and edges uv_i for all $i \in \{1, 2, 3, \dots\}$. Let $\mathcal{F}_\infty$ be the family of all subdivisions of F_∞; see Figure 1.2(a). Let $\mathcal{F}^\Delta_\infty$ be the family of graphs obtained from F_∞ by replacing every rim vertex with a triangle, and then arbitrarily subdividing each edge not in such a triangle; see Figure 1.2(b).



(a) $\mathcal{F}_\infty$

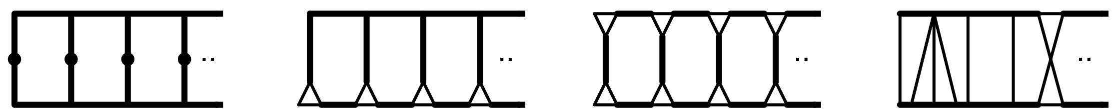
(b) $\mathcal{F}^\Delta_\infty$

Figure 1.2: Two types of fan-like structure

Note that the labeling of vertices in Figures 1.2(a) and 1.2(b) will be used in Section 3.

We also need ladder-like structures, illustrated in Figure 1.3. The ladder L_∞ consists of two disjoint induced rays, $P = p_1p_2\dots$ and $Q = q_1q_2\dots$, called *rails* and edges p_iq_i for each $i \in \{1, 2, 3, \dots\}$. Let $\mathcal{L}_\infty$ be the family of graphs obtained from L_∞ by subdividing each p_iq_i at least once and each of the rail edges an arbitrary number, possibly zero, of times; see Figure 1.3(a). Let $\mathcal{L}^\Delta_\infty$ be the family of graphs obtained from

L_∞ by replacing every vertex of one rail with a triangle, and then arbitrarily subdividing each edge not in such a triangle; see Figure 1.3(b). Let $\mathcal{L}_\infty^{\nabla\Delta}$ be the family of graphs obtained from L_∞ by replacing every vertex with a triangle, and the arbitrarily subdividing each edge not in such a triangle; see Figure 1.3(c). An *infinite slim ladder* is a locally finite graph consisting of two disjoint induced rays $P = p_1p_2\dots$ and $Q = q_1q_2\dots$ called *rails* and infinitely many edges p_iq_j, including p_1q_1, called *rungs*, such that all crosses (defined as for finite ladders) are trivial. In particular, L_∞ is an infinite slim ladder, and another example is shown in Figure 1.3(d). Infinite slim ladders can be regarded as a one-way infinite version of clean ladders.



(a) $\mathcal{L}_\infty$ (b) $\mathcal{L}_\infty^{\Delta}$ (c) $\mathcal{L}_\infty^{\nabla\Delta}$ (d) An infinite slim ladder

Figure 1.3: Ladder-like structures

Theorem 1.8 ([2, (1.7)]). *Every infinite 2-connected graph contains K_∞, an infinite slim ladder, or a member of $\Theta_\infty \cup \mathcal{F}_\infty \cup \mathcal{F}_\infty^{\Delta} \cup \mathcal{L}_\infty \cup \mathcal{L}_\infty^{\Delta} \cup \mathcal{L}_\infty^{\nabla\Delta}$ as an induced subgraph.*

As shown in [2], Theorem 1.7 on finite graphs can also be proved using Theorem 1.8 on infinite graphs.

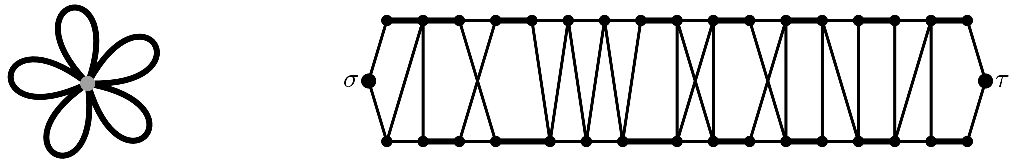
In Ramsey-type theorems, we want the list of unavoidable graphs (except complete graphs) to be minimal in terms of their connectivity. Since clean ladders and infinite slim ladders are not in general minimally 2-edge-connected, we use a different kind of ladder, super-clean pinched ladders, which we define below. Also, since 2-edge-connected graphs may have cutvertices, our unavoidable subgraphs include graphs with long chains of blocks, or many blocks meeting at a vertex.

An *r-flower*, where $r \geq 1$ or $r = \infty$, consists of r edge-disjoint induced cycles that have a single common vertex; see Figure 1.4(a). In our characterization, r-flowers and elements of Θ_r can be regarded as generalizations of a star $K_{1,r}$.

A *pinched ladder* is a triple (L, P, Q) where (1) L is a graph and P and Q are two paths in L, (2) $V(L) = V(P) \cup V(Q)$, (3) $P \cap Q$ consists of two vertices σ and τ, the *initial* and *final* vertices of L, respectively, and (4) either P is a single edge e and $Q = L - \{e\}$ (or vice versa), or P and Q are both induced paths. If either P or Q is a single edge then the pinched ladder is a cycle. The *rails* of a pinched ladder are $P(\sigma, \tau)$ and $Q(\sigma, \tau)$ (one of which may be empty). The rungs, crosses, and embedded fans of a pinched ladder are defined in terms of the rails $P(\sigma, \tau)$ and $Q(\sigma, \tau)$, in the same way as for a ladder. The edges incident with σ or τ (or both) are *terminal edges*. Every edge of L is a rail edge, a rung, or a terminal edge. The two vertices of L adjacent to the initial vertex are called *quasi-initial* vertices.

An *embedded fan* is an induced subgraph of a ladder or pinched ladder consisting of a vertex u on one rail, the *apex*, with at least two neighbors on the other rail, the subpath of the second rail between the first and last neighbors of v, the *rim*, and all edges between the apex and the rim. An embedded fan is *trivial* if it is a triangle. We frequently abbreviate "embedded fan" to "fan" if the meaning is clear from context.

We can now describe a modified version of a clean ladder that is minimally 2-edge-connected. A *super-*



(a) A 5-flower

(b) A super-clean pinched ladder

Figure 1.4: Graphs from Theorem 1.9

clean pinched ladder is a pinched ladder in which all crosses and embedded fans are trivial. See Figure 1.4(b). A super-clean pinched ladder has induced subgraphs that consist of (1) a cross and any other rungs induced by its endpoints, or (2) a maximal sequence of trivial fans, where consecutive elements intersect in a rung (these zigzag between the rails). All subgraphs of these two types are vertex-disjoint.

A *block* B of a graph G is a maximal connected subgraph of G with no cutvertex. Each block is a single vertex, a cutedge, or is 2-connected, so in a 2-edge-connected graph all blocks are 2-connected. The *block-cutvertex tree* of a graph G is a tree T where each cutvertex u of G is a vertex of T, for each block B_i of G there is a vertex v_i of T, and $uv_i \in E(T)$ whenever $u \in V(B_i)$. Every leaf of T corresponds to a block in G.

A *chain of blocks* of *length* n, or just *chain of* n *blocks*, is a graph with n blocks whose block-cutvertex tree is a path. In this case we call the cutvertices *joining vertices*. We can refer to a *chain of cycles* or *chain of triangles* if every block is a cycle or triangle, respectively. A *chain of super-clean pinched ladders* H is a chain of blocks with blocks $L_1, L_2, \ldots, L_n$ such that each L_i is a super-clean pinched ladder with initial vertex u_i and final vertex u_{i+1}. Thus, $u_2, u_3, \ldots, u_n$ are the joining vertices. We call u_1 and u_{n+1} the *initial* and *final* vertices of H, respectively. See Figure 1.5. Large super-clean pinched ladders and long chains of super-clean pinched ladders can be thought of as generalizations of a long path P_r. Lemma 2.5 below shows that all 2-edge-connected graphs contain induced subgraphs that are chains of super-clean pinched ladders.

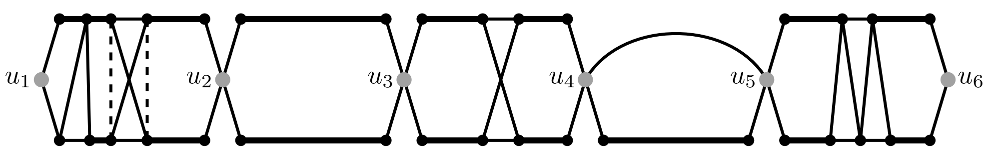


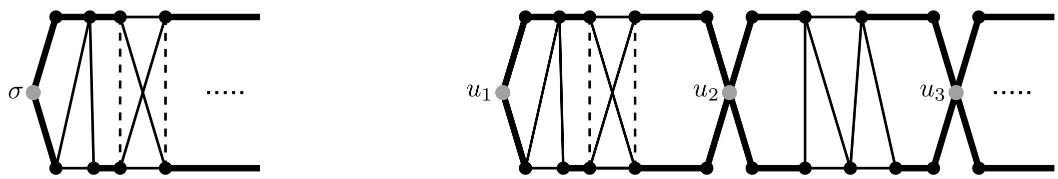
Figure 1.5: A chain of 5 super-clean pinched ladders

Our first result describes the unavoidable induced subgraphs of finite 2-edge-connected graphs.

Theorem 1.9. *For every integer $r \geq 3$, there is an integer $f_{1.9}(r)$ such that every 2-edge-connected graph of order at least $f_{1.9}(r)$ contains K_r, an r-flower, a super-clean pinched ladder of order at least r, a chain of r super-clean pinched ladders, or a member of the family Θ_r as an induced subgraph.*

We also prove the infinite analogue of Theorem 1.9. Since graphs in $\mathcal{F}_\infty$ and $\mathcal{F}_\infty^\Delta$ are not minimally

2-edge-connected, we replace them by an ∞-flower. Since an infinite slim ladder (see Figure 1.3(d)) is not in general minimally 2-edge-connected, we instead have two possibilities. One is an *infinite super-clean pinched ladder*, which is a triple (L, P, Q) such that (1) L is a locally finite graph and P and Q are rays in L, (2) $V(L) = V(P) \cap V(Q)$, (2) $P \cap Q$ is a single vertex σ, the *initial vertex* of L, (4) P and Q are induced paths, and (5) relative to the *rails* $P - \{\sigma\}$ and $Q - \{\sigma\}$ there are infinitely many rungs, and all crosses and embedded fans are trivial. See Figure 1.6(a). The second possibility is a *one-way infinite chain of finite super-clean pinched ladders*, which is a graph H with blocks $L_1, L_2, L_3, \dots$ such that each L_i is a finite super-clean pinched ladder with initial vertex u_i and final vertex u_{i+1}. Thus, $u_2, u_3, u_4, \dots$ are the cutvertices of H, which we call *joining vertices*. We call u_1 the *initial* vertex of H. See Figure 1.6(b).



(a) A one-way infinite super-clean pinched ladder (b) A one-way infinite chain of super-clean pinched ladders

Figure 1.6: Graphs from Theorem 1.10

Our second main result is the following.

Theorem 1.10. *Every infinite 2-edge-connected graph contains one of the following as an induced subgraph: K_∞, an ∞-flower, a one-way infinite chain of finite super-clean pinched ladders, a one-way infinite super-clean pinched ladder, or a member of the family $\Theta_\infty \cup \mathcal{L}_\infty \cup \mathcal{L}_\infty^{\Delta} \cup \mathcal{L}_\infty^{\nabla\Delta}$.*

In Sections 2 and 3, we prove Theorems 1.9 and 1.10, respectively. In Section 4, we describe other unavoidable structures for 2-edge-connected graphs, including subgraphs, topological minors, minors, Eulerian subgraphs, induced topological minors, and induced minors. Section 5 contains some final remarks.

2 Proof of the finite theorem

In this section, we will prove Theorem 1.9.

Lemma 2.1. *For all integers $p, q \geq 3$ there exists an integer $f_{2.1}(p, q)$ such that every 2-edge-connected graph of order at least $f_{2.1}(p, q)$ has a block-cutvertex tree of order at least p or a block of order at least q.*

Proof. Suppose the graph G has blocks $B_1, B_2, \dots, B_k$. Then $|V(G)| = 1 + \sum_{i=1}^{k}(|V(B_i)| - 1)$, so if $k \leq p-1$ and $|V(B_i)| \leq q - 1$ for each i we have at most $1 + (p-1)(q-2)$ vertices. Thus, $|V(G)| \geq f_{2.1}(p, q) = 2 + (p-1)(q-2)$ guarantees that either $k \geq p$ or $|V(B_i)| \geq q$ for some i. □

Lemma 2.1 implies that we can consider two cases: either G has a large block or G has a large block-cutvertex tree. In the former case we apply Theorem 1.7 and Lemma 2.3, and we address the latter in Lemmas 2.4 to 2.6.

We will now define a super-cleaning process to obtain a large super-clean pinched ladder or a long chain of super-clean pinched ladders from a large clean ladder. The idea is to destroy nontrivial embedded fans by deleting the interior vertices of their rims; this creates cutvertices that are the joining vertices of the chain we are building. However, this is complicated by the fact that two nontrivial fans may intersect in various ways. The last rung of one fan may be the first rung of another (with apex on the opposite rail); two crosses that share a pair of vertices give two fans where the apex of each is an interior rim vertex of the other; and two fans with apices on the same rung share a rim edge if the last edge of one crosses the first edge of the other.

Algorithm 2.2 (Super-cleaning process)**.** Let (L, P, Q) be a clean ladder with $P = p_1 p_2 \dots p_\ell$, and $Q = q_1 q_2 \dots q_m$. Each vertex will have one of three statuses: **unknown**, **keep**, or **delete**. Initially we assign status **unknown** to all vertices. The status of each vertex will be changed exactly once, to either **keep** or **delete**. The vertices with final status **keep** determine the induced subgraph we want. Decisions are processed monotonically along P and Q: after each step each of P and Q has a sequence of consecutive vertices whose status is **keep** or **delete**, the last one of which has status **keep** (except possibly after the last step), followed by a sequence of consecutive **unknown** vertices.

To complete the initialization we choose an initial vertex for the chain and make it have degree 2. If p_1 has degree 2, let p_1 be the initial vertex. Otherwise, $\deg_L(p_1) \geq 3$. Similarly, if q_1 has degree 2, then let q_1 be the initial vertex. Otherwise, $\deg_L(q_1) \geq 3$. Since (L, P, Q) is a clean ladder, it follows that $\deg_L(p_1) = \deg_L(q_1) = 3$, otherwise L has a nontrivial cross. In this case, let p_1 be the initial vertex of the chain and assign **delete** to q_1. In all cases the initial vertex now has two **unknown** neighbors, one on P and one on Q. Assign **keep** to the initial vertex and these two neighbors (which become the quasi-initial vertices of the first pinched ladder).

Let $U_Q(p_i)$ be the set of **unknown** neighbors of p_i on Q, and $U_P(q_j)$ the set of **unknown** neighbors of q_j on P. We say p_i is *open* if either p_i is **unknown**, or p_i has status **keep** and $U_Q(p_i) \neq \emptyset$. Similarly, q_j is *open* if either q_j is **unknown**, or q_j has status **keep** and $U_P(q_j) \neq \emptyset$. We say that a rung $p_i q_j$ of L is *open* if neither p_i nor q_j is assigned **delete** or has been designated as the initial vertex or a joining vertex of the chain, and at least one of p_i or q_j is open. A vertex or rung that is not open is *closed.* If p_i is closed, then either p_i is assigned **delete**, or p_i is assigned **keep** and $U_Q(p_i) = \emptyset$. A similar conclusion applies if q_j is closed.

We now describe a single iteration of the algorithm. Assign a *value* $h(p_i q_j) = i + j$ to each rung $p_i q_j$. Since there are no nontrivial crosses, there are at most two rungs with any given value, and any two rungs of the same value form a trivial cross. Select the open rung(s) with the smallest value, if one exists. We will consider three cases. We continue iterating until some vertex has been designated as a final vertex for the chain of super-clean pinched ladders, at which point all vertices will be assigned **keep** or **delete**.

Case 1. Suppose there is only one open rung $p_i q_j$ with smallest value and $\{p_i q_{j+1}, p_{i+1} q_j\}$ is not a cross. We proceed in the following order.

1.1. Assign **keep** to all **unknown** p_a and q_b with $a \leq i$ or $b \leq j$.

1.2. If $i = \ell$, assign **delete** to all q_k with $j < k \leq m$. In this case, $p_i = p_\ell$ is the final vertex in the chain of super-clean pinched ladders, and the algorithm terminates.

1.3. If $j = m$, assign **delete** to all p_k with $i < k \leq \ell$. In this case, $q_j = q_m$ is the final vertex in the chain of super-clean pinched ladders, and the algorithm terminates.

1.4. If $U_Q(p_i) \cup U_P(q_j) = \emptyset$, then $p_i q_j$ is now closed, and we conclude this iteration. We may therefore assume that $U_Q(p_i) \cup U_P(q_j) \neq \emptyset$.

1.5. If $U_Q(p_i) \neq \emptyset$, then p_i is the apex of an embedded fan, which may be trivial or nontrivial. Since p_i and q_j are not vertices in a cross, it follows that $U_P(q_j) = \emptyset$.

(a) If $U_Q(p_i)$ contains a vertex distinct from q_{j+1}, then p_i is the apex of a nontrivial embedded fan. Let q_k be the neighbor of p_i with maximum k, and assign **keep** to q_k and p_{i+1} and **delete** to q_a for $j < a < k$. So, $p_i q_j$ is now closed, and we conclude this iteration. Note that p_i becomes the initial vertex of a new pinched ladder, and hence a joining vertex of the chain we are building; the quasi-initial vertices of the new ladder are p_{i+1} and q_k.

(b) If $U_Q(p_i) = \{q_{j+1}\}$, then p_i is the apex of a trivial embedded fan. We assign **keep** to q_{j+1}. So, $p_i q_j$ is closed, and we conclude this iteration.

1.6. If we have reached this point, $U_Q(p_i) = \emptyset$ and $U_P(q_j) \neq \emptyset$, so q_j is the apex of a fan.

(a) If $U_P(q_j)$ contains a vertex distinct from p_{i+1}, then q_j is the apex of a nontrivial embedded fan. Let p_k be the neighbor of q_j with maximum k, and assign **keep** to p_k and q_{j+1} and **delete** to p_a for $i < a < k$. So, $p_i q_j$ is now closed, and we conclude this iteration. Note that q_j becomes a joining vertex.

(b) If $U_P(q_j) = \{p_{i+1}\}$, then q_j is the apex of a trivial embedded fan. Assign **keep** to p_{i+1}. So, $p_i q_j$ is closed, and we conclude this iteration.

This completes the case that there is only one rung with smallest value and $\{p_i q_{j+1}, q_j p_{i+1}\}$ is not a cross.

Case 2. Suppose that either there is one open rung of smallest value $p_i q_j$ and $\{p_i q_{j+1}, p_{i+1} q_j\}$ is a cross, or there are two open rungs $p_i q_{j+1}$ and $p_{i+1} q_j$ with the same smallest value. Since the only crosses are trivial and rungs of value less then $i + j$ are closed, it follows that $U_Q(p_i) \subseteq \{q_j, q_{j+1}\}$ and $U_P(q_j) \subseteq \{p_i, p_{i+1}\}$. We proceed in the following order.

2.1. Assign **keep** to all **unknown** p_a and q_b with $a \leq i$ or $b \leq j$.

2.2. If $i + 1 = \ell$, assign **delete** to all vertices q_k where $j + 1 \leq k \leq m$. In this case $p_{i+1} = p_\ell$ is the final vertex of the chain, and the algorithm terminates.

2.3. Similarly, if $j + 1 = m$, assign **delete** to all vertices p_k where $i + 1 \leq k \leq \ell$. In this case, $q_{j+1} = q_m$ is the final vertex of the chain, and the algorithm terminates. We may therefore assume that none of the vertices in the cross are final.

2.4. If $U_Q(p_{i+1}) \cup U_P(q_{j+1}) \subseteq \{p_{i+1}, q_{j+1}\}$, then assign **keep** to p_{i+1} and q_{j+1}. Now $p_i q_{j+1}$ and $q_j p_{i+1}$ are closed, as are $p_i q_j$ or $p_{i+1} q_{j+1}$ if they exist, and we conclude this iteration.

2.5. If $U_Q(p_{i+1})$ contains a vertex distinct from q_{j+1}, then there is a nontrivial embedded fan with apex p_{i+1}. Let q_k be the neighbor of p_{i+1} with maximum k, assign **keep** to q_k, p_{i+1}, and p_{i+2}, and assign

delete to q_a for $j < a < k$. Now p_iq_{j+1} and q_jp_{i+1} are closed, as are p_iq_j or $p_{i+1}q_{j+1}$ if they exist, and we conclude this iteration. Note that p_{i+1} becomes a joining vertex.

2.6. If we have reached this point, $U_Q(p_{i+1}) \subseteq \{q_{j+1}\}$ and $U_P(q_{j+1})$ contains a vertex distinct from p_{i+1}, so there is a nontrivial embedded fan with apex q_{j+1}. Let p_k be the neighbor of q_{j+1} with maximum k, assign **keep** to p_k, q_{j+1}, and q_{j+2}, and assign **delete** to p_a for $i < a < k$. Now p_iq_{j+1} and q_jp_{i+1} are closed, as are p_iq_j or $p_{i+1}q_{j+1}$ if they exist, and we conclude this iteration. Note that q_{j+1} becomes a joining vertex.

This completes Case 2.

Case 3. The final case is where there is no open rung. In particular, $p_\ell q_m$ is closed, so neither p_ℓ nor q_m is **unknown**. The last known vertex on each rail of L is always **keep** if the algorithm has not yet terminated. Thus, p_ℓ and q_m are **keep**, but neither has been designated as the final vertex for the chain. This can only happen if both p_ℓ and q_m are quasi-initial vertices for the final pinched ladder, assigned **keep** either during initialization or in 1.5(a), 1.6(a), 2.5, or 2.6. The final pinched ladder is just a triangle. We designate p_ℓ as the final vertex of the chain and terminate the algorithm.

This completes the description of the super-cleaning process.

Let G be the graph obtained by applying the super-cleaning process to (L, P, Q). Every cross in G comes from a cross in (L, P, Q). So, all crosses in G are trivial.

The algorithm guarantees that G has no nontrivial fans as follows. First, observe that every rung of G is also a rung of L, because the rails of each pinched ladder in G are subpaths of the rails P and Q of L. The *initial rung* of a nontrivial embedded fan F in G is the rung of F with smallest value as a rung of L. Suppose e is a rung of L that remains as an edge in G. The algorithm ensures that e is not the initial edge of a nontrivial fan in G because one of the following is true at the end of the step (initialization, or one of the iterations) during which we decide to keep e (by making both of its ends **keep**).

(1) We know that L has no nontrivial fan with e as initial rung, and hence neither does G (p_iq_j in Case 2).
(2) All nontrivial fans in L with e as the initial edge are destroyed during the step (p_iq_j in Case 1, and p_iq_{j+1}, $p_{i+1}q_{j+1}$, $p_{i+1}q_{j+1}$ in Case 2).
(3) The edge e becomes a terminal edge of a new pinched ladder, not a rung, in G (the rung incident with the initial vertex during initialization, p_iq_k in 1.5(a), q_jp_k in 1.6(a), $p_{i+1}q_k$ in 2.5, $q_{j+1}p_k$ in 2.6).
(4) At the end of the step e is either closed, and hence not the initial rung of a fan, or the unique open rung of L of smallest value, and (1) or (2) above will apply to e during the next iteration (p_iq_{j+1} in 1.5(b), q_jp_{i+1} in 1.6(b); also, a rung between the two quasi-initial vertices of the first pinched ladder from the initialization, $p_{i+1}q_k$ in 1.5(a), $q_{j+1}p_k$ in 1.6(a), $p_{i+2}q_k$ in 2.5, and $q_{j+2}p_k$ in 2.6).

Since no rung of L that is kept in G can be the initial rung of a nontrivial fan in G, G has no nontrivial fans.

Since G has no nontrivial crosses and no nontrivial fans, G is a chain of super-clean pinched ladders. Note that this chain may have only one pinched ladder.

We can now prove that if (L, P, Q) is a large clean ladder, then L contains either an r-flower, a large super-clean pinched ladder, or a large chain of super-clean pinched ladders as an induced subgraph.

Lemma 2.3. *Let $r \geq 3$ be an integer. If (L, P, Q) is a clean ladder of order at least $f_{2.3}(r)$, then L contains a super-clean pinched ladder of order at least r, a chain of r super-clean pinched ladders, or an r-flower as an induced subgraph.*

Proof. If $r = 3$, then $f_{2.3}(3) = 3$ satisfies the conditions, so we may assume that $r \geq 4$. We will assume L has none of the induced subgraphs listed above and obtain an upper bound on the order of L. So assume L has no induced subgraph that is a super-clean pinched ladder of order at least r, a chain of r super-clean pinched ladders, or an r-flower. Let H be the chain of pinched super-clean ladders obtained by running the super-cleaning process (Algorithm 2.2) on L.

Consider a vertex $u \in V(P)$ with at least two neighbors $v_1, v_2, \ldots, v_k \in V(Q)$, in order along Q. Let d_i be the distance between v_i and v_{i+1} along Q. If $k \geq 3r - 1$ then the subgraph of L induced by u and the sets $V(Q[v_{3i+1}, v_{3i+2}])$ for $0 \leq i \leq r - 1$ is an r-flower. Therefore, $k \leq 3r - 2$. If $d_i \geq r - 2$ for some i, then the subgraph of L induced by u and $V(Q[v_i, v_{i+1}])$ is a cycle of length at least r, which is a super-clean pinched ladder of order at least r. Therefore, $d_i \leq r - 3$ for each i, $1 \leq i \leq k - 1$. It follows that the embedded fan induced by $\{u, v_1, v_2, \ldots, v_k\}$ has a rim of length at most $(3r - 3)(r - 3)$, and hence has at most $R = (3r - 3)(r - 3) + 1$ rim vertices. A similar argument applies if we start with $u \in V(Q)$ and its neighbors on P.

Now consider the super-cleaning process. We potentially delete vertices of L in three situations: when choosing the initial vertex, when processing a nontrivial embedded fan, and when choosing the final vertex. When we choose the initial vertex we assign **keep** to at least three vertices and **delete** to at most one vertex. When we process a nontrivial fan we assign **keep** to at least two vertices (the quasi-initial vertices of the new pinched ladder whose initial vertex is the apex of the fan) and we assign **delete** to the interior vertices of the rim of the fan, which from above means to at most $R - 2$ vertices. Thus we keep at least the fraction $2/R$ of the vertices to which we assign a new status. When we choose the final vertex of the chain we possibly do not assign **keep** to any vertices, and we assign **delete** to the interior and final rim vertices of a fan whose apex is the designated final vertex of the chain, which means to at most $R - 1$ vertices. We keep all vertices not covered by these three situations. Therefore,

$$|V(H)| \geq (2/R)\left(|V(L)| - 1 - (R - 1)\right) + 3 = (2/R)|V(L)| + 1.$$

Now, by our assumption, H has at most $r - 1$ blocks and each block has order at most $r - 1$, so $|V(H)| \leq 1 + (r-1)(r-2)$. Combining this with the above inequality gives $1 + (r-1)(r-2) \geq (2/R)|V(L)| + 1$ from which $|V(L)| \leq R(r - 1)(r - 2)/2$.

So if L has order at least $f_{2.3}(r) = \max\{3, \lceil (R(r - 1)(r - 2) + 1)/2 \rceil\}$, where $R = (3r - 3)(r - 3) + 1$, then L must have one of the listed induced subgraphs. ☐

A long path in a block-cutvertex tree means that the graph has a long chain of blocks.

Lemma 2.4. *Let t be a positive integer. If the block-cutvertex tree T of a graph G contains a path of order $f_{2.4}(t)$, then G contains an induced chain of at least t blocks.*

Proof. Let $f_{2.4}(t) = 2t$. Let P be an induced path of order $f_{2.4}(t)$ in T. Then alternating vertices of P represent blocks of G. So there are at least t vertices of P representing blocks of G. Since G is simple and 2-edge-connected, each block in the chain contributes at least one unique vertex to the chain of blocks. Hence it has length at least t, and the conclusion follows. □

Lemma 2.5. *Let G be a 2-edge-connected graph with distinct vertices u and v. Then G has a chain of super-clean pinched ladders with initial vertex u and final vertex v as an induced subgraph.*

Proof. Since G is 2-edge-connected there are two edge-disjoint uv-paths. Let P and Q be two such paths such that (1) $|E(P)| + |E(Q)|$ is minimum, and subject to that (2) $|V(P) \cup V(Q)|$ is minimum. If one of P or Q is just the edge uv, then (1) implies that $P \cup Q$ is an induced cycle, which is the required subgraph. So we may assume this is not the case, and then (1) implies that P and Q are induced paths. Vertices of $V(P) \cap V(Q)$ occur along P in the same order as they occur along Q, otherwise we could find two paths with fewer edges, contradicting (1). Let the elements of $V(P) \cap V(Q)$ be $u_1, u_2, \ldots, u_{n+1}$ in order along P (or Q), so that $u_1 = u$ and $u_{n+1} = v$. Any edge from $P(u_i, u_{i+1})$ to $Q(u_j, u_{j+1})$ has $i = j$, otherwise we could find two paths with fewer edges, contradicting (1). Let $P_i = P[u_i, u_{i+1}]$, $Q_i = Q[u_i, u_{i+1}]$, and let L_i be the subgraph induced by $V(P_i) \cup V(Q_i)$. Then (L_i, P_i, Q_i) is a pinched ladder. A nontrivial cross in L_i allows us to find two paths with fewer edges, contradicting (1). A nontrivial embedded fan in L_i allows us to find two paths that either have fewer edges, contradicting (1), or have the same number of edges but one fewer vertex, contradicting (2). Thus, each L_i is a super-clean pinched ladder, and so the subgraph of G induced by $V(P) \cup V(Q)$ is a chain of super-clean pinched ladders, as desired. □

While Lemma 2.5 gives existence of chains of super-clean pinched ladders, it does not give us control over their size, and so we still need the super-cleaning process in Algorithm 2.2. A proof similar to that of Lemma 2.5 but using a pair of internally disjoint paths shows that in a 2-connected graph there is an induced clean pinched ladder ('clean' meaning all crosses are trivial) with given initial and final vertex.

Lemma 2.6. *Let t be a positive integer. If a 2-edge-connected graph G contains an induced subgraph that is a chain of t blocks, then G contains a chain of at least t super-clean pinched ladders as an induced subgraph.*

Proof. Let G' be an induced chain of t blocks in G. Choose a vertex u in the first block of G' that is not a cutvertex of G', and a similar vertex v in the last block of G'. By Lemma 2.5, G', and hence G, has an induced subgraph H that is a chain of super-clean pinched ladders with initial vertex u and final vertex v. Each block of H lies inside a block of G', and H contains an edge of every block of G' since it contains a uv-path. Hence H has at least t blocks. □

We can now prove the finite theorem.

Theorem 1.9. *For every integer $r \geq 3$, there is an integer $f_{1.9}(r)$ such that every 2-edge-connected graph of order at least $f_{1.9}(r)$ contains K_r, an r-flower, a super-clean pinched ladder of order at least r, a chain of r super-clean pinched ladders, or a member of the family Θ_r as an induced subgraph.*

Proof. Let $n_1 = f_{2.3}(r)$, $n_2 = \max\{f_{2.4}(r), f_{1.7}(n_1)\}$, $n_3 = f_{1.6}(n_2)$, and $f_{1.9}(r) = f_{2.1}(n_3, n_2)$. Then $f_{1.9}(r) \geq n_3 \geq n_2 \geq n_1 \geq r$. Note that n_2 contributes to $f_{1.9}(r)$ in two ways because we can get a large block in two ways: as a large block initially, or as a large block coming from a block vertex of high degree in the block-cutvertex tree. Let G be a 2-edge-connected graph of order at least $f_{1.9}(r)$, and let T be the block-cutvertex tree of G.

Lemma 2.1 implies that either G has a block of order at least n_2 or T has order at least n_3. If a block has order at least n_2, then Theorem 1.7 implies that G contains an induced member of $\{K_{n_1}\} \cup \Theta_{n_1} \cup \Lambda_{n_1}$. Then G contains as an induced subgraph either a member of $\{K_r\} \cup \Theta_r$ and the conclusion holds, or a member of Λ_{n_1}, in which case Lemma 2.3 implies that G contains a super-clean pinched ladder of order at least r, a chain of r super-clean pinched ladders, or an r-flower, as desired.

We may therefore assume that G does not have a block of order at least n_2, so T has order at least n_3. Theorem 1.6 implies that T contains as an induced subgraph either K_{1,n_2} or a path of order at least n_2. Suppose that T contains an induced K_{1,n_2}. Since G does not have a block of order at least n_2, it follows that the vertex, say v, of high degree in T is a cutvertex of G. So v is in at least n_2 blocks. Since G is 2-edge-connected, it follows that each block contains an induced cycle containing v. Thus G contains an n_2-flower, and thus an r-flower, and the conclusion follows.

We may therefore assume that T contains a path of order at least $n_2 \geq f_{2.4}(r)$. Lemma 2.4 implies that G contains an induced chain of r blocks. Then Lemma 2.6 implies that G contains an induced chain of at least r super-clean pinched ladders, and the conclusion follows. □

3 Proof of the infinite theorem

In this section we prove Theorem 1.10. This proof will utilize the super-cleaning process described in Section 2. When applied to an infinite slim ladder this produces a sequence of finite induced subgraphs (induced by the vertices with **keep** status after each iteration) whose union is an infinite induced subgraph. The super-cleaning process is deterministic, so we may apply it to infinite graphs without invoking the Axiom of Choice.

First, we prove the following lemma.

Lemma 3.1. *Let G be an infinite 2-edge-connected graph. Then G contains an induced infinite block, a vertex that is in infinitely many blocks, or an induced one-way infinite chain of finite blocks.*

Proof. If G contains an infinite block, then the conclusion follows. We may therefore assume that G does not contain an infinite block. Since G is infinite, it follows that G contains infinitely many finite blocks. Thus, the block-cutvertex tree T of G is infinite, connected, and every vertex of infinite degree corresponds to a cutvertex of G. Since T is bipartite, connected, and infinite, Theorem 1.5 implies that T contains as an induced subgraph either $K_{1,\infty}$ or an induced ray.

If T contains an induced $K_{1,\infty}$, then G contains a cutvertex that is in infinitely many blocks, and the conclusion follows. We may therefore assume that T contains an induced ray. This ray corresponds to an induced one-way infinite chain of finite blocks in G, and the conclusion follows. □

Since members of $\mathcal{F}_\infty$ and $\mathcal{F}_\infty^\Delta$ are not minimally 2-edge-connected, we show that each member contains an induced ∞-flower.

Lemma 3.2. *Each member of one of the families $\mathcal{F}_\infty$ or $\mathcal{F}_\infty^\Delta$ contains an induced ∞-flower.*

Proof. Suppose G is a member of $\mathcal{F}_\infty$, labeled as in Figure 1.2(a). Let R be the rim of G and for each positive integer let Q_i be the uv_i-path all of whose interior vertices have degree 2. The subgraph of G induced by $\bigcup_{i=0}^{\infty} V(Q_{3i+1} \cup R[v_{3i+1}, v_{3i+2}] \cup Q_{3i+2})$ is an ∞-flower.

We may therefore assume that G is a member of $\mathcal{F}_\infty^\Delta$, labeled as in Figure 1.2(b). For each positive integer let Q_i be the uv_i-path all of whose interior vertices have degree 2, and let R be the path $G - \bigcup_{i=1}^{\infty} V(Q_i)$. The subgraph of G induced by $\bigcup_{i=0}^{\infty} V(Q_{2i+1} \cup R[x_{2i+1}, w_{2i+2}] \cup Q_{2i+2})$ is an ∞-flower. □

Lemma 3.3. *An infinite slim ladder contains either a one-way infinite chain of finite super-clean pinched ladders or a one-way infinite super-clean pinched ladder as an induced subgraph.*

Proof. If the infinite slim ladder has only finitely many nontrivial embedded fans, then we can delete an initial part of the infinite slim ladder to destroy all nontrivial embedded fans. We apply the initialization step of the super-cleaning process (Algorithm 2.2) to choose an initial vertex, possibly deleting one vertex in the process, and we then have a one-way infinite super-clean pinched ladder. Otherwise, the infinite slim ladder contains infinitely many nontrivial embedded fans, and the super-cleaning process produces infinitely many joining vertices. Then we have a one-way infinite chain of finite super-clean pinched ladders. □

We can now prove Theorem 1.10.

Theorem 1.10. *Every infinite 2-edge-connected graph contains one of the following as an induced subgraph: K_∞, an ∞-flower, a one-way infinite chain of finite super-clean pinched ladders, a one-way infinite super-clean pinched ladder, or a member of the family $\Theta_\infty \cup \mathcal{L}_\infty \cup \mathcal{L}_\infty^\Delta \cup \mathcal{L}_\infty^{\nabla\Delta}$.*

Proof. Lemma 3.1 implies that G contains either an infinite block, a vertex that is in infinitely many blocks, or a one-way infinite chain of finite blocks.

If G contains an infinite block, then Theorem 1.8 implies that G contains one of the following as an induced subgraph: K_∞, an infinite slim ladder, or a member of one of the following families: Θ_∞, $\mathcal{F}_\infty$, $\mathcal{F}_\infty^\Delta$, $\mathcal{L}_\infty$, $\mathcal{L}_\infty^\Delta$, or $\mathcal{L}_\infty^{\nabla\Delta}$. If G contains K_∞ or a member of Θ_∞, $\mathcal{L}_\infty$, $\mathcal{L}_\infty^\Delta$, or $\mathcal{L}_\infty^{\nabla\Delta}$, then the theorem holds. If G contains a member of $\mathcal{F}_\infty$ or $\mathcal{F}_\infty^\Delta$, then Lemma 3.2 implies that G contains an ∞-flower, and the conclusion follows. If G contains an infinite slim ladder, then Lemma 3.3 implies that G contains either an infinite chain of finite super-clean pinched ladders or a one-way infinite super-clean pinched ladder, and the conclusion follows.

We may therefore assume that G does not contain an infinite block. Suppose that G has a cutvertex v that is in infinitely many blocks B_1, B_2, $\ldots$. Let C_i be a shortest cycle in B_i that contains v. Then $\bigcup_{i\in\mathbb{N}} C_i$ is an ∞-flower, and the conclusion follows.

We may therefore assume that G is locally finite and has a one-way infinite chain of finite blocks, B'_1, $B'_2, \ldots$. Since each block is finite, we may use Lemma 2.5 to find a chain of super-clean ladders between the joining vertices of each block B'_i for $i \geq 2$, and the union of these is an infinite chain of finite super-clean pinched ladders, and the conclusion follows. □

4 Other unavoidable structures

In this section we apply our results on unavoidable induced subgraphs (Theorems 1.9 and 1.10) to provide results on other unavoidable substructures. The proofs are mostly straightforward, so we leave them to the reader, giving only occasional comments.

4.1 Unavoidable subgraphs, topological minors, and minors

In this subsection we state results on unavoidable subgraphs, topological minors, and minors in 2-edge-connected graphs. We assume the reader is familiar with these orderings. Our results also imply the existence of unavoidable Eulerian subgraphs.

Recall that $\mathcal{K}_{2,r}$ is the subset of Θ_r consisting of subdivisions of $K_{2,r}$, and we define $\mathcal{K}_{2,\infty}$ similarly. An r-flower or ∞-flower is *triangular* if each of its cycles is a triangle. Theorem 1.9 immediately implies the following.

Theorem 4.1. *For every integer $r \geq 3$, there is an integer $f_{4.1}(r)$ such that every 2-edge-connected graph of order at least $f_{4.1}(r)$ has the following.*

(a) *A subgraph that is an r-flower, a cycle of order at least r, a chain of r cycles, or a member of the family $\mathcal{K}_{2,r}$.*
(b) *A topological minor (and hence minor) that is a triangular r-flower, C_r, a chain of r triangles, or $K_{2,r}$.*

We can also obtain a result for multigraphs from Theorem 4.1 by subdividing each edge of a multigraph to get a simple graph, then translating the unavoidable substructures into multigraph substructures. For multigraphs the number of edges is the appropriate measure of largeness, rather than order. The topological minor and minor orderings are slightly different for multigraphs, because we do not necessarily delete parallel edges created by contractions.

Let D_r be the r-edge dipole consisting of r parallel edges between two vertices, let $S_{2,r}$ be the graph obtained from $K_{1,r}$ by doubling each edge, i.e., replacing each edge with a parallel class of two edges, and let $P_{2,r}$ be the graph obtained from P_r by doubling each edge. We obtain D_r, $S_{2,r}$, and $P_{2,r}$ from $K_{2,r}$, a triangular r-flower, and a chain of r triangles, respectively, by contracting vertices of degree 2.

Corollary 4.2. *For every integer $r \geq 2$, there is an integer $f_{4.2}(r)$ such that every 2-edge-connected multigraph with at least $f_{4.2}(r)$ edges has the following.*

(a) *A subgraph that is an r-flower, a cycle of order at least r, a chain of r cycles, D_r, or a member of the family $\mathcal{K}_{2,r}$.*
(b) *A topological minor (and hence minor) that is $S_{2,r}$, C_r, $P_{2,r}$, or D_r.*

We now consider Eulerian subgraphs. Motivated by a question involving rainbow walks between all pairs of vertices in a graph, Goddard and LaVey [13] proved that for each fixed t, there are only finitely many 2-edge-connected graphs whose maximum closed trail length is at most t. This can be restated as follows.

Theorem 4.3 (Goddard and LaVey [13, Lemma 4.1])**.** *For every integer $s \geq 2$, there is an integer $f_{4.3}(s)$ such that every 2-edge-connected graph of order at least $f_{4.3}(s)$ has an Eulerian subgraph of order at least s.*

By taking Theorem 4.1(a) with even $r = 2s$, we obtain the following strengthening of Theorem 4.3.

Corollary 4.4. *For every integer $s \geq 2$ there is an integer $f_{4.4}(s)$ such that every 2-edge-connected graph of order at least $f_{4.4}(s)$ has one of the following Eulerian subgraphs: a $2s$-flower, a cycle of order at least s, a chain of s cycles, or a member of the family $\mathcal{K}_{2,2s}$.*

There is also a multigraph version of Corollary 4.4, which uses number of edges rather than order, and includes D_{2s} as an additional possible Eulerian subgraph.

We also get a result for infinite graphs from Theorem 1.10. Recall that $\mathcal{L}_\infty$ consists of subdivisions of the infinite ladder L_∞ where each rung is subdivided at least once. Let $\mathcal{L}^0_\infty$ be the family of graphs obtained from L_∞ by subdividing each of the rail edges an arbitrary number, possibly zero, of times (but without subdividing any rungs). Note that L_∞ is a member of $\mathcal{L}^0_\infty$. In the infinite case the minor result is different from the topological minor result, because L_∞ has a triangular ∞-flower (and also a one-way infinite chain of triangles) as a minor.

Theorem 4.5. *Every infinite 2-edge-connected graph has the following.*

(a) *A subgraph that is an ∞-flower, a one-way infinite chain of cycles, or a member of $\mathcal{K}_{2,\infty} \cup \mathcal{L}^0_\infty \cup \mathcal{L}_\infty$.*
(b) *A topological minor that is a triangular ∞-flower, a one-way infinite chain of triangles, L_∞, or $K_{2,\infty}$.*
(c) *A minor that is a triangular ∞-flower, a one-way infinite chain of triangles, or $K_{2,\infty}$.*

If we consider multigraphs instead of simple graphs, we get the following result by a similar process to the finite case. Note that a multigraph is infinite if it has an infinite number of edges or vertices. The multigraph $P_{2,\infty}$ is obtained from P_∞ by doubling each edge, $S_{2,\infty}$ is obtained from $K_{1,\infty}$ by doubling each edge, and D_∞ denotes a dipole with a countably infinite number of edges.

Corollary 4.6. *Every infinite 2-edge-connected multigraph has the following.*

(a) *A subgraph that is an ∞-flower, a one-way infinite chain of cycles, D_∞, or a member of $\mathcal{K}_{2,\infty} \cup \mathcal{L}^0_\infty \cup \mathcal{L}_\infty$.*
(b) *A topological minor that is $S_{2,\infty}$, $P_{2,\infty}$, L_∞, or D_∞.*
(c) *A minor that is $S_{2,\infty}$, $P_{2,\infty}$, or D_∞.*

An infinite graph is *Eulerian* if it has a two-way infinite trail that uses every edge. We can guarantee the existence of an infinite Eulerian subgraph with restricted structure.

Corollary 4.7. *Every infinite 2-edge-connected graph contains an Eulerian subgraph that is a two-way infinite path, an ∞-flower, an element of $\mathcal{K}_{2,\infty}$, or a one-way infinite chain of cycles.*

For multigraphs we must add D_∞ as an additional possibility. Note that a two-way infinite path is not 2-edge-connected, but it is the infinite analogue of a cycle because it is 2-regular and connected. We must include the two-way infinite path because it is the only infinite Eulerian subgraph of some 2-edge-connected infinite ladders (including all members of $\mathcal{L}^0_\infty$).

4.2 Unavoidable induced topological minors and induced minors

In this subsection we consider two less common orderings, the induced topological minor and induced minor orderings (for simple graphs only, not for multigraphs). We say that H is an *induced minor* of G if a graph isomorphic to H can be obtained from G by vertex deletions and edge contractions. If, moreover, each contracted edge is incident with a vertex of degree 2 then we say H is an *induced topological minor* of G, which is equivalent to an induced subgraph of G being isomorphic to a subdivision of H. The orderings induced subgraph, induced topological minor, induced minor, and minor form a ranked sequence of orderings, from strongest to weakest. Also, the induced topological minor ordering is stronger than the topological minor ordering, giving a third ranked sequence of orderings, namely induced subgraph, induced topological minor, topological minor, minor, again from strongest to weakest.

A super-clean pinched ladder is *(topologically) irreducible* if it is a triangle or the only vertices of degree 2 are the initial and final vertices. Super-clean pinched ladders in a chain can be made irreducible by contracting edges incident with vertices of degree 2, and can be reduced to triangles by contracting arbitrary edges. However, we cannot make single large super-clean pinched ladders irreducible, as that may decrease their order in an uncontrolled way. But if we are allowed to contract arbitrary edges, we can contract both edges in the span of each cross, which reduces the number of vertices by a factor of at most $\frac{1}{2}$, and does not introduce any nontrivial embedded fans. Therefore, Theorem 1.9 yields the following.

Theorem 4.8. *For every integer $r \geq 3$, there is an integer $f_{4.8}(r)$ such that every 2-edge-connected graph of order at least $f_{4.8}(r)$ has the following.*

(a) *An induced topological minor that is K_r, a triangular r-flower, a super-clean pinched ladder of order at least r, a chain of r irreducible super-clean pinched ladders, or $K_{1,1,r}$.*
(b) *An induced minor that is K_r, a triangular r-flower, a super-clean pinched ladder with no crosses of order at least r, a chain of r triangles, or $K_{1,1,r}$.*

In the infinite case we say a one-way infinite super-clean pinched ladder is *(topologically) irreducible* if the only vertex of degree 2 is the initial vertex. An infinite super-clean pinched ladder has infinitely many rungs, so we can make it irreducible by contracting rail edges incident with vertices of degree 2 without affecting the fact that we have an infinite graph. Define L^{Δ}_∞ and $L^{\nabla\Delta}_\infty$ to be the graphs obtained by replacing all vertices on one or both rails of L_∞, respectively, by triangles. Then all members of $\mathcal{L}_\infty \cup \mathcal{L}^{\Delta}_\infty \cup \mathcal{L}^{\nabla\Delta}_\infty$ are subdivisions of L_∞, L^{Δ}_∞, or $L^{\nabla\Delta}_\infty$. Moreover, each of L_∞, L^{Δ}_∞, $L^{\nabla\Delta}_\infty$, or an irreducible one-way infinite super-clean pinched ladder has a triangular ∞-flower as an induced minor. Thus, Theorem 1.10 gives the following.

Theorem 4.9. *Every infinite 2-edge-connected graph has the following.*

(a) *An induced topological minor that is K_∞, a triangular ∞-flower, a one-way infinite chain of irreducible super-clean pinched ladders, an irreducible one-way infinite super-clean pinched ladder, $K_{1,1,\infty}$, L_∞, L_∞^Δ, or $L_\infty^{\nabla\Delta}$.*

(b) *An induced minor that is K_∞, a triangular ∞-flower, a one-way infinite chain of triangles, or $K_{1,1,\infty}$.*

5 Conclusion

The next natural question would be to determine the unavoidable induced subgraphs, or even unavoidable subgraphs, for 3-edge-connected or 3-connected graphs. However, these sets cannot have a simple structure. Consider an arbitrary 3-connected (equivalently 3-edge-connected) cubic graph G. The deletion of a vertex or edge in G results in a graph that is not 3-edge-connected because there would be at least one vertex of degree two in $G - v$. Thus, the list of unavoidable 3-edge-connected (or 3-connected) induced subgraphs or subgraphs must include every large cubic graph.

There is another ordering of graphs that is a weakening of the induced subgraph, subgraph and topological minor orderings, but incomparable with the minor ordering, namely the *(weak) immersion* ordering. For immersions it makes most sense to deal with multigraphs and consider edge-connectivity rather than (vertex-)connectivity. It is easy to show that the only large unavoidable immersions for 2-edge-connected multigraphs are cycles C_r (this follows from our Corollary 4.2(b)). Barnes [3] determined the unavoidable immersions for 3-edge-connected multigraphs, and Ding and Qualls [12] determined them for 4-edge-connected multigraphs. Ding and Qualls [private communication] have also determined the unavoidable immersions for 2- and 3-edge-connected infinite multigraphs. We note that the unavoidable immersions for 2-edge-connected infinite multigraphs (namely $S_{2,\infty}$, $P_{2,\infty}$, and L_∞) can also be obtained from our Corollary 4.6(b).

Unavoidable substructure results for matroids, or involving orderings other than those we have already discussed, or involving different notions of connectivity, are also known, and we mention a few of these. Ding, Oxley, Oporowski, and Vertigan [9, 10] determined the unavoidable large minors for 3-connected binary and general matroids, respectively. There is a *parallel minor* ordering for both graphs and matroids that strengthens the minor ordering in a different direction from topological minors. C. Chun, Ding, Oporowski, and Vertigan [5] determined the unavoidable parallel minors for k-connected graphs for $k \leq 3$, and for internally 4-connected graphs, and C. Chun and Oxley [6] determined the unavoidable parallel minors for 3-connected regular matroids. C. Chun and Ding [4] obtained results on large unavoidable topological minors and parallel minors in infinite graphs based on the idea of 'loose connectivity'.

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