Triangular embeddings of complete graphs (neighborly maps) with 12 and 13 vertices

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Abstract

In this paper we describe the generation of all nonorientable triangular embeddings of the complete graphs K_{12} and K_{13}. (The 59 nonisomorphic orientable triangular embeddings of K_{12} were found in 1996 by Altshuler, Bokowski and Schuchert, and K_{13} has no orientable triangular embeddings.) There are $182,200$ nonisomorphic nonorientable triangular embeddings for K_{12}, and $243,088,286$ for K_{13}. Triangular embeddings of complete graphs are also known as neighborly maps and are a type of twofold triple system. We also use methods of Wilson to provide an upper bound on the number of simple twofold triple systems of order n, and thereby on the number of triangular embeddings of K_n. We discuss applications of our results to flexibility of embedded graphs.

Mathematics Subject Classification: 05B07, 05C10, 05C30, 05C85, 52B70, 57M15, 57M20, 57Q15

1. Introduction

In this paper we describe the construction of complete lists of triangular embeddings of K_{12} and K_{13}. We discuss bounds on the number of triangular embeddings of K_n, and mention some

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ways in which our lists of embeddings may be applied.

Triangular embeddings of complete graphs on surfaces, or *complete triangulations*, have played a central role in topological graph theory since they were used by Ringel, Youngs and others, as summarized in [**35**], to prove the Map Color Theorem conjectured by Heawood [**24**]. By a *surface* we mean a compact connected 2-manifold without boundary.

When a complete triangulation of a given surface exists, it has various extreme properties. The complete graph has the largest chromatic number of any graph embeddable on that surface, which was the relevant property for the Map Color Theorem. If the surface is not the sphere, the complete graph also has the largest minimum degree and largest connectivity. The embedding, like all triangulations of simple graphs, is a minimum (orientable or nonorientable) genus embedding. It is also a minor-minimal 3-representative embedding, and minor-minimal polyhedral (3-connected and 3-representative) embedding.

In general, complete triangulations seem to be a good place to look for extremal embeddings of various kinds. We were motivated by questions on flexibility of graph embeddings, which we discuss in more detail in Section 7. However, there are many other problems for which complete triangulations provide useful information, particularly as sources of possible counterexamples for conjectures.

From a design theory perspective, complete triangulations are a type of twofold triple system. In general, twofold triple systems are equivalent to *pseudo-embeddings* of a complete graph on a surface, which may have singular vertices, where the faces do not occur in a single cycle around a vertex, but in several cycles. Complete triangulations are of interest not only to graph theorists and design theorists, but also to polyhedral theorists, who call them neighborly maps or neighborly 2-manifolds [**1**, **2**, **3**]. They are the 2-dimensional case of ‘tight’ triangulations [**27**].

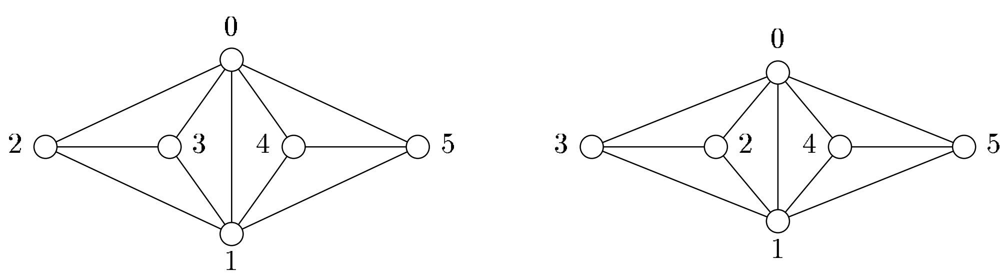

Figure 1. Distinct but isomorphic embeddings

When discussing numbers of objects, it is necessary to be clear about what is meant by ‘different’ objects. If two embeddings with the same vertex set V have different sets of faces, or if two collections of subsets of a given set V contain different subsets, we will say they are *distinct*. Two distinct embeddings, or collections of subsets, may be *isomorphic* by a permutation of the set V, which carries the faces, or subsets, of the first to the second. There are usually many more distinct embeddings, or collections of subsets, than nonisomorphic ones. For example, in Figure 1 we see two embeddings that are distinct, since the first has 013 as a face but the second does not; however, they are clearly isomorphic.

2. Background and previous work

As shown by Ringel, Youngs and others [**35**], K_n has at least one triangular embedding on the orientable surface S_h (a sphere with $h \geq 0$ handles added) if and only if $n \equiv 0, 3, 4$ or $7 \pmod{12}$ and $h = (n-3)(n-4)/12$, and on the nonorientable surface N_k (a sphere with $k \geq 0$ crosscaps added) if and only if $n \equiv 0, 1, 3$ or $4 \pmod 6$, $k = (n-3)(n-4)/6$, and $(n, k) \neq (7, 2)$.

For a long time it was not known whether complete triangulations of a given surface are unique up to isomorphism. In 1980 Lins [**30**] showed that two examples of embeddings of K_{12} in S_6 due to Youngs [**37**] are nonisomorphic. In the mid-1990's Arocha, Bracho and Neumann-Lara [**4**] produced nonisomorphic triangular embeddings of K_{30} in N_{117}, and Lawrencenko, Negami and White [**29**] produced nonisomorphic triangular embeddings of K_{19} in S_{20}.

For small complete graphs, it is not too hard to show that the embeddings of K_4 in S_0, K_6 in N_1 and K_7 in S_1 are unique up to isomorphism. All triangular embeddings of K_9 in N_5 and K_{10} in N_7 were found by Altshuler and Brehm [**3**] and rediscovered by Bracho and Strausz [**10**]. Altshuler, Bokowski and Schuchert [**2**] found all 59 embeddings of K_{12} in S_6, and Altshuler [**1**] constructed $40,615$ embeddings of K_{12} in N_{12}, but did not claim this list was complete.

Two basic approaches have been used to construct large families of complete triangulations of a given order n, and thereby to obtain lower bounds on the number of such triangulations. Korzhik and Voss [**25**] constructed large families by showing that current graphs that lead to complete triangulations can be modified in certain ways to obtain many nonisomorphic complete triangulations. By modifying current graphs used in the Map Color Theorem they found $2^{(n-4)/6}$ and $2^{(n-7)/6}$ nonisomorphic orientable triangular embeddings of K_n when $n \equiv 4$ and 7, respectively, (mod 12). They also [**26**] constructed large families of orientable minimal genus embeddings of complete graphs in cases where they are not triangulations. Grannell, Griggs and Širáň, sometimes collaborating with Bonnington, used a surgical approach to obtain large families of complete triangulations [**9**, **18**, **20**]. Their best results are obtained by cutting and sewing together embeddings of K_n and $K_{m,m,m}$ to obtain embeddings of $K_{m(n-1)+1}$; the strongest result [**20**] is that for $n \equiv 15$ or $43 \pmod{84}$ there are $2^{an^2 - O(n \log n)}$ orientable 2-face-colorable embeddings of K_n, where $a = \log_2(720)/294 \approx 0.032285$. As we shall see in Section 3, the best upper bound on the number of triangulations of K_n has the form $(cn)^{n^2/3}$, so there is a substantial gap between the lower and upper bounds.

Complete triangulations that are 2-face-colorable are of particular interest to design theorists, because each color class of faces forms a Steiner triple system. If there is a 2-face-colorable complete triangulation with one color class isomorphic to T_0 and the other to T_1, then the pair (T_0, T_1) is said to be *biembeddable*. The Map Color Theorem [**35**] orientable construction for $n \equiv 3 \pmod{12}$ provides a 2-face-colorable complete triangulation. Grannell, Griggs and Siran construct 2-face-colorable complete triangulations beginning with design theory considerations in [**19**], and from a topological graph theory viewpoint in [**18**]. The large families of complete triangulations constructed by the same authors and Bonnington [**9**] come from the construction in [**18**] and are all 2-face-colorable. Even larger families of 2-face-colorable complete triangulations were found by generalizing this construction in [**20**], where another construction was also given.

Steiner triple systems of order n (abbreviated STS(n)s) exist for all $n \equiv 1$ or $3 \pmod 6$. For small orders, $n = 3$, 7 and 9, there is a unique (up to isomorphism) Steiner triple system, which has a unique (up to isomorphism) biembedding with itself. Grannell, Griggs and Knor [**17**] generated all $38,608$ 2-face-colorable triangular embeddings of K_{13}. There are two nonisomorphic STS(13)s: a cyclic one C, and a noncyclic one N. Grannell, Griggs and Knor [**17**] showed that all pairs (C, C), (C, N) and (N, N) have biembeddings, and generated all of the $38,608$ 2-face-colorable triangular embeddings of K_{13}. There are 80 nonisomorphic STS(15)s. Besides the Map Color Theorem construction for $n \equiv 3 \pmod{12}$, Bennett, Grannell and Griggs [**5**, **6**, **7**] provide some information on their biembeddings, although much is still unknown. In [**7**] it is shown that there is a pair of STS(15)s that do not have an orientable biembedding (although it is not known whether they have a nonorientable one).

3. Representation of triangulations and an upper bound

One issue in generating combinatorial objects is how to represent them. If we have a collection C of subsets of $V = \{0, 1, \ldots, n-1\}$, it is natural to just write down the subsets. However, each subset of size i can be written $i!$ ways, and the subsets can be written in any order. To define a unique representation, we may insist that in each subset the points are in increasing order, and the subsets are listed in lexicographically increasing order; we call this the *lexicographic form*. Thus, for example, the Fano plane may be represented as the collection of triples

$$013, 124, 235, 346, 450, 561, 602$$

which has lexicographic form

$$013, 026, 045, 124, 156, 235, 346.$$

Doyen and Vallette [**12**] observed that for Steiner triple systems the lexicographic form gives rise to a concise representation of, and provides an upper bound on the number of, Steiner triple systems. We state the result in a more general form.

Observation 1. *Suppose we are given a fixed graph G, and a nonnegative integer weight $w(e)$ for each $e \in E(G)$. Let T be a collection of triangles in G such that each edge $e \in E(G)$ belongs to exactly $w(e)$ triangles of T. Suppose the lexicographic form of T is $a_0b_0c_0$, $a_1b_1c_1$, …, $a_{m-1}b_{m-1}c_{m-1}$. Then T is completely determined by the sequence $c_0, c_1, \ldots, c_{m-1}$.*

Proof. For each i, a_ib_i is just the first edge e of G in the lexicographic ordering of edges (2-subsets) that is contained in less than $w(e)$ of the triangles $a_0b_0c_0, a_1b_1c_1, \ldots, a_{i-1}b_{i-1}c_{i-1}$. ∎

For example, the Fano plane above is completely described by the sequence $3, 6, 5, 4, 6, 5, 6$.

For Steiner triple systems we take $G = K_n$ and $w(e) = 1$ for each edge e. For triangular embeddings (or, more generally, pseudo-embeddings) of a graph G we take $w(e) = 2$ for each edge e. As mentioned earlier, triangular pseudo-embeddings of K_n are equivalent to twofold triple systems.

For a given weight assignment w, the number of triangles we need is $m = \frac{1}{3}\sum_{e\in E(G)} w(e)$. Each c_i can take on at most $n-2$ distinct values (it cannot be equal to a_i or b_i) and so a bound on the number of distinct collections T is $(n-2)^m$. For Steiner triple systems $m = n(n-1)/6$, giving Doyen and Valette's bound that the number of distinct Steiner triple systems of order n is at most $(n-2)^{n(n-1)/6}$. For twofold triple systems, or triangular pseudo-embeddings (and hence embeddings) of K_n, we have $m = n(n-1)/3$, so there are at most $(n-2)^{n(n-1)/3}$ distinct ones. Wilson [**36**] found a slightly better upper bound of $(e^{-1}n)^{n^2/6}$ on the number of Steiner triple systems, and by adapting his methods we can also improve the bound on the number of triangular embeddings, as follows.

Theorem 2. *There are at most $(e^{-1/2}n)^{n^2/3}$ distinct simple (no repeated blocks) twofold triple systems of order n. Therefore, there are at most $(e^{-1/2}n)^{n^2/3}$ distinct triangular embeddings of K_n.*

Proof. We work in K_n with each edge having weight 2. When we remove a triple we reduce the weight of each of its edges by 1. We say a triple is *present* in a weighted graph if all of its edges have positive weight.

If a twofold triple system exists, $n(n-1)/3$ is an integer which must be even, so $b = n(n-1)/6$ is an integer. Let T be a twofold triple system, consisting of $2b = n(n-1)/3$ triangles in our doubly weighted K_n, with lexicographic form $S_0, S_1, \ldots, S_{b-1}, S_b, \ldots S_{2b-1}$. As above, the first b triples may be chosen in at most $(n-2)^b \le n^{n^2/6}$ ways.

Let n_S be the number of ways to choose the remaining b triples $S_b, S_{b+1}, \ldots, S_{2b-1}$. Let n_R be the number of sequences of triples $R_0, R_1, \ldots, R_{b-1}$ such that R_t, $0 \le t \le b-1$, is a triple present after we remove $S_0, S_1, \ldots, S_{b-1}, R_0, R_1, \ldots, R_{t-1}$. Since we are dealing with simple triple systems, each sequence $S_b, S_{b+1}, \ldots, S_{2b-1}$ corresponds to exactly $b!$ sequences $R_0, R_1, \ldots, R_{b-1}$, and so $n_S \le n_R/b!$. R_t is a triple in a weighted graph with total weight $3(b-t)$, and so the graph of edges of positive weight has at most $3(b-t)$ edges. We now refer the reader to Wilson [**36**, Lemma 1 and Theorem 1] for the proof that there are at most $\sqrt{6}(b-t)^{3/2}$ possibilities for each R_t, and hence $n_R/b! \leq (e^{-1}n)^{n^2/6}$.

Therefore, there are at most $n^{n^2/6}n_R/b! \leq n^{n^2/6}(e^{-1}n)^{n^2/6} = (e^{-1/2}n)^{n^2/3}$ simple twofold triple systems of order n. ∎

Note that the above bound concerns distinct embeddings. We would expect there to be fewer nonisomorphic embeddings, by a factor of roughly $n!$ (assuming that almost all embeddings have nontrivial automorphism group). However, $n!$ is of a smaller order of magnitude than $(e^{-1/2}n)^{n^2/3}$ and so the correction is not significant. We therefore take $(e^{-1/2}n)^{n^2/3}$ as our upper bound on the number of nonisomorphic triangular embeddings of K_n.

We venture the following:

Conjecture 3. *The number of distinct triangular embeddings of K_n is $n^{n^2/3+o(n^2)}$.*

For Steiner triple systems, Wilson [**36**] (assuming the van der Waerden Permanent Conjecture, later proved by Egorychev [**14**] and Falikman [**15**], for the lower bound) proved that the number of distinct STS(n)s lies between $((3^{-3/2}e^{-2}-\epsilon)n)^{n^2/6}$ and $(e^{-1}n)^{n^2/6}$ for any positive ϵ. Thus, it may be possible to prove something more precise than Conjecture 3.

4. The algorithm

When generating combinatorial objects, one major problem is to avoid generating many isomorphic versions of a given object. One way to avoid this difficulty is to define a *canonical form* for the objects, so that in each isomorphism class exactly one object is defined to be the *canonical* representative of that class. Read [**34**] pioneered the idea of designing an algorithm that will output only canonical objects, and hence generate each object only once up to isomorphism. Moreover, if there are ways of checking in the middle of the generation process whether the partial object currently constructed can lead to a canonical object, then the generation process can be made more efficient by cutting off the search when this is not the case.

So, in order to efficiently generate triangulations of complete graphs, we would like to choose a suitable canonical form. An easily defined canonical form for T is the lexicographically least lexicographic form of any triangulation isomorphic to T. We could have used this in our computations; it is possible to check partial objects at certain points to see if they can lead to a canonical object. However, the calculations would have been more complicated than for the canonical form we actually used.

To motivate what we did, we observe that canonical forms can be easier to find if we make use of invariant properties of substructures. For example, in constructing a canonical form for graphs we may insist that the vertex labeled 0 have maximum degree in the graph. This limits our choices and so reduces the amount of work needed to find the canonical form.

For embeddings, one can define invariants on vertices such as the *fingerprint vector* of Altshuler [**1**] for vertices of complete triangulations. However, it is more effective to look at flags rather than vertices. A *flag* in an embedded graph is a designation of one side of one end of an edge. Given two isomorphic embeddings, the image of one flag is sufficient to determine the entire isomorphism [**30**, Prop. 2.1]. If we can define an invariant for flags that allows us to single out one or a small number of flags in a given embedding, then a canonical form can be defined by starting with such a flag.

In triangulations, a flag is equivalent to an ordered triple (u, v, w) where uvw is a triangle (this designates the side belonging to uvw of the end at u of the edge uv). To each such flag (u, v, w) of a triangulation T of K_n we associate a permutation $p = p(u, v, w)$ of the set $\{0, 1, 2, \ldots, n-5\}$, as follows. Let the neighbors of u be $v, w, x_0, x_1, \ldots, x_{n-5}, t$ in order around u. Let the neighbors of v be $u, w, y_0, y_1, \ldots, y_{n-5}, t$ in order around v. Then $\{x_0, x_1, \ldots, x_{n-5}\} = \{y_0, y_1, \ldots, y_{n-5}\}$, and there is a permutation $p = [p_0p_1 \ldots p_{n-5}]$ such that $y_i = x_{p_i}$. Note that $p = p(u, v, w)$ does not depend on the labeling of the vertices, so it is an invariant of the flag (u, v, w).

For example, consider $p(3, 7, 8)$ for the situation shown in Figure 2. The x_i and y_i are as shown, and $y_0 = 4 = x_4$ so $p_0 = 4$, $y_1 = 2 = x_0$ so $p_1 = 0$, etc., giving $p(3, 7, 8) = [405231]$. Note that $p(v, u, w) = p(u, v, w)^{-1}$, so $p(7, 3, 8) = [153402]$. Also, let p^* denote the permutation obtained from p by $p^*_i = (n-5) - p_{n-5-i}$ (we reverse p and then take the 'opposite' of each element). If t is the other vertex occuring in a triangle with u and v, we have $p(u, v, t) = p(u, v, w)^*$. Thus, $p(3, 7, 1) = [423051]$ and, taking the inverse, $p(7, 3, 1) = [351204]$.

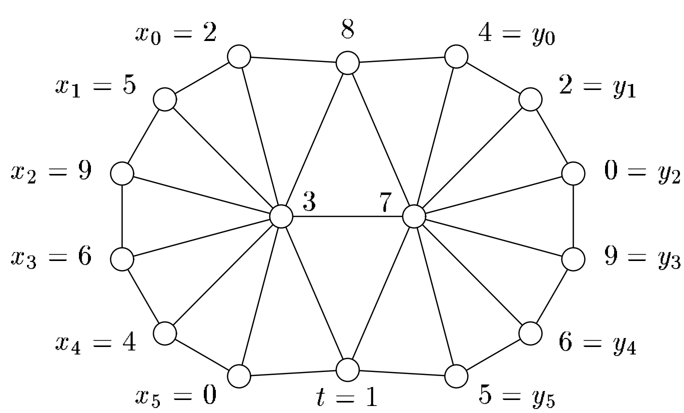

Figure 2. Example of permutation, $p(3, 7, 8)$

Now we define our canonical form. Let T be a complete triangulation with vertices labeled $\{0, 1, \ldots, n-1\}$. We say T is *weakly canonical* if

(i) around vertex 0 the other vertices appear in the order $1, 2, \ldots, n-1$; and

(ii) $p(0, 1, 2)$ is lexicographically smallest over all $p(u, v, w)$ for (u, v, w) an ordering of a triple of T.

We say T is *canonical* if in addition

(iii) subject to (i) and (ii), the lexicographic form of T is lexicographically smallest.

Condition (i) here just means that the choice of which vertices to label 0, 1 and 2 determines the labeling of all the other vertices. Condition (ii) is what makes our algorithm efficient: it is easy to check at various points during the algorithm, and has proved very effective in practice at pruning the search tree. Condition (iii) is comparatively unimportant in practice, but is necessary to ensure that there is at most one canonical triangulation in each isomorphism class. Our algorithm actually generates all weakly canonical triangulations, and checks condition (iii) only once a complete embedding has been generated.

Our algorithm proceeds as follows. We construct all the faces incident with vertex 0, which must be 012, 023, …, $0(n-2)(n-1)$, $0(n-1)1$. Then we begin choosing faces incident to vertex 1. In choosing a face uvw to add at any point, we must ensure that it does not complete a cycle of faces of length less than $n-1$ around any of u, v or w. When vertex 1 is complete (i.e., has $n-1$ faces around it), we determine $p(0, 1, 2)$, and compare it to $p(1, 0, 2)$, $p(0, 1, n-1)$ and $p(1, 0, n-1)$ to make sure $p(0, 1, 2)$ is minimum. We then move on to choose faces incident with vertex 2 until vertex 2 is complete, and so on. Whenever a newly added face completes a vertex u (which may or may not be the vertex we are working to complete) we check all permutations of the forms $p(u, v, w)$ and $p(v, u, w)$ where v is a previously completed vertex, to make sure they are not smaller than $p(0, 1, 2)$. (The relations $p(v, u, w) = p(u, v, w)^{-1}$ and $p(u, v, t) = p(u, v, w)^*$, noted above, are useful here.) If we succeed in completing all vertices up to $n-1$ we have a weakly canonical embedding, which we then check for condition (iii). If the embedding is canonical, we check its orientability, and output it.

In the case of $n = 12$ or 13, the number of weakly canonical embeddings is not significantly larger than the number of canonical embeddings. For $n = 12$ there are $182, 259$ canonical and $190, 574$ weakly canonical ones, an excess of just 4.6%. For $n = 13$ there are $243, 088, 286$ canonical and $245, 783, 224$ weakly canonical ones, an excess of just 1.1%. Thus, weakly canonical embeddings have a high probability of being canonical, and generating weakly canonical embeddings is a practical and efficient strategy.

5. Computational issues

To verify our results, the embeddings were generated twice, using two separately written programs. The programs both shared the same outline given above, but differed in implementation details and in how new faces were chosen to add. Our first program added triangles in lexicographic order, while the second always added a triangle at least one of whose edges belonged to a previously added triangle. The second program will be easy to modify to enforce orientability if we wish to tackle the generation of the orientable triangular embeddings of K_{15}.

The computations for K_{12} took slightly less than 30 minutes on a PC running at 2 GHz. The computations for K_{13} were done the first time on one dual-processor 300 MHz and one single processor 500 MHz Compaq Alpha machine, and took over five weeks. The second time we used a large 'Beowulf'-style cluster of PCs, and it took $10\frac{1}{4}$ hours running on 60 dual Xeon processor PCs running at 2 GHz (so the total CPU time was roughly $51\frac{1}{4}$ days).

A nontrivial consideration for the roughly 243 million complete triangulations of order 13 was how to store them. Listing each vertex of each triangle would require $n(n-1) = 156$ integers for each embedding; if we store two integers per byte this would require nearly 18 GB, which even today is a nontrivial amount of space. Observation 1 and condition (i) come to our rescue here. By Observation 1 we need only store the final vertex of each triangle, and by condition (i) we do not need to store the triangles incident with vertex 0. Therefore, we need only store $(n-1)(n-3)/3 = 40$ integers for each embedding, which can be done in 4.5 GB. Rather than storing the data in a compact binary form, we actually opted to store them in human-readable form, requiring 40 bytes plus 7 spacing bytes per embedding, for a total of 10.6 GB, and then compress them, which resulted in just under 2 GB of data.

To obtain copies of the files of triangulations, please contact the first author.

6. Numbers of embeddings

Our programs generate both orientable and nonorientable embeddings at the same time, so we generated the orientable embeddings of K_{12}, already described in [**2**], as well as the nonorientable ones. We present the orientable ones here with some information on automorphism groups to show that they confirm the information given in [**2**].

For each graph and surface we state the number of nonisomorphic embeddings and their automorphism group orders, from which it is possible to compute the number of distinct (i.e., 'labeled') embeddings.

Theorem 4. *The triangular embeddings of K_{12} and K_{13} have the following properties:*

Orientable triangular embeddings of K_{12}, on S_6:

59 nonisomorphic embeddings;

Automorphism groups: 39 of order 1, 9 of order 2, 1 of order 3, 7 of order 4,
2 of order 6, 1 of order 12;

$25,079,040$ distinct embeddings.

Nonorientable triangular embeddings of K_{12}, on N_{12}:

$182,200$ nonisomorphic embeddings;

Automorphism groups: $181,508$ of order 1, 577 of order 2, 84 of order 3, 12 of order 4, 16 of order 6, 3 of order 11;

$87,097,071,398,400$ distinct embeddings.

Nonorientable triangular embeddings of K_{13}, on N_{15}:

$243,088,286$ nonisomorphic embeddings;

Automorphism groups: $243,079,445$ of order 1, 6972 of order 2, 1867 of order 3, 1 of order 13, 1 of order 39;

$1,513,686,343,383,244,800$ distinct embeddings.

∎

There are too many nonorientable embeddings with nontrivial automorphism groups for us to give details here. However, the most symmetric embedding found is the one of K_{13} with 39 automorphisms. It is isomorphic to a known current graph embedding of K_{13} (see, for example, [**21**, Figure 4.24]).

As mentioned previously, our results were generated twice by two independent programs. We also checked the embeddings of K_{13} for 2-face-colorability; our calculations agreed with [**17**] that there are $38,608$ of them.

7. Applications

As we have said, triangular embeddings of complete graphs are likely to have extreme properties and are good places to look for counterexamples to conjectures about embeddings. This has already been shown by an example in polyhedral theory. One of the 59 orientable embeddings of K_{12} on S_6 found by Altshuler, Bokowski and Schuchert [**2**] was used by Bokowski and Guedes de Oliveira [**8**] to disprove an old conjecture of Grünbaum [**13**; **22**, Ex. 13.2.3] about the existence of 'geometric embeddings' of orientable triangulations.

Triangular embeddings of complete graphs are minor-minimal 3-representative embeddings, which means that they are useful for checking conjectures for 3-representative graphs, such as Zha's Separating Cycle Conjecture (see [**38**]), which states that every 3-representative embedding on S_g or N_g, $g \geq 2$, has a noncontractible cycle in the graph that separates the surface. There are various coloring problems due to Grünbaum [**23**], Fisk [**16**], and Kündgen and Ramamurthi [**28**] where these lists may provide useful examples. Altshuler [**1**] defines various graphs whose vertices are the nonisomorphic triangular embeddings of K_n on a given surface, with edges representing operations that transform one embedding into another; our lists can be used to investigate these graphs and formulate general conjectures about when and how one such embedding may be transformed into another.

Our original motivation for looking at triangulations of complete graphs was because we thought they might provide a way to construct highly flexible embeddings. Mohar and Robertson [**31**] showed that for every surface Σ there is a constant $\xi(\Sigma)$ such that every 3-connected graph has at most $\xi(\Sigma)$ distinct 3-representative embeddings on Σ. For triangulations this also follows from a result of Chen and Lawrencenko [**11**] and Negami, Nakamoto and Tanuma [**33**]. An explicit value for Mohar and Robertson's $\xi(\Sigma)$ can be derived from their arguments, but it is huge. Thus, it is of interest to determine lower bounds on $\xi(\Sigma)$ by constructing graphs with many distinct embeddings on a given surface.

We can determine lower bounds on $\xi(\Sigma)$ using the large families of complete triangulations discussed in Section 2. The best of these, from [**20**], gives $720^{2h/49-O(\sqrt{h})} \approx (1.308)^{h-O(\sqrt{h})}$ distinct embeddings of a complete graph on S_h for certain values of h.

Mohar and Robertson [**31**] improved on this by using triangular embeddings of K_7 to construct very flexible orientable embeddings. There are 48 distinct embeddings of K_7 that contain a fixed triangle, say 012. Take a triangulation T of the sphere and glue one K_7 along 012 to each of h triangles of T. The result is a graph with 48^h distinct embeddings on S_h. By using K_6 to ensure nonorientability, they also found graphs that have at least $6 \times 48^{(k-1)/2}$ distinct triangular embeddings on N_k for k odd.

The orientable embeddings of K_{12} on S_6 turn out not to give any improvement in these results. However, for nonorientable embeddings we can use embeddings for K_{13} to improve on the construction using K_6 and K_7. Use the same basic idea, but glue on embeddings of K_{13} with a fixed triangle to the spherical triangulation. Any given triangle from the $\binom{n}{3} = n(n-1)(n-2)/6$ possibilities is equally likely to be one of the $n(n-1)/3$ triangles in a given embedding, so each triangle appears in $2/(n-2)$ of the distinct embeddings. So the number of embeddings of K_{13} with a fixed triangle is $2/11$ of 1.514×10^{18}, or 2.752×10^{17}. If we glue p copies of K_{13}, we obtain a graph with at least $(2.752 \times 10^{17})^p$ distinct embeddings on N_{15p}, or letting $k = 15p$, a graph with at least $(14.54)^k$ embeddings on N_k when k is divisible by 15. We may compare this to $6 \times 48^{(k-1)/2} \approx 0.8660 \times (6.928)^k$ from [**31**].

If Conjecture 3 is correct, the number of triangular embeddings of K_n on S_h or N_k for suitable h or k would be $h^{4h+o(h)}$ or $k^{2k+o(k)}$, respectively: then one large complete graph on a surface actually would provide more flexibility than many small ones.

8. Conclusion

The next value of n for which K_n has triangular embeddings is $n = 15$. It is probably infeasible to generate all triangular embeddings of K_{15} in N_{22}. However, based on the fact that there are very few orientable triangular embeddings of K_{12} compared to the nonorientable ones, it may be possible to generate all orientable embeddings of K_{15}, in S_{11}. Brendan McKay (personal communication) estimates that there may be about 7×10^9 such embeddings; finding them may be a feasible (if still long-term) project on a large many-processor system.

It may also be worthwhile to generate minimum genus embeddings of small complete graphs in the cases where they are not triangulations: nonorientable embeddings of K_n for $n = 5, 7, 8, 11$, and orientable embeddings for $n = 5, 6, 8, 9, 10, 11, 13$. Possibly some results might also be obtained for K_{14}.

There are many questions of asymptotic enumeration that can be explored in relation to complete triangulations. It would be desirable to have good asymptotic estimates for the number of distinct simple twofold triple systems of order n. For complete triangulations of order n we would like estimates for the total number of distinct ones, for those that are orientable, for those that are 2-face-colorable, and for those that are both 2-face-colorable and orientable. There are obvious questions about proportions; for example, what proportion of all distinct complete triangulations are orientable? It would be useful to prove that almost all of a given type of complete triangulation have trivial automorphism group, to relate the number of distinct and nonisomorphic ones.

There are interesting questions related to Steiner triple systems. Most notably, is every pair of Steiner triple systems of the same order biembeddable on some surface? As noted above, we know that we cannot require the surface to be orientable [**7**].

If we wish to explore other useful classes of graph embeddings, it might perhaps be worthwhile to generate quadrangular embeddings of small complete bipartite graphs $K_{m,n}$, for whatever values of m and n seem computationally feasible.

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