Uniqueness of Impartial Edge-colourings

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ABSTRACT

A (*proper*) *k-edge-colouring* of a multigraph assigns one of k colours to each edge, so that no two edges of the same colour are incident. Such a colouring is *impartial* if the numbers of edges of any two colours differ by at most 1. This paper considers the question of which multigraphs have an impartial k-edge-colouring which is unique (up to renaming of colours). It is shown that, for $k \geq 4$, no multigraph with at least $k+3$ edges, and no graph with at least $k+1$ edges, has a unique impartial k-edge-colouring. The proof relies on a characterization of multigraphs with unique k-edge-colourings for $k \geq 4$, which follows from A. G. Thomason's 1978 result on uniquely edge-colourable graphs.

1. Definitions and notation.

The purpose of this paper is to examine the uniqueness of certain edge-colourings of multigraphs. A multigraph G is a triple $(V(G), E(G), \psi_G)$, where $V(G)$ is a finite set of *vertices*, $E(G)$ is a finite set of *edges*, and ψ_G is an *incidence function* mapping each edge to a 2-element subset of $V(G)$. Two edges e, f are *parallel* if $\psi_G(e) = \psi_G(f)$. For any $u, v \in V(G)$, let uv denote ${\psi_G}^{-1}(\{u, v\})$, the set of edges joining u to v. If $|uv| = 1$ then uv (or its single element) is called a *simple edge*; if $|uv| \geq 2$ then uv is a *multiple edge*. A multigraph is *simple* if it has no multiple edges; simple multigraphs are just called graphs. Every multigraph has an *underlying graph* obtained by identifying all elements of each multiple edge.

The word *subgraph* will be used to mean a submultigraph. A subgraph H of a multigraph G is *isolated* if no vertex of H is adjacent to a vertex in $V(G) - V(H)$; isolated subgraphs are unions of components of G. G is *basic* if it has no isolated vertices. Since isolated vertices do not affect edge-colouring properties, we shall often restrict our attention to basic multigraphs.

The *order* of a multigraph G is $\nu(G) = |V(G)|$, and the *size* of G is $\epsilon(G) = |E(G)|$. For $v \in V(G)$, let $\delta(v) = \{e \in E(G)$: e is incident with $v\}$, and $N(v) = \{u \in V(G)$: u is adjacent to $v\}$. If u, v are nonadjacent vertices of G, then $G(u{=}v)$ denotes the multigraph obtained by identifying u and v.

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A *(proper) k-edge-colouring* of a multigraph G is a function $\gamma: E(G) \to \Gamma$, $|\Gamma| = k$, such that $\gamma(e) \neq \gamma(f)$ for any two edges e, f incident with a common vertex. Together G and γ form a *k-edge-coloured multigraph* $G[\gamma]$. The *chromatic index* of G, denoted $\chi'(G)$, is the least integer k for which G has a k-edge-colouring. Let $L(x_1, ..., x_n; \gamma) = \gamma^{-1}(\{x_1, ..., x_n\})$ denote the set of edges coloured $x_1, x_2, ..., x_{n-1}$ or x_n under γ. Let $l(x_1, ..., x_n; \gamma) = |L(x_1, ..., x_n; \gamma)|$ be the number of edges coloured $x_1, x_2, ..., x_{n-1}$ or x_n under γ, and let $G[x_1, ..., x_n; \gamma]$ be the subgraph of G induced by $L(x_1, ..., x_n; \gamma)$. The colouring γ is said to be *impartial* if $|l(x; \gamma) - l(y; \gamma)| \leq 1$ for all $x, y \in \Gamma$.

Given an edge-colouring γ of a multigraph G, a vertex v of G is said to be either x^0 or x^1 under γ if it has respectively 0 or 1 edges coloured x incident with it. If v is both x^i and y^j, where $i, j \in \{0, 1\}$, then v is said to be $x^i y^j$.

If γ is any edge-colouring of G, then for any two colours x, y the subgraph $G[x,y; \gamma]$ must consist of components which are paths or even cycles, whose edges are alternately coloured x and y. These components may therefore be described as *odd* or *even xy-paths*, or *even xy-cycles*, where the odd or even refers to the number of edges. An odd xy-path must have one more edge of one colour than the other; if there is one more x than y edge it is called *x-dominated*. If H is an isolated subgraph of $G[x,y; \gamma]$, then we may exchange the colours x, y on the edges of H to obtain a new edge-colouring; this operation is referred to as *inversion* of H.

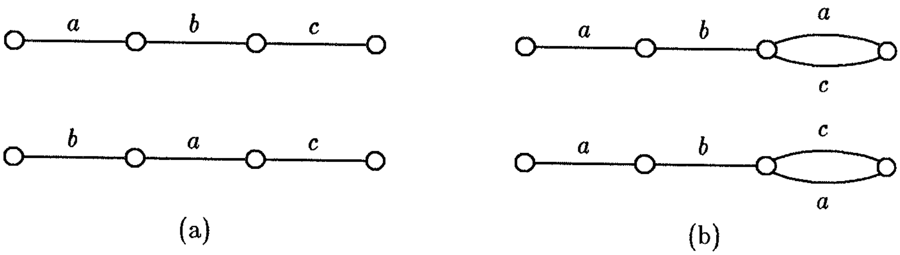

Figure 1.1

Two k-edge-colourings $\gamma_1: E(G) \to \Gamma_1$ and $\gamma_2: E(G) \to \Gamma_2$ are *equivalent* if $\gamma_1(e) = \gamma_1(f)$ if and only if $\gamma_2(e) = \gamma_2(f)$, for all $e, f \in E(G)$. Note that the two colourings in Figure 1.1 (a) are equivalent, but the two in Figure 1.1 (b) are not. Whenever the word "unique" (or its derivatives) is used to describe edge-colourings, we always mean unique up to equivalence. This paper considers the uniqueness of impartial k-edge-colourings of multigraphs for $k \geq 4$. This subject is dealt with in Section 3, after some preliminaries in Section 2 regarding impartial edge-colourings and the uniqueness of general k-edge-colourings.

2. Impartial and unique edge-colourings.

Impartial edge-colourings arise naturally in certain scheduling problems. Their existence was first investigated by de Werra and McDiarmid, who independently proved the following result.

Lemma 2.1. ([2], [6]): *Let G be a multigraph with a k-edge-colouring γ. Then there exists a sequence of k-edge-colourings $\gamma = \gamma_0, \gamma_1, \ldots, \gamma_s$ such that*

(a) *for each i, $0 \leq i \leq s-1$, γ_{i+1} is obtained by inverting an odd $x_i y_i$-path of $G[\gamma_i]$;*

(b) *γ_s is impartial, but no γ_i is impartial for $i < s$.*

Proof. If γ is impartial there is nothing to prove, so assume that $\gamma_0 = \gamma$ is not impartial. Choose colours x_0, y_0 so that $l(x_0; \gamma_0)$ is as large as possible, and $l(y_0; \gamma_0)$ is as small as possible; since γ_0 is not impartial, $l(x_0; \gamma_0) > \epsilon(G)/k > l(y_0; \gamma_0)$. Because there are more x_0 than y_0 edges, some component of $G[x_0, y_0; \gamma_0]$ must be an x_0-dominated odd $x_0 y_0$-path. Inverting this path gives a new k-edge-colouring γ_1 with

$$l(x_0; \gamma_1) = l(x_0; \gamma_0) - 1, \qquad l(y_0; \gamma_1) = l(y_0; \gamma_0) + 1.$$

Thus, γ_1 is closer to being impartial (in a way which can be made rigorous) than γ_0. If γ_1 is not impartial, we may take x_1, y_1 with $l(x_1; \gamma_1)$ maximum and $l(y_1; \gamma_1)$ minimum, and invert an x_1-dominated odd $x_1 y_1$-path to obtain γ_2, which is even closer to impartiality. Continuing in this fashion, we arrive at an impartial colouring γ_s after a finite number of steps.

Now consider unique k-edge-colourings. Note that if a multigraph G has a unique k-edge-colouring γ, then γ must be impartial, for by Lemma 2.1 there always exists an impartial k-edge-colouring if any k-edge-colouring exists.

Uniquely edge-colourable graphs have been studied by various people. It is easy to see that a multigraph G has a unique 1-edge-colouring if and only if it is sK_2, for some $s \geq 1$, and a unique 2-edge-colouring if and only if it is a path or an even cycle. Not very much is known about graphs with unique 3-edge-colourings, although infinite families of 3-regular uniquely 3-edge-colourable graphs have been found. A very brief survey of work on this topic before 1978 appears in [4]; more recent papers are [1], [3] and [5]. For $k \geq 4$, the following result on uniquely k-edge-colourable graphs was published by Thomason in 1978.

Theorem 2.2. ([4]): *Let G be a basic graph with $k = \chi'(G) \geq 4$. Then G has a unique k-edge-colouring if and only if G is isomorphic to $K_{1,k}$.*

Note that this result only deals with graphs G having $k = \chi'(G)$. In order to characterise all graphs with a unique k-edge-colouring, $k \geq 4$, the following two lemmas can be used.

Lemma 2.3. *Let G be a basic multigraph with $\epsilon(G) \leq k$. Then G has a unique k-edge-colouring if and only if the underlying graph of G is K_3 or $K_{1,s}$ for some s.*

Proof. Suppose G has a unique k-edge-colouring. If there are edges e, f of G which are not incident, then G can be given two inequivalent k-edge-colourings, one in which e and f receive the same colour, and one in which they do not. Therefore, any two edges of G are incident, and hence the underlying graph of G is either K_3 or $K_{1,s}$ for some s.

Conversely, all multigraphs with k or fewer edges and underlying graph K_3 or $K_{1,s}$ have a unique k-edge-colouring.

Lemma 2.4. *Let G be a multigraph with a unique k-edge-colouring, and suppose that $\epsilon(G) \geq k$. Then $\chi'(G) = k$.*

Proof. By the remarks following Lemma 2.1, the unique k-edge-colouring γ of G must be impartial. Since $\epsilon(G) \geq k$, every colour therefore appears on at least one edge. Therefore, for any $k' < k$, G has no k'-edge-colouring, for such a colouring would yield a k-edge-colouring of G in which $k - k'$ colours appear on no edges. Hence $\chi'(G) = k$.

From these lemmas and Theorem 2.2 we can deduce the following characterisation of graphs with unique k-edge-colourings, for $k \geq 4$.

Corollary 2.5. *G is a basic graph with a unique k-edge-colouring, $k \geq 4$, if and only if G is isomorphic to either K_3 or $K_{1,s}$, $1 \leq s \leq k$.*

Proof. If $\epsilon(G) \leq k$ then this holds by Lemma 2.3. If $\epsilon(G) \geq k$, then $\chi'(G) = k$ by Lemma 2.4, and the theorem holds by Theorem 2.2.

Thus we have, for $k \geq 4$, a complete description of all graphs with a unique k-edge-colouring. All such colourings must be impartial. Therefore, as a generalisation it is natural to ask for a characterisation of all graphs having a unique impartial k-edge-colouring. In Section 3 we show that, apart from the obvious case of graphs with k or fewer edges, there are no graphs with unique impartial k-edge-colourings for $k \geq 4$. However, the proofs in Section 3 require us to examine not just graphs, but also multigraphs. We need a characterisation of multigraphs with unique k-edge-colourings. This is given by Corollary 2.5 in conjunction with the following proposition.

Proposition 2.6. *Let G be a basic nonsimple multigraph, and $k \geq 1$. Then G has a unique k-edge-colouring if and only if one of the following is true.*

(a) $\epsilon(G) \leq k$ *and the underlying graph of G is K_3 or $K_{1,s}$ for some s;*

(b) $\epsilon(G) = k+1$ *and G is as shown in Figure 2.1 (a), (b) or (c);*

(c) $\epsilon(G) = k+2$ *and G is as shown in Figure 2.1 (d) or (e).*

(Note that in Figure 2.1, each pair of parallel edges joined by dots represents two or more parallel edges.)

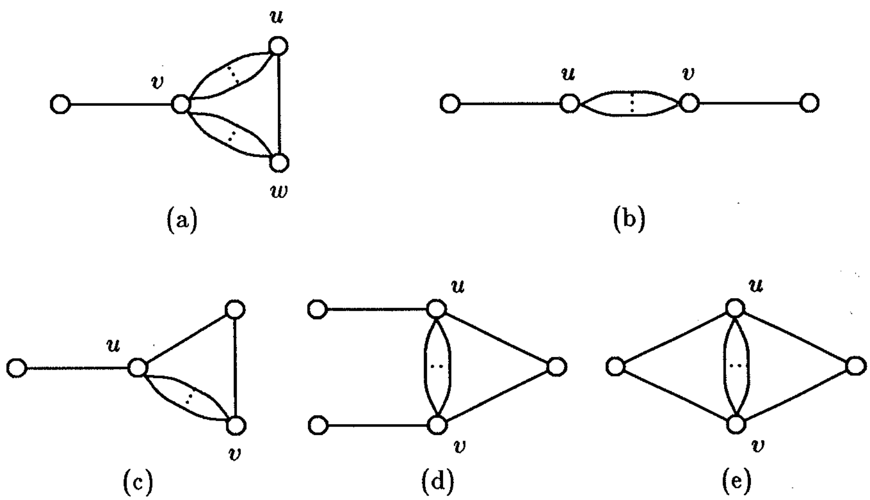

Figure 2.1

Proof. Suppose that G has a unique k-edge-colouring. If $\epsilon(G) \leq k$ then, by Lemma 2.3, (a) holds. Therefore, assume that $\epsilon(G) \geq k+1$. By Lemma 2.4, $\chi'(G) = k < \epsilon(G)$. Let γ be the unique k-edge-colouring of G.

Suppose that xy is any multiple edge of G. Let $e \in xy$, and $\gamma(e) = a$. Then no other edge may be coloured a under γ, because if one were, then another, inequivalent, k-edge-colouring could be obtained by exchanging the colours of e and some other member of xy. Thus, all edges must be incident to at least one of x or y: if some edge f were not, then we could recolour it with a to obtain a k-edge-colouring inequivalent to γ.

Since G is nonsimple there is some multiple edge uv. From above, all edges are incident with u or v. Suppose first that G has another multiple edge; without loss of generality we may assume it to be vw. Then every edge not in $uv \cup vw$ must be either incident with v or a member of uw, to satisfy the requirement that it be incident with at least one end of each multiple edge. Since $\chi'(G) < \epsilon(G)$, there must be at least one edge in

$\delta(v) - uv - vw$, and at least one edge in uw. However, since γ is unique, there must be exactly one edge in each of $\delta(v) - uv - vw$ and uw, and G is as in Figure 2.1 (a).

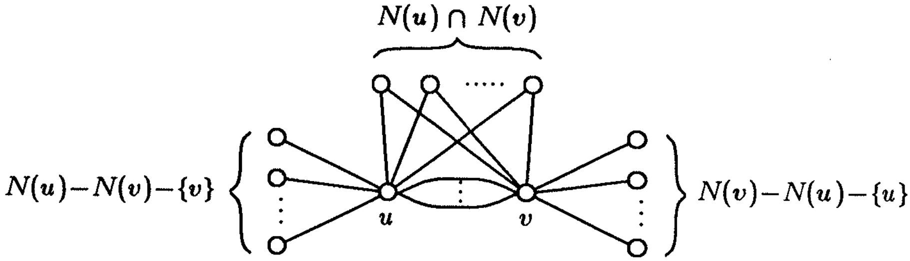

Figure 2.2

Now suppose that uv is the only multiple edge of G. G must be as shown in Figure 2.2. Let $p = |N(u) \cap N(v)|$, $q = |N(u) - N(v) - \{v\}|$ and $r = |N(v) - N(u) - \{u\}|$. Without loss of generality, assume that $q \geq r$. Recall that $k = \chi'(G)$, and that $\chi'(G) < \epsilon(G)$. If $p = 0$, then $\chi'(G) = \epsilon(G)$ if either of q or r is 0, and so $q, r \geq 1$; but there are two inequivalent $\chi'(G)$-edge-colourings if either $q \geq 2$ or $r \geq 2$. Hence when $p = 0$, $q = r = 1$, and G is as in Figure 2.1 (b). If $p = 1$, then $\chi'(G) = \epsilon(G)$ if $q = r = 0$, and so $q \geq 1$; if $q \geq 2$ then there are two inequivalent $\chi'(G)$-edge-colourings. Thus when $p = 1$, $q = 1$ and $r = 0$ or 1, and so G is as in Figure 2.1 (c) or (d). If $p = 2$, then there are inequivalent $\chi'(G)$-edge-colourings unless $q = r = 0$, showing that G is as in Figure 2.1 (e). If $p \geq 3$ then G does not have a unique $\chi'(G)$-edge-colouring.

To complete the proof of the proposition, note that all the multigraphs described in (a), (b) and (c) do have a unique k-edge-colouring.

For each $k \geq 1$, let X_k denote the set of multigraphs described in (b) and (c) of Proposition 2.6 and depicted in Figure 2.1. Corollary 2.5 and Proposition 2.6 together prove the following.

Theorem 2.7. *Suppose that $k \geq 4$, and that G is a basic multigraph with a unique k-edge-colouring. Then either*

(a) *$\epsilon(G) \leq k$ and the underlying graph of G is K_3 or $K_{1,s}$ for some s; or*

(b) *$G \in X_k$.*

3. Unique impartial edge-colourings.

It is clear that any multigraph with k or fewer edges must have a unique impartial k-edge-colouring, since impartiality requires that each edge have a different colour. Also, since the multigraphs in X_k have a unique k-edge-colouring, they have a unique impartial k-edge-colouring. In this section we show that for $k \geq 4$ there are no other basic multigraphs with a unique impartial k-edge-colouring.

Let U_k denote the set of all basic multigraphs with a unique impartial k-edge-colouring. Note that $U_1 = \{sK_2: s \geq 1\}$ and that U_2 consists of all paths, even cycles and unions of two disjoint odd paths. Nothing is known about U_3 except that it contains all basic multigraphs with 3 or fewer edges, and all basic multigraphs with a unique 3-edge-colouring.

Lemma 3.1. *Suppose that $k \geq 4$. Let G be a multigraph such that $\epsilon(G) \geq k+1$ and $G \in U_k - X_k$. Then G has impartial and nonimpartial k-edge-colourings α and β respectively, which differ by an odd path inversion.*

Proof. Since $\epsilon(G) \geq k+1$ and $G \notin X_k$, by Theorem 2.7 G has at least two inequivalent k-edge-colourings. Since $G \in U_k$, at least one of these must be nonimpartial. Given such a nonimpartial γ, by Lemma 2.1 there exists a sequence of k-edge-colourings $\gamma = \gamma_0, \gamma_1, \ldots, \gamma_{s-1}, \gamma_s$ such that each γ_i and γ_{i+1} differ by an odd path inversion, $\alpha = \gamma_s$ is impartial and $\gamma_{s-1} = \beta$ is nonimpartial.

Our argument proceeds by examination of three distinct cases, depending on whether $\epsilon(G) \equiv 1$, 2, or anything else, modulo k. For the first two cases we require some results about 3-edge-colourings.

Lemma 3.2. *Let G be a multigraph with $\epsilon(G) = 3s+1$, $s \geq 1$. If G has a 3-edge-colouring β with*

$$l(a;\beta) = l(b;\beta) = s+1, \quad l(c;\beta) = s-1$$

then $G \notin U_3$.

Proof. Since $l(a;\beta) > l(c;\beta)$, there must be an a-dominated odd ac-path in $G[\beta]$. By inverting this path we obtain an impartial 3-edge-colouring α with

$$l(a;\alpha) = l(c;\alpha) = s, \quad l(b;\alpha) = s+1.$$

Also, by inverting a b-dominated odd bc-path in $G[\beta]$ we get another impartial 3-edge-colouring α' with

$$l(a;\alpha') = s+1, \quad l(b;\alpha') = l(c;\alpha') = s.$$

If α and α' are equivalent, then $L(b;\alpha)$ must be the same as $L(a;\alpha')$. But $L(b;\alpha) = L(b;\beta)$ while $L(a;\alpha') = L(a;\beta)$, and since $s \geq 1$ these sets are nonempty. Therefore $L(b;\alpha) \neq L(a;\alpha')$, α and α' are inequivalent, and $G \notin U_3$.

Lemma 3.3. *Let G be a multigraph with $\epsilon(G) = 3s+2$, $s \geq 1$. If G has a 3-edge-colouring β with*

$$l(a;\beta) = s+2, \quad l(b;\beta) = l(c;\beta) = s$$

then $G \notin U_3$.

Proof. By inverting an a-dominated odd ab-path in $G[\beta]$, we obtain an impartial 3-edge-colouring α with

$$l(a;\alpha) = l(b;\alpha) = s+1, \quad l(c;\alpha) = s.$$

Also, by inverting an a-dominated odd ac-path in $G[\beta]$ we get another impartial 3-edge-colouring α' with

$$l(a;\alpha') = l(c;\alpha') = s+1, \quad l(b;\alpha') = s.$$

If α and α' are equivalent, then $L(c;\alpha)$ must be the same as $L(b;\alpha')$. But $L(c;\alpha) = L(c;\beta)$ while $L(b;\alpha') = L(b;\beta)$, and since $s \geq 1$ these sets are nonempty. Therefore $L(c;\alpha) \neq L(b;\alpha')$, α and α' are inequivalent, and $G \notin U_3$.

Lemmas 3.2 and 3.3 will be applied to 3-edge-colourings derived from certain k-edge-colourings. For $k \geq 3$ a k-edge-colouring β is said to be (a, b, c)*-semiimpartial* if for some $s \geq 1$,

(a) $l(x;\beta) = s$ or $s+1$ for all colours $x \neq a, b$ or c; and

(b) either

  (i) $l(a;\beta) = l(b;\beta) = s+1$, $l(c;\beta) = s-1$; or

  (ii) $l(a;\beta) = s+2$, $l(b;\beta) = l(c;\beta) = s$.

Corollary 3.4. *Suppose that* $k \geq 3$. *Let* G *be a multigraph with a* k*-edge-colouring* β *which is* (a, b, c)*-semiimpartial for some colours* a, b, c. *Then* $G \notin U_k$.

Proof. Since β is (a, b, c)-semiimpartial, the restriction of β to $G[a,b,c;\beta]$ satisfies the conditions of either Lemma 3.2 or Lemma 3.3. Thus $G[a,b,c;\beta]$ has two inequivalent impartial 3-edge-colourings γ and γ'; note that for any $x \in \{a, b, c\}$, $l(x;\gamma)$ and $l(x;\gamma')$ are either s or $s+1$. Define k-edge-colourings α, α' of G by

$$\alpha(e) = \begin{cases} \gamma(e) & e \in L(a,b,c;\beta), \\ \beta(e) & e \notin L(a,b,c;\beta), \end{cases} \qquad \alpha'(e) = \begin{cases} \gamma'(e) & e \in L(a,b,c;\beta), \\ \beta(e) & e \notin L(a,b,c;\beta). \end{cases}$$

Then α and α' are inequivalent k-edge-colourings of G, and for any colour x, $l(x;\alpha)$ and $l(x;\alpha')$ are either s or $s+1$. Thus α and α' are impartial, and hence $G \notin U_k$.

Now the cases $\epsilon(G) \equiv 1$ and 2 modulo k can be dealt with.

Proposition 3.5. *Suppose that* $k \geq 4$, *and that* G *is a multigraph with* $\epsilon(G) = ks+1$, $s \geq 1$. *If* $G \in U_k$ *then* $G \in X_k$.

Proof. Suppose that $G \in U_k - X_k$. Let α and β be the k-edge-colourings of G given by Lemma 3.1, and assume that α and β use colours $a, b, c, d, \ldots$ Without loss of generality, we may assume that

$$l(a;\alpha) = s+1, \quad l(b;\alpha) = l(c;\alpha) = l(d;\alpha) = \ldots = s.$$

There are essentially two different possibilities for β.

(1) $l(a;\beta) = l(b;\beta) = s+1,\ l(c;\beta) = s-1,\ l(d;\beta) = \ldots = s.$

Then β is (a, b, c)-semiimpartial, and thus by Corollary 3.4 $G \notin U_k$, a contradiction.

(2) $l(a;\beta) = s+2,\ l(b;\beta) = s-1,\ l(c;\beta) = l(d;\beta) = \ldots = s.$

Then $G[\beta]$ must contain an a-dominated odd ac-path, which when inverted gives a colouring β' with

$$l(a;\beta') = l(c;\beta') = s+1, \quad l(b;\beta') = s-1, \quad l(d;\beta') = \ldots = s.$$

But β' is (a, c, b)-semiimpartial, and thus again $G \notin U_k$, a contradiction.

Since we obtain a contradiction in both cases, if $G \in U_k$ then $G \in X_k$.

Proposition 3.6. *Suppose that $k \geq 4$, and that G is a multigraph with $\epsilon(G) = ks+2$, $s \geq 1$. If $G \in U_k$ then $G \in X_k$.*

Proof. Suppose that $G \in U_k - X_k$. Let α and β be the k-edge-colourings of G given by Lemma 3.1, using colours $a, b, c, d, \ldots$. Without loss of generality, we may assume that

$$l(a;\alpha) = l(b;\alpha) = s+1, \quad l(c;\alpha) = l(d;\alpha) = \ldots = s.$$

There are essentially three different possibilities for β.

(1) $l(a;\beta) = l(b;\beta) = l(c;\beta) = s+1,\ l(d;\beta) = s-1,\ l(x;\beta) = s$

for all $x \neq a, b, c, d$.

Then β is (a, b, d)-semiimpartial, and so by Corollary 3.4 $G \notin U_k$, a contradiction.

(2) $l(a;\beta) = s+2,\ l(b;\beta) = s+1,\ l(c;\beta) = s-1,\ l(d;\beta) = \ldots = s.$

Then an a-dominated odd ad-path in $G[\beta]$ may be inverted to obtain a colouring β' with

$$l(a;\beta') = l(b;\beta') = l(d;\beta') = s+1, \quad l(c;\beta') = s-1, \quad l(x;\beta') = s$$

for all $x \neq a, b, c, d$.

Since β' is (a, b, c)-semiimpartial, $G \notin U_k$ by Corollary 3.4, a contradiction.

(3) $l(a;\beta) = s+2,\ l(b;\beta) = l(c;\beta) = l(d;\beta) = \ldots = s.$

Since β is (a, b, c)-impartial, $G \notin U_k$ and a contradiction is obtained.

Since a contradiction occurs in all three cases, if $G \in U_k$ then $G \in X_k$.

To deal with the remaining case, namely $\epsilon(G) \not\equiv 1$ or 2 modulo k, it suffices to examine one special subcase: $k = 4$ and $\epsilon(G) \equiv 0$ modulo 4. The following assumption will be used frequently.

Assumption A: $G \in U_4$ and $\epsilon(G) = 4s$, $s \geq 2$ (note that $G \notin X_4$ since $\epsilon(G) > 6$). Moreover G has 4-edge-colourings α and β as given by Lemma 3.1, using colours a, b, c, d, and β is obtained from α by inverting a b-dominated odd ab-path. Thus

$$l(a;\alpha) = l(b;\alpha) = l(c;\alpha) = l(d;\alpha) = s,$$
$$l(a;\beta) = s+1, \quad l(b;\beta) = s-1, \quad l(c;\beta) = l(d;\beta) = s.$$

Lemma 3.7. *Under Assumption A, for any two distinct colours x, y, $G[x,y;\alpha]$ consists of*

(a) *an even path; or*

(b) *an even cycle; or*

(c) *two odd paths.*

Proof. If $G[x,y;\alpha]$ is not one of (a), (b) or (c), then it has a proper isolated subgraph H containing an equal number of x and y edges. Inverting H gives an impartial k-edge-colouring for G which is inequivalent to α, a contradiction.

Corollary 3.8. *Under Assumption A, $G[a,b;\alpha]$ consists of one a-dominated and one b-dominated odd ab-path, and $G[a,b;\beta]$ consists of two a-dominated odd ab-paths.*

Proof. By Assumption A, $G[a,b;\alpha]$ contains an odd ab-path; therefore by Lemma 3.7 $G[a,b;\alpha]$ consists of two odd ab-paths. Since $l(a;\alpha) = l(b;\alpha) = s$, one must be a-dominated and the other b-dominated. In $G[a,b;\beta]$ the b-dominated path of $G[a,b;\alpha]$ is inverted and becomes a-dominated.

Corollary 3.9. *Given Assumption A, $G[\alpha]$ has exactly two a^1b^0 and exactly two a^0b^1 vertices.*

Proof. Any vertex not in $G[a,b;\alpha]$ is a^0b^0, and any internal vertex of the two odd ab-paths is a^1b^1. The two ends of the a-dominated path are a^1b^0, and the two ends of the b-dominated path are a^0b^1.

Lemma 3.10. *Under Assumption A, each of $G[b,c;\beta]$ and $G[b,d;\beta]$ consists of a single path of length $2s-1$.*

Proof. Consider $G[b,c;\beta]$. Since $l(c;\beta) = s > s-1 = l(b;\beta)$, it contains a c-dominated odd bc-path P. Assume that $P \neq G[b,c;\beta]$. Then

$L(b,c;\beta) - E(P)$ is nonempty and contains an equal number of b and c edges. Thus there exist edges $e \in L(c;\beta) \cap E(P)$ and $f \in L(c;\beta) - E(P)$. Note that $\alpha(e) = \beta(e) = c$ and $\alpha(f) = \beta(f) = c$. Invert P to obtain a new colouring β' with

$$l(a;\beta') = s+1, \quad l(b;\beta') = s-1, \quad l(c;\beta') = l(d;\beta') = s.$$

In $G[\beta']$ there is an a-dominated odd ac-path Q, which can be inverted to obtain an impartial colouring α'. Now since $e \in E(P)$, $b = \beta'(e) = \alpha'(e)$, and since $f \notin E(P)$, $\beta'(f) = c$ and thus $\alpha'(f) = a$ (if $f \in E(Q)$) or c (otherwise). Thus $\alpha'(e) \neq \alpha'(f)$, while $\alpha(e) = \alpha(f)$. Hence α and α' are inequivalent, a contradiction.

Therefore the assumption that $P \neq G[b,c;\beta]$ was incorrect, and $P = G[b,c;\beta]$ is a path. Similarly, $G[b,d;\beta]$ is a path.

Corollary 3.11. *Under Assumption A, G has exactly $2s-2$ vertices of degree 4.*

Proof. Any vertex of degree 4 must be incident with one of the $s-1$ edges of $L(b;\beta)$. On the other hand, any endvertex of an edge in $L(b;\beta)$ must be incident with an edge of $L(a;\beta)$ by Corollary 3.8, and edges of $L(c;\beta)$ and $L(d;\beta)$ by Lemma 3.10, and thus has degree 4. So the vertices of degree 4 are exactly the $2s-2$ endvertices of edges in $L(b;\beta)$.

Henceforth, let R be the set of vertices of degree 4 in G, and let $\overline{R}$ denote $V(G) - R$. For any $x \in \{a, b, c, d\}$, each of the $2s-2$ vertices of R must be x^1 in $G[\alpha]$. Since there are a total of $2s$ vertices that are x^1 in $G[\alpha]$, there must be exactly two vertices of $\overline{R}$ which are x^1 under α; the other vertices of $\overline{R}$ are x^0.

Lemma 3.12. *Suppose that $k \geq 1$, and that G is a multigraph with a unique impartial k-edge-colouring α. If $\alpha(\delta(u))$ and $\alpha(\delta(v))$ are disjoint for some distinct u, $v \in V(G)$, then α is also a unique impartial k-edge-colouring of $G(u{=}v)$.*

Proof. Note that u and v are nonadjacent since for any edge e incident with both, $\alpha(e) \in \alpha(\delta(u)) \cap \alpha(\delta(v))$. Therefore $G(u{=}v)$ exists. Since $\alpha(\delta(u))$ and $\alpha(\delta(v))$ are disjoint, α is an impartial proper k-edge-colouring of $G(u{=}v)$. Any impartial k-edge-colouring of $G(u{=}v)$ inequivalent to α is also an impartial k-edge-colouring of G inequivalent to α, and hence no such colouring exists.

Lemma 3.13. *Given Assumption A, there exist distinct u, $v \in \overline{R}$ with $\alpha(\delta(u))$ and $\alpha(\delta(v))$ disjoint.*

Proof. Form a multigraph F with $V(F) = \overline{R}$, $E(F) = \{a, b, c, d\}$, and with $\psi_F(x)$ being the set of two x^1 vertices in $\overline{R}$ for each colour x. Then two vertices u, v of F are adjacent if and only if both are x^1 for some x, which is true if and only if $\alpha(\delta(u))$ and $\alpha(\delta(v))$ intersect. Thus, it suffices to show that F contains two nonadjacent vertices. By Corollary 3.9, $\nu(F) \geq 4$, while $\epsilon(F) = 4 < \binom{\nu(F)}{2}$; therefore F contains two nonadjacent vertices, as required.

Theorem 3.14. *Let G be a multigraph with $\epsilon(G) = 4s$, $s \geq 2$. Then $G \notin U_4$.*

Proof. Assume this is false. Let G be a basic multigraph of minimum order which satisfies the conditions of the theorem, but which has a unique impartial 4-edge-colouring α. By Lemma 3.13 there exist distinct $u, v \in V(G)$ with $\alpha(\delta(u))$ and $\alpha(\delta(v))$ disjoint. Therefore, by Lemma 3.12 $G(u{=}v)$ has a unique impartial 4-edge-colouring. But since $G(u{=}v)$ is basic, satisfies the conditions of the theorem and has smaller order than G, this contradicts the choice of G.

Corollary 3.15. *Let G be a multigraph with $\epsilon(G) = 4s+3$, $s \geq 1$. Then $G \notin U_4$.*

Proof. Consider the multigraph $H = G \cup K_2$ obtained by adding a single isolated edge to G. It is not difficult to see that $G \in U_4$ if and only if $H \in U_4$. But by Theorem 3.14 $H \notin U_4$, and thus $G \notin U_4$.

Theorem 3.16. *Suppose that $k \geq 4$. Let G be a multigraph with $\epsilon(G) = ks + m$, $s \geq 1$ and $3 \leq m \leq k$. Then $G \notin U_k$.*

Proof. Suppose that $G \in U_k$. Let α be the unique impartial k-edge-colouring of G, and assume that α uses colours $a, b, c, d, \ldots$, where

$$l(a;\alpha) \geq l(b;\alpha) \geq l(c;\alpha) \geq l(d;\alpha) \geq \ldots .$$

Then, since $m \geq 3$, $l(a;\alpha) = l(b;\alpha) = l(c;\alpha) = s+1$, and $l(d;\alpha) = s+1$ or s. Since α is unique, the restriction of α to the edges of $G[a,b,c,d;\alpha]$ must be a unique impartial 4-edge-colouring of $G[a,b,c,d;\alpha]$. However, by Theorem 3.14 or Corollary 3.15, $G[a,b,c,d;\alpha] \notin U_4$, which is a contradiction.

Thus, from our initial comments in this section, Propositions 3.5 and 3.6, and Theorem 3.16, we obtain our main result.

Theorem 3.17. *Suppose that $k \geq 4$. Then a basic multigraph G has a unique impartial k-edge-colouring if and only if $\epsilon(G) \leq k$ or $G \in X_k$.*

Corollary 3.18. *Suppose that $k \geq 4$. No multigraph with at least $k+3$ edges, and no graph with at least $k+1$ edges, has a unique impartial k-edge-colouring.*

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