2-connected spanning subgraphs with low maximum degree in locally planar graphs

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In this paper, we prove that there exists a function $a : \mathbb{N}_0 \times \mathbb{R}_+ \to \mathbb{N}$ such that for each ε, if G is a 4-connected graph embedded on a surface of Euler genus k such that the face-width of G is at least $a(k, \varepsilon)$, then G has a 2-connected spanning subgraph with maximum degree at most 3 such that the number of vertices of degree 3 is at most $\varepsilon|V(G)|$. This improves results due to Kawarabayshi, Nakamoto and Ota [11], and Böhme, Mohar and Thomassen [4].

Key Words: Spanning subgraph, surface, representativity, degree restriction.

1. INTRODUCTION

All graphs in this paper are simple, with no loops or multiple edges. A *closed surface* means a connected compact 2-dimensional manifold without boundary. We denote the orientable and nonorientable closed surfaces of genus g by S_g and N_g, respectively. For a closed surface F^2, let $\chi(F^2)$ denote the Euler characteristic of F^2. The number $k = 2 - \chi(F^2)$ is called the *Euler genus* of F^2. Let F^2_k denote a closed surface of Euler genus k. It is well-known that for every even $k \geq 0$, either $F^2_k = S_{k/2}$ or $F^2_k = N_k$, and for every odd k, $F^2_k = N_k$. If a graph G is embedded on

*Research supported by National Science Foundation grant DMS-0070613 and National Security Agency grant H98230-04-1-0110. The United States Government is authorized to reproduce and distribute reprints notwithstanding any copyright notation herein.

† Research partly supported by the Japan Society for the Promotion of Science for Young Scientists, by Japan Society for the Promotion of Science, Grant-in-Aid for Scientific Research, by Sumitomo Foundation and by Inoue Research Award for Young Scientists.

a surface so that every noncontractible closed curve intersects G at least k times, we say the embedding is *k-representative*. The *face-width* or *representativity* is the smallest nonnegative integer k for which the embedding is k-representative.

In 1931 Whitney [21] showed that 4-connected planar triangulations are hamiltonian, and in 1956, Tutte [20] proved that every 4-connected planar graph is hamiltonian. Almost thirty years later, Thomassen [18] (see also [5]) gave a short proof of Tutte's theorem and extended it to show that every 4-connected planar graph is hamiltonian-connected, i.e., for any two distinct vertices u, v, there is a hamiltonian path from u to v. There are many results inspired by these theorems of Whitney, Tutte and Thomassen. While we cannot survey all such results, we mention some that motivate the present paper.

Thomas and Yu [17] extended Tutte's theorem to projective-planar graphs and proved that every 4-connected projective-planar graph is hamiltonian. However, Archdeacon, Hartsfield, and Little [1] proved that for each k there exists a k-connected triangulation of some orientable surface having face-width k in which every spanning tree has a vertex of degree at least k. In particular, such graphs are far from having hamiltonian cycles. So a fixed connectivity or face-width or both, independent of the surface, will not suffice for hamiltonicity on arbitrary surfaces.

If the surface is fixed and the face-width is large enough, then the situation is different. The first results in this direction were by Thomassen [19], who examined a generalization of hamiltonicity. A *k-tree* is a spanning tree of maximum degree at most k; this generalizes the idea of a hamilton path, which is a 2-tree. Barnette [2] showed that every 3-connected planar graph has a 3-tree. Thomassen [19] showed that local planarity provides a similar result. He proved that a triangulation of a fixed orientable surface with large face-width has a 4-tree. Ellingham and Gao [6] modified the method of [19] to prove that a 4-connected triangulation of a fixed orientable surface with large face-width has a 3-tree.

These results were improved by examining another generalization of hamiltonicity. A *k-walk* is a spanning closed walk that uses every vertex at most k times; this generalizes the idea of a hamilton cycle, which is a 1-walk. Jackson and Wormald [9] noted that if a k-walk exists, then a $(k+1)$-tree exists. Gao and Richter [8] improved Barnette's result by showing that every 3-connected planar graph has a 2-walk. Yu [22] improved the results of Thomassen and Ellingham and Gao by showing that on a fixed surface, a 3-connected graph of large face-width has a 3-walk, and a 4-connected graph of large face-width has a 2-walk: the surface can be orientable or nonorientable, and the graph need not be a triangulation. Yu [22] also verified a conjecture of Thomassen [19] that every 5-connected triangulation of large face-width on a fixed surface is hamiltonian. Kawarabayashi [10] improved the conclusion here to hamiltonian-connected. Yu [22] posed the question of whether every 5-connected graph (not just triangulation) of large face-width on a fixed surface is hamiltonian, which is still unresolved. Thomassen [19] showed that for every surface of Euler genus greater than 2 there are 4-connected triangulations of arbitrarily large face-width that are not hamiltonian, so this would be best possible.

One way to tighten results on the existence of k-trees or k-walks is to bound the number of vertices of high degree, or visited more than once. Kawarabayashi, Nakamoto and Ota improved Thomassen's result on 4-trees and Yu's result on 3-walks as follows (the bounds are best possible).

THEOREM 1.1 ([11]). *For every non-spherical closed surface F^2 of Euler genus k, there exists a positive integer $N(F^2)$ such that every 3-connected $N(F^2)$-representative graph on F^2 has a 4-tree with at most $\max\{2k-5,0\}$ vertices of degree 4, and a 3-walk in which at most $\max\{2k-4,0\}$ vertices are visited 3 times.*

A further way to generalize hamiltonicity is as follows. A *k-covering* (sometimes called a *k-trestle*) of a graph G is a spanning 2-connected subgraph of G with maximum degree at most k. Hence a 2-covering is exactly a hamiltonian cycle. The first result in this area was by Barnette [3], who showed that every 3-connected planar graph has a 15-covering; this was improved by Gao [7], who showed that every 3-connected graph on a surface with non-negative Euler characteristic has a 6-covering. Barnette showed this would be best possible. For arbitrary surfaces, Sanders and Zhao [16] showed that 3-connected graphs on a fixed surface F^2 have a $K(F^2)$-covering, where K is bounded by a linear function of the genus.

It is possible to obtain a result for graphs of large face-width on a fixed surface, and at the same time bound the number of vertices of high degree. Kawarabayashi, Nakamoto and Ota proved the following (the bounds "$4k-8$" and "$2k-4$" are best possible).

THEOREM 1.2 ([11]). *For every non-spherical closed surface F^2 of Euler genus k, there exists a positive integer $N(F^2)$ such that every 3-connected $N(F^2)$-representative graph on F^2 has an 8-covering with at most $\max\{4k-8,0\}$ vertices of degree 7 or 8, among which at most $\max\{2k-4,0\}$ have degree 8.*

The bound "8" in Theorem 1.2 is not best possible. Kawarabayashi, Nakamoto and Ota improved this to 7, at the cost of increasing the number of vertices of large degree, as follows (the bound "$6k-12$" is best possible).

THEOREM 1.3 ([12]). *For every non-spherical closed surface F_k^2 of Euler genus $k \geq 2$, there exists a positive integer $M(F^2)$ such that every 3-connected $M(F^2)$-representative graph on F^2 has a 7-covering with at most $6k-12$ vertices of degree 7.*

However, for each closed surface F_k^2 with $k > 2$, there exists a triangulation with arbitrarily large face-width having no 6-covering.

Now let us focus on 4-connected case. Recently, Böhme, Mohar and Thomassen proved the following.

THEOREM 1.4 ([4]). *There exists a function $a : \mathbb{N}_0 \times \mathbb{R}_+ \to \mathbb{N}$ such that for each $\varepsilon > 0$, if G is a 4-connected graph embedded on a closed surface of Euler genus k such that the face-width of G is at least $a(k,\varepsilon)$, then G has a 4-covering such that the number of vertices of degree 3 or 4 is at most $\varepsilon|V(G)|$.*

Kawarabayashi, Nakamoto and Ota were able to provide a linear bound on the number of vertices of degree 4.

THEOREM 1.5 ([11]). *For every non-spherical closed surface F^2 of Euler genus k, there exists a positive integer $N(F^2)$ such that every 4-connected $N(F^2)$-repre-*

sentative graph on F^2 has a 4-covering with at most $\max\{4k-6, 0\}$ *vertices of degree* 4*.*

But the bound "4" in the above theorem is not best possible. The purpose of this paper is to prove that the bound "4" can be improved to 3.

THEOREM 1.6. *There exists a function $a : \mathbb{N}_0 \times \mathbb{R}_+ \to \mathbb{N}$ such that for each $\varepsilon > 0$, if G is a 4-connected graph embedded on a closed surface of Euler genus k such that the face-width of G is at least $a(k, \varepsilon)$, then G has a 3-covering (2-connected spanning subgraph with maximum degree at most 3) such that the number of vertices of degree 3 is at most $\varepsilon|V(G)|$.*

But perhaps the bound on the number of vertices of degree 3 in the above theorem is not best possible. The natural conjecture is the following.

CONJECTURE 1.1 ([11]). *For every non-spherical closed surface F_k^2 of Euler genus k, there exists a positive integer $M(F^2)$ such that every 4-connected $M(F^2)$-representative graph on F^2 has a 3-covering with at most ck vertices of degree 3, where c is a constant which does not depend on k.*

The bound "3" here would be best possible, as shown by Thomassen's nonhamiltonian 4-connected triangulations of large face-width, mentioned earlier. If true, Conjecture 1.1 implies a conjecture of Mohar [13] which says for every non-spherical closed surface F_k^2 of Euler genus k, there exists a positive integer $M(F^2)$ such that every 4-connected $M(F^2)$-representative graph on F^2 has a 3-tree with at most ck vertices of degree 3, where c is a constant which does not depend on k.

However, Conjecture 1.1 seems to be difficult because it is closely related to the conjecture of Nash-Williams [14] that every 4-connected graph in the torus is hamiltonian. So far, we know from Sanders and Zhao [16] that every 4-connected graph in the torus or in the Klein bottle has a 3-covering.

2. DEFINITIONS AND PRELIMINARY RESULTS

If P is a path containing vertices u and v, let $P[u, v]$ denote the subpath of P between u and v. If C is a cycle with a particular assumed direction, let $C[u, v]$ denote the subpath of C from u to v in the given direction.

A *disk graph* is a graph H embedded in a closed disk, such that a cycle Z of H bounds the disk. We write $\partial H = Z$. An *internally* 4*-connected disk graph* or *I4CD graph* is a disk graph H such that from every internal vertex v ($v \in V(H) - V(\partial H)$) there are four paths, pairwise disjoint except at v, from v to ∂H.

A *cylinder graph* is a graph H embedded in a closed cylinder, such that two disjoint cycles Z_0, Z_1 of H bound the cylinder. We write $\partial H = Z_0 \cup Z_1$. An *internally* 4*-connected cylinder graph* or *I4CC graph* is a cylinder graph H such that from every internal vertex v there are four paths, pairwise disjoint except at v, from v to ∂H. Note that an I4CC graph is not necessarily connected: Z_0 and Z_1 may lie in different components.

If G is an embedded graph and Z is a contractible cycle of G bounding a closed disk, then the embedded subgraph consisting of all vertices, edges and faces in that closed disk is a *disk subgraph* of G. Similarly, if Z_0 and Z_1 are disjoint homotopic cycles bounding a closed cylinder, then the embedded subgraph H consisting of all vertices, edges and faces in that closed cylinder is a *cylinder subgraph* of G. We write $H = Cyl_G[Z_0, Z_1]$ or just $H = Cyl[Z_0, Z_1]$. If the surface is a torus or Klein bottle and Z_0, Z_1 are nonseparating, then this notation is ambiguous, but it should be clear from context which one of the two possible cylinders we mean. We define $Cyl(Z_0, Z_1]$ to be the graph $Cyl[Z_0, Z_1] - V(Z_0)$, and define $Cyl[Z_0, Z_1)$ and $Cyl(Z_0, Z_1)$ similarly.

The following is easy to prove.

Lemma 2.1. *Suppose G is a 4-connected embedded graph. Any disk subgraph of G bounded by a cycle of length at least 4 is I4CD, and any cylinder subgraph of G is I4CC.*

Suppose G is an embedded graph. If $\mathcal{R} = \{R_0, R_1, \ldots, R_m\}$ is a collection of pairwise disjoint homotopic cycles with $R_i \subseteq Cyl[R_0, R_m]$ for each i, and $\mathcal{S} = \{S_0, S_1, \ldots, S_{n-1}\}$ is a collection of disjoint paths with $S_j \subseteq Cyl[R_0, R_m]$ for each j, such that $R_i \cap S_j$ is a nonempty path (possibly a single vertex) for each i and j, then we say that $(\mathcal{R}, \mathcal{S})$ is a *cylindrical mesh* in G.

In two places in the proof of Theorem 1.6 (Steps 3 and 6) we will need to move two consecutive cycles in a cylindrical mesh closer together, so that there are no vertices between them. An arbitrary homotopic shifting of a cycle may not preserve the existence of a mesh, so we need the following technical lemma.

Lemma 2.2. *Suppose N is an I4CC graph with $\partial N = R_0 \cup R_1$ that has a cylindrical mesh $(\{R_0, R_1\}, \{S_0, S_1, \ldots, S_{n-1}\})$.*

(i) *In N there are disjoint cycles R'_0 and R'_1 homotopic to R_0 (with R'_0 closer to R_0) and pairwise disjoint paths $S'_0, S'_1, \ldots, S'_{n-1}$, such that $Cyl(R'_0, R'_1)$ is empty, each S'_j has the same ends as S_j, and $R'_i \cap S'_j$ is a nonempty path for each i and j.*

(ii) *Moreover, if every component of $Cyl(R_0, R_1)$ has at most two neighbors on R_0, we may take $R'_0 = R_0$.*

Proof. (i) Embed N in the plane with R_1 as the outer face and R_0 as an inner face, with $S_0, S_1, \ldots S_{n-1}$ directed outwards from R_0 to R_1, and with all cycles directed clockwise. The proof is by induction on the number of vertices of $Cyl(R_0, R_1)$. If there are none we are finished. Otherwise, let T be a component of $Cyl(R_0, R_1)$. Since N is I4CC, T has at least two neighbors on one of R_0 or R_1.

Assume first that T has two neighbors on R_0. The graph A consisting of R_0, T, and all edges joining T to R_0 has a block B containing R_0 and at least one vertex of T.

Suppose that some S_i has a subpath with both ends in B but containing an edge not in B. This path has a subpath P whose ends are in B and all of whose edges and internal vertices are not in B. If an internal vertex of P belongs to R_1, then $R_1 \cap S_i$ is not a path, a contradiction, so $V(P) \cap V(R_1) = \emptyset$. If both ends of P are in R_0, then $R_0 \cap S_i$ is not a path, a contradiction, so at least one end of P is in T. It follows that all internal vertices of P belong to $V(T) - V(B)$, and all edges of P belong to $E(A) - E(B)$. Thus, $B \cup P$ is a 2-connected subgraph of A larger than

B, contradicting the fact that B is a block of A. Hence, every subpath of every S_i with both ends in B lies completely in B.

Let R_0^* be the outer cycle of B. (The subgraph of N between R_0 and R_0^* may contain vertices not in A or B, from components of $Cyl(R_0, R_1)$ other than T, but this does not affect our argument.) For each i, let r_i be the first vertex of S_i, let s_i be the first vertex of S_i that belongs to R_0^*, let t_i be the last vertex of S_i that belongs to B (t_i is necessarily also the last vertex of S_i on R_0^*), and let u_i be the last vertex of S_i. From above, each $S_i[r_i, t_i]$ lies entirely in B.

If $s_i \neq t_i$, then by planarity, one of $R_0^*[s_i, t_i]$ or $R_0^*[t_i, s_i]$ lies on the same side of $S_i[s_i, t_i]$ as the interior of R_0, and the other lies on the opposite side. Let Z_i denote the one on the opposite side, or let $Z_i = s_i = t_i$ if $s_i = t_i$. By planarity $S_j[r_j, t_j]$ does not intersect Z_i for any $j \neq i$. If Z_i intersects Z_j then at least one of $s_i \in V(Z_j)$, $t_i \in V(Z_j)$, $s_j \in V(Z_i)$ or $t_j \in V(Z_i)$ must hold, which contradicts the fact that $S_j[r_j, t_j] \cap Z_i$ and $S_i[r_i, t_i] \cap Z_j$ are empty. Therefore, the paths $S_i^* = Z_i \cup S_i[t_i, u_i]$ for $0 \leq i \leq n-1$ are pairwise disjoint, with $R_0^* \cap S_i^* = Z_i$ and $R_1 \cap S_i^* = R_1 \cap S_i$ both being paths for each i.

Since $Cyl(R_0^*, R_1)$ has fewer vertices than $Cyl(R_0, R_1)$, we may apply induction to $Cyl[R_0^*, R_1]$, R_0^*, R_1, S_0^*, …, S_{n-1}^*, to obtain R_0', R_1', and paths S_0'', …, S_{n-1}''. Let $S_i' = S_i[r_i, s_i] \cup S_i''$ for each i, then the required conclusion holds.

Similarly, if T has two neighbors on R_1 then we may construct an R_1^* and apply induction to $Cyl[R_0, R_1^*]$.

(ii) If every component of $N - V(R_0 \cup R_1)$ has at most two neighbors on R_0, then in the above T always has at least two neighbors on R_1, and we can always construct R_1^* rather than R_0^*. The components of $Cyl(R_0^*, R_1)$ are subgraphs of the components of $Cyl(R_0, R_1)$, and so also have at most two neighbors on R_0. Thus, by induction we may take $R_0' = R_0$. ∎

3. PROOF OF THEOREM 1.6

We divide the proof into ten steps. Since 4-connected graphs on the plane (and hence on the sphere) or projective plane are hamiltonian [17, 20], we assume F^2 has Euler genus at least 2.

Step 1. Cylindrical meshes on handles. Let G and H be graphs, both embedded on the closed surface F^2. We say that H is a *surface minor* of G if the embedding of H can be obtained from the embedding of G by a sequence of contractions and deletions of edges. The following deep result by Robertson and Seymour will be used to guarantee that G contains certain cylindrical meshes.

Lemma 3.1 (Robertson and Seymour [15]). *Let M be a fixed graph embedded on a closed surface F^2. Then, there exists a positive integer $R(M)$ such that if G has an $R(M)$-representative embedding on F^2, then G has M as a surface minor.*

Suppose F^2 has Euler genus $2g$ or $2g+1$, where $g \geq 1$. Let $q \geq 2$ be an integer so that $1/q \leq \varepsilon$. We can find a connected graph M embedded on F^2 that contains g pairwise disjoint copies of $Q = P_{7q+1} \times C_{40}$ ("×" denotes Cartesian product), in such a way that deleting the vertices of one C_{40} in each of the g copies results in a planar or projective-planar graph. Take the representativity of G to be at least $\max\{4, R(M)\}$, with $R(M)$ from Lemma 3.1. Then G has M as a surface minor, with pairwise disjoint subgraphs $Q_1, Q_2, \ldots, Q_g$ of G contracting to the copies of Q in M. Each Q_i has pairwise disjoint cycles $R_{i0}, R_{i1}, \ldots, R_{i,7q}$ (in that order) and paths $S_{i0}, S_{i1}, \ldots, S_{i,39}$ (in that cyclic order) such that each R_{ij} contracts to one of the C_{40} in a copy of Q, each S_{ik} contracts to one of the P_{7q+1} in a copy of Q, and $(\{R_{ij}|0 \leq j \leq 7q\}, \{S_{ik}|0 \leq k \leq 39\})$ is a cylindrical mesh in G. Deleting the vertices of one R_{ij} for each i from G results in a planar or projective-planar graph.

Step 2. Small cylinders. For each i, $1 \leq i \leq g$, choose $m_i \in \{0, 1, \ldots, q-1\}$ so as to minimize $|V(Cyl(R_{i,7m_i}, R_{i,7m_i+7}))|$. Then $|\bigcup_{i=1}^{g} V(Cyl(R_{i,7m_i}, R_{i,7m_i+7}))| < |V(G)|/q \leq \varepsilon|V(G)|$. We will construct a 3-covering all of whose degree 3 vertices lie in this set. To simplify our notation, we assume without loss of generality that $m_i = 0$ for each i, so we will be concerned with $Cyl[R_{i0}, R_{i7}]$ for each i.

Step 3. Empty spaces for cutting. For each i, $1 \leq i \leq g$, define $X_{2i-1} = R_{i0}$, $Y_{2i-1} = R_{i1}$, $Z_{2i-1} = R_{i2}$, $Z_{2i} = R_{i5}$, $Y_{2i} = R_{i6}$, and $X_{2i} = R_{i7}$. By Lemma 2.1 we may apply Lemma 2.2 (i) to each cylinder $Cyl[Y_j, Z_j]$, $1 \leq j \leq 2g$, modifying the paths $S_{\lceil j/2 \rceil, k}$, $0 \leq k \leq 39$, as specified by Lemma 2.2 to preserve the existence of a cylindrical mesh. Thus, we may assume that $Cyl(Y_j, Z_j)$ is empty for each j.

Step 4. Cut G into a planar or projective-planar subgraph and g cylinder subgraphs. Define $H = G - \bigcup_{i=1}^{g} V(Cyl[Z_{2i-1}, Z_{2i}])$, then H has g cylindrical faces, each bounded by Y_{2i-1} and Y_{2i} for some i. By cutting around each such cylindrical face, and filling in the resulting pair of holes with two disks, we obtain an embedding of H in the plane or projective plane, in which each cycle Y_j, $1 \leq j \leq 2g$, bounds a face. Now $V(G)$ is partitioned by H and $Cyl[Z_{2i-1}, Z_{2i}]$, $i \leq i \leq g$. These are all 2-connected graphs, because if there were a cutvertex, either it would be a cutvertex in G, or there would be a nonseparating simple closed curve intersecting G only at the cutvertex, contradicting the fact that G is 4-connected and 4-representative. For similar reasons, any 2-cut or 3-cut S in H must contain at least two vertices of some Y_j. Moreover, $H - S$ has exactly two components, one of which is a subgraph of $Cyl(X_j, Y_j]$.

Now for $1 \leq j \leq 2g$, add a vertex v_j in each face of H bounded by Y_j, joining v_j to each vertex of Y_j that is adjacent in G to a vertex of Z_j. Let H' be the resulting graph embedded in the plane or projective plane. Since H is 2-connected, so is H'. Consider any minimal cutset S' of H' with $|S'| \leq 3$. If S' contains no v_j, it is a cutset in H, using two vertices of some Y_j. Let T be the component of $H - S'$ contained in $Cyl(X_j, Y_j]$. Since G is 4-connected, v_j and T are part of the same component of $H' - S'$. But then there is a nonseparating simple closed curve intersecting G only at S', contradicting the fact that G is 4-representative. Therefore S' contains some v_j. Then $S = S' - \{v_j\}$ is a cutset in H, so $|S| = 2$, and both vertices of S belong to some Y_k. Since S' is minimal, v_j is adjacent to vertices in more than one component of $H' - S'$, so $k = j$. Thus, we have proved that H' is 3-connected, and any 3-cut S' in H' consists of some v_j and two vertices on Y_j. Moreover, $H' - S'$ has exactly two components, one of which is a subgraph of $Cyl(X_j, Y_j]$.

Step 5. Tutte cycle. A *Tutte cycle* C in a graph G is a cycle so that every component of $G - V(C)$ has at most three neighbors on C. If C' is a cycle in G, then a *Tutte cycle with respect to* C' in G is a Tutte cycle C with the added property that any component of $G - V(C)$ containing a vertex of C' has at most two neighbors on C. We construct a Tutte cycle in H' to form the skeleton of our 3-covering of G. Some care is required to avoid getting a 3-cycle, or a cycle restricted to the disk subgraph of H' bounded by X_j for some j.

Since $q \geq 2$, there is $w \in V(G)$ at distance at least two from $\bigcup_{i=1}^{g} Cyl[X_{2i-1}, X_{2i}]$. Let ww_1, ww_2, ..., ww_k be the edges around w in cyclic order, where $k \geq 4$. Since the embedding of G is 3-representative, there is a cycle W in G, and hence in H', containing $w_1, w_2, \ldots, w_k$ in that order, bounding a closed disk containing all faces incident with w. The cycle $W' = ww_1 \cup W[w_1, w_3] \cup w_3w$ is a face of $G - ww_2$ and also of the planar or projective-planar embedding of $H' - ww_2$. Since $H' - ww_2$ is 2-connected, by [20] (if H' is planar) or [17] (if H' is projective-planar) we can find a Tutte cycle C with respect to W' in $H' - ww_2$ through ww_3. If $w_2 \notin V(C)$, let A denote the component of $H' - ww_2 - V(C)$ containing w_2, which has at most two neighbors on C.

Suppose C is a 3-cycle. Then C is a cycle in G. Since G is 4-representative and 4-connected, C is contractible and does not separate G. In other words, C is a face of G, so it must be ww_3ww_4w. But now A contains the successor of w_4 on W, the predecessor of w_3 on W, and w_1 which is adjacent to w, so A has three neighbors on C, a contradiction. Therefore, C is not a 3-cycle.

If $w_2 \notin V(C)$, restoring ww_2 to $H' - ww_2$ adds at most one neighbor on C to the component A, which therefore has at most three neighbors on C. Thus, C is a Tutte cycle in H'.

Let T be a component of $H' - V(C)$. Since C is a Tutte cycle in H' and H' is 3-connected, T has a set S' of exactly three neighbors on C. Since C is not a 3-cycle, S' is a cutset. From above, S' consists of v_j and two vertices of Y_j, for some j, and $H' - S'$ has exactly two components: T, and another component T' that contains $C - S'$. Moreover, one of T or T', call it T_1, is a subgraph of $Cyl(X_j, Y_j]$. By choice of w, w is not adjacent to a vertex of S', so $w \in V(C - S')$. However, $w \notin \bigcup_{i=1}^{g} V(Cyl[X_{2i-1}, X_{2i}])$, so w, and hence $C - S'$, are not in T_1. Thus, $T_1 = T$, so that T is a subgraph of $Cyl(X_j, Y_j]$.

Such a T cannot contain any vertex v_k, so C contains all vertices $v_1, v_2, \ldots, v_{2g}$.

Step 6. Absorb vertices not used by C into the cylinders. Let $\mathcal{T}$ denote the set of components of $H' - V(C)$, and for each j, $1 \leq j \leq 2g$, let $\mathcal{T}_j$ be the set of such components that are adjacent in H' to v_j. From above, $\mathcal{T} = \bigcup_{j=1}^{2g} \mathcal{T}_j$, and each $T \in \mathcal{T}_j$ is adjacent to two vertices y_T, $y'_T \in V(Y_j)$, where we may assume that $Y_j[y_T, y'_T] \cap V(T) \neq \emptyset$. There is a face f_T in $Cyl[X_j, Y_j]$ incident with y_T, y'_T and at least one vertex of T.

Form G' from G by adding in the face f_T the edge $y_Ty'_T$, if it is not already an edge of G, for every $T \in \mathcal{T}$. For each j, $1 \leq j \leq 2g$, let Y'_j be the cycle in G' obtained from Y_j by replacing the segment $Y_j[y_T, y'_T]$ by the edge $y_Ty'_T$ for each $T \in \mathcal{T}_j$; then $V(Y'_j) = V(Y_j) \cap V(C)$. Modify each path S_{ik} to obtain S'_{ik} in G' by replacing any segment $Y_j[y_T, y'_T] \subseteq S_{ik}$ by the edge $y_Ty'_T$. Then $(\{X_{2i-1}, Y'_{2i-1}, Z_{2i-1}, R_{i3}, R_{i4}, Z_{2i}, Y'_{2i}, X_{2i}\}, \{S'_{ik} | 0 \leq k \leq 39\})$ forms a cylindrical mesh in G' for each i.

For each j, the components of each $Cyl_{G'}(Y'_j, Z_j)$ are precisely the elements of $\mathcal{T}_j$, each of which is adjacent to two vertices of Y'_j. Thus, Lemma 2.1 allows us to apply Lemma 2.2 (ii) for each j to find Z'_j (not changing Y'_j) such that $Cyl_{G'}(Y'_j, Z'_j)$ is empty, modifying the paths $S'_{\lceil j/2 \rceil, k}$, $0 \leq k \leq 39$, appropriately, so that for each i, $1 \leq i \leq g$, $(\{X_{2i-1}, Y'_{2i-1}, Z'_{2i-1}, R_{i3}, R_{i4}, Z'_{2i}, Y'_{2i}, X_{2i}\}, \{S'_{ik} | 0 \leq k \leq 39\})$ forms a cylindrical mesh in G'. Each Z'_j is a cycle in G as well as in G' (since it contains no edge $y_T y'_T$), and every vertex of G is either in C or belongs to a cylinder subgraph $Cyl[Z'_{2i-1}, Z'_{2i}]$.

Step 7. Two large subgraphs in each cylinder. For each j, let $r_j, r'_j \in V(Y'_j)$ denote the neighbors of v_j in C. Then in G or G', each r_j is adjacent to s_j and each r'_j is adjacent to s'_j, where $s_j, s'_j \in V(Z'_j)$. If $s_j \neq s'_j$, let $W_j = \{r_j, r'_j\}$ and $V_j = \{s_j, s'_j\}$. If $s_j = s'_j$, then we let x_j and x'_j denote the vertices closest to s_j in either direction along Z'_j that have a neighbor in Y'_j, and we let w_j and w'_j, respectively, be those neighbors. In this case, let $W_j = \{r_j, r'_j, w_j, w'_j\}$ and $V_j = \{s_j = s'_j, x_j, x'_j\}$.

We now claim that for each i, $1 \leq i \leq g$, $Cyl[Z'_{2i-1}, Z'_{2i}]$ has disjoint disk subgraphs L_{2i-1}, L_{2i} with the following properties.

- (i) $L_{2i-1} \cap Z'_{2i-1}$, $L_{2i-1} \cap Z'_{2i}$, $L_{2i} \cap Z'_{2i-1}$, $L_{2i} \cap Z'_{2i}$ are all paths with at least one edge;
- (ii) for $j = 2i-1$ and $2i$, every neighbor of W_j on Z'_j (including every vertex of V_j) belongs to L_j;
- (iii) for $j = 2i-1$ and $2i$, no vertex of Y'_j is adjacent to both components of $Z'_j - V(L_{2i-1} \cup L_{2i})$; and
- (iv) subject to (i), (ii) and (iii), $|V(L_{2i-1} \cup L_{2i})|$ is as large as possible.

We prove this for $i = 1$; the proof for general i is similar. We need only find L_1 and L_2 satisfying (i), (ii) and (iii).

Define $R'_{11} = Y'_1$, $R'_{12} = Z'_1$, $R'_{13} = R_{13}$, $R'_{14} = R_{14}$, $R'_{15} = Z'_2$ and $R'_{16} = Y'_2$. For each j, $1 \leq j \leq 5$, and for each $k \in \mathbf{Z}_{40}$, let U_{jk} denote the disk subgraph of G' bounded by subpaths of R'_{1j}, $R'_{1,j+1}$, S'_{1k} and $S'_{1,k+1}$ that does not contain vertices of any other paths of the cylindrical mesh. We call U_{jk} a *cell* of the mesh. Let $[i,j]$ denote the set $\{i, i+1, \ldots, j\}$ either as an interval in the integers, or as a cyclic interval in $\mathbf{Z}_{40} = \{0, 1, \ldots, 39\}$—it will be clear from context which is intended. Let $U_{j,[k_1,k_2]}$ denote $\bigcup_{k \in [k_1,k_2]} U_{jk}$ and $U_{[j_1,j_2],[k_1,k_2]}$ denote $\bigcup_{j \in [j_1,j_2]} \bigcup_{k \in [k_1,k_2]} U_{jk}$.

Let $U_{1,[a,a+\alpha]}$ be a contiguous block of cells that contains V_1, such that α is as small as possible. Then $\alpha \leq 20$. The neighbors of V_1 on R'_{11}, including W_1, lie in $U_{1,[a-1,a+\alpha+1]}$. Therefore, the neighbors of W_1 on $R'_{12} = Z'_1$ lie in $U_{1,[a-2,a+\alpha+2]} \cap R'_{12} \subseteq U_{2,[a-3,a+\alpha+3]} \cap R'_{12}$. Similarly, there are b and $\beta \leq 20$ such that the neighbors of W_2 on $R'_{15} = Z'_2$ lie in $U_{4,[b-3,b+\beta+3]} \cap R'_{15}$.

Now $L^1_2 = U_{4,[b-3,b+\beta+3]}$ and $L^2_2 = U_{[2,4],[a-7,a-5]}$ together use up at most $27+3 = 30$ of the 40 cells U_{4j}, in one or two contiguous blocks. Therefore there is a block of at least 5 contiguous unused cells. Hence, we can choose c so that $U_{4,[c,c+2]}$ is a block of 3 cells disjoint from $L^1_2 \cup L^2_2$. If $[b-3, b+\beta+3] \cup [a-7, a-5]$ is a cyclic interval in $\mathbf{Z}_{40}$, define $L^3_2 = \emptyset$; otherwise, define L^3_2 to be whichever of $U_{4,[b+\beta+4,a-8]}$ or $U_{4,[a-4,b-4]}$ does not intersect $U_{4,[c,c+2]}$. Let $L^1_1 = U_{2,[a-3,a+\alpha+3]}$ and $L^2_1 = U_{[2,4],[c,c+2]}$. If $[a-3, a+\alpha+3] \cup [c, c+2]$ is a cyclic interval in $\mathbf{Z}_{40}$, define $L^3_1 = \emptyset$; otherwise, define $L^3_1 = U_{2,[a+\alpha+4,c-1]}$.

Then $L_1 = L^1_1 \cup L^2_1 \cup L^3_1$ and $L_2 = L^1_2 \cup L^2_2 \cup L^3_2$ are both unions of contiguous blocks of cells, using cyclic intervals of cells along $R'_{12} = Z'_1$ and $R'_{15} = Z'_2$, giving (i). Property (ii) is immediate from our construction. For (iii), consider any v on $R'_{11} = Y'_1$. Since v belongs to at most two cells U_{1j}, the neighbors of v on $R'_{12} = Z'_1$ lie in $U_{1,[d,d+1]}$ for some d. Since both $L^1_1 \cup L^3_1$ and L^2_2 use at least three contiguous blocks U_{2j}, it is not possible for $U_{1,[d,d+1]}$ to intersect both components of $Z'_1 - V(L_1 \cup L_2) = R'_{12} - V((L^1_1 \cup L^3_1) \cup L^2_2)$. A similar argument applies to vertices of Y'_2.

Step 8. The remainder of each cylinder. Now we show that for each i, $Cyl[Z'_{2i-1}, Z'_{2i}]$ contains four additional subgraphs M_{jl}, $j = 2i-1$ or $2i$ and $l = 1$ or 2, each of which intersects $L_{2i-1} \cup L_{2i}$ at exactly two vertices $u_{j,2l-1}, u_{j,2l}$ of Z'_j. We begin with the case $i = 1$.

There are vertices $u_{11}, u_{12}, u_{13}, u_{14}$ in order along Z'_1, and $u_{21}, u_{22}, u_{23}, u_{24}$ in order along Z'_2, such that $\partial L_1 = Z'_1[u_{14}, u_{11}] \cup Z'_2[u_{24}, u_{21}] \cup P_4 \cup P_1$ and $\partial L_2 = Z'_1[u_{12}, u_{13}] \cup Z'_2[u_{22}, u_{23}] \cup P_2 \cup P_3$, where each P_k is a path from u_{1k} to u_{2k} internally disjoint from $Z'_1 \cup Z'_2$. Write $Q_{jk} = Z'_j[u_{jk}, u_{j,k+1}]$ (subscripts added modulo 4).

We first claim that u_{11} and u_{12} lie on a common face of G'. Consider the boundaries of the faces containing u_{11}. If they do not contain u_{12}, then there must exist a path joining $Q_{11} - \{u_{11}, u_{12}\}$ and $(P_1 \cup Q_{21} \cup P_2) - \{u_{11}, u_{12}\}$. This contradicts the maximality of $|V(L_1 \cup L_2)|$.

Thus, we can add an edge $u_{11}u_{12}$ (if it does not already exist) through this face. In the same way, we can add an edge $u_{21}u_{22}$. Consider the disk subgraph U_1 bounded by P_1, P_2 and $u_{11}u_{12}$, $u_{21}u_{22}$. If U_1 contains an inner vertex v, then since G is 3-connected, there exist three disjoint paths joining v to the boundary of U_1. However, this also contradicts the maximality of $|V(L_1 \cup L_2)|$. Thus, U_1 has no interior vertices.

Similarly, U_2 has no interior vertices, where U_2 is bounded by P_3, P_4 and $u_{13}u_{14}$, $u_{23}u_{24}$ (we add these edges as before).

If Q_{11} is the single edge $u_{11}u_{12}$, define $M_{11} = Q_{11}$. Otherwise, let M_{11} denote the disk subgraph bounded by $Q_{11} \cup \{u_{11}u_{12}\}$. Let q_{11} denote the vertex of $Q_{11} - u_{11}$ closest to u_{11} that has a neighbor p_{11} on Y'_1, and let q_{12} denote the vertex of $Q_{11} - u_{12}$ closest to u_{12} that has a neighbor p_{12} on Y'_1. Since G is 4-connected, $q_{11} \neq q_{12}$, and we may assume that $p_{11} \neq p_{12}$.

In a similar way we can construct M_{21} bounded by $Q_{21} \cup \{u_{21}u_{22}\}$, M_{12} bounded by $Q_{13} \cup \{u_{13}u_{14}\}$, and M_{22} bounded by $Q_{23} \cup \{u_{23}u_{24}\}$. More generally, for every j and l, $1 \leq j \leq 2g$ and $1 \leq l \leq 2$, we can construct M_{jl} and, if appropriate, $q_{j,2l-1}$, $p_{j,2l-1}$, $q_{j,2l}$, $p_{j,2l}$. By property (iii) of Step 7, p_{j1}, p_{j2}, p_{j3}, p_{j4} are all distinct, and by property (ii) of Step 7 none of these vertices belong to W_j. Because the disk subgraphs U_{2i-1}, U_{2i} have no interior vertices, every vertex of $Cyl[Z'_{2i-1}, Z'_{2i}]$ belongs to exactly one of L_{2i-1}, L_{2i}, or $M_{jl} - \{u_{j,2l-1}, u_{j,2l}\}$, $j = 2i-1$ or $2i$ and $l = 1$ or 2.

Step 9. Spanning each L_j and M_{jl}. In [16], Sanders and Zhao proved the following theorem. They stated it for "2-connected graphs without any interior component 3-cuts" but these are exactly our I4CD graphs.

THEOREM 3.7 (Sanders and Zhao [16], Lemma 6.2). *Let G be an I4CD graph and let x, y be two distinct vertices in ∂G. Then G has a 3-covering K such that $E(\partial G) \subseteq E(K)$ and $\deg_K(x) = 2$, $\deg_K(y) = 2$.*

For each L_j we construct two subgraphs which together include all vertices of L_j, and connect L_j to C. First suppose that $s_j \neq s'_j$. Let D'_j denote whichever of $Z'_j[s_j, s'_j]$ and $Z'_j[s'_j, s_j]$ lies in L_j, and let $D_j = D'_j \cup \{r_j s_j, r'_j s'_j\}$. By Lemma 2.1 we may apply Theorem 3.7 to L_j to obtain a 3-covering E_j in which s_j, s'_j have degree 2. Note that D_j and E_j share the path D'_j. Now suppose that $s_j = s'_j$. Let D_j be the path $r_j s_j r'_j$. The graph $L_j \cup \{x_j s_j, x'_j s_j\}$ is a disk subgraph of the 4-connected embedded graph $G \cup \{x_j s_j, x'_j s_j\}$ and so is I4CD by Lemma 2.1. Apply Theorem

3.7 to this graph to obtain a 3-covering E'_j in which s_j has degree 2, which contains $x_j s_j$ and $x'_j s_j$. Let $E_j = (E'_j - \{s_j\}) \cup \{w_j x_j, w'_j x'_j\}$.

Now for each M_{jl} we construct a subgraph which includes all vertices of $M_{jl} - \{u_{j,2l-1}, u_{j,2l}\}$, and which connects this subgraph to C. First, if M_{jl} is just a single edge $u_{j,2l-1}u_{j,2l}$, let $F_{jl} = \emptyset$. Now suppose M_{jl} is not a single edge. The graph $M_{jl} \cup \{u_{j,2l-1}q_{j,2l-1}, u_{j,2l}q_{j,2l}\}$ is a disk subgraph of the 4-connected embedded graph $G \cup \{u_{j,2l-1}u_{j,2l}, u_{j,2l-1}q_{j,2l-1}, u_{j,2l}q_{j,2l}\}$, so it is I4CD by Lemma 2.1. Apply Theorem 3.7 to this graph to obtain a 3-covering F'_{jl} in which $u_{j,2l-1}$, $u_{j,2l}$ have degree 2, which contains $u_{j,2l-1}u_{j,2l}$, $u_{j,2l-1}q_{j,2l-1}$ and $u_{j,2l}q_{j,2l}$. Let $F_{jl} = (F'_{jl} - \{u_{j,2l-1}, u_{j,2l}\}) \cup \{p_{j,2l-1}q_{j,2l-1}, p_{j,2l}q_{j,2l}\}$.

Step 10. Join everything together and verify 2-connectedness. The proof of the following lemma is straightforward.

LEMMA 3.2. *Let G_1 and G_2 be 2-connected graphs, and suppose we form G from G_1 and G_2 in one of the following ways.*

(i) *Identify a path on at least two vertices in G_1 with a path of the same length in G_2.*

(ii) *Take a path $u_0u_1 \ldots u_k$ in G_2, such that all of $u_1, u_2, \ldots, u_{k-1}$ have degree 2, and let $G = G_1 \cup (G_2 - \{u_1, u_2, \ldots, u_{k-1}\}) \cup \{u_0v, u_kw\}$ where v and w are distinct vertices of G_1.*

Then G is 2-connected.

Let $C' = C - \{v_1, v_2, \ldots, v_{2g}\}$. We claim that $C' \cup \bigcup_{j=1}^{2g}(D_j \cup E_j \cup F_{j1} \cup F_{j2})$ is the required 3-covering. By construction it spans all vertices of G, and has at most $\varepsilon|V(G)|$ vertices of degree greater than 2. It does not use any of the edges we added to G in Step 6 or Step 8. By the last paragraph of Step 8, we do not create any vertices of degree greater than 3. We use Lemma 3.2 to verify that it is 2-connected. By our construction, $C' \cup \bigcup_{j=1}^{2g} D_j$ is a cycle. For each j, we may apply Lemma 3.2 (i) with $G_2 = E_j$ if $s_j \neq s'_j$, or Lemma 3.2 (ii) with $G_2 = E'_j$ if $s_j = s'_j$, to show that we retain 2-connectedness when we add E_j. For each j and l, we may also apply Lemma 3.2 (ii) with $G_2 = F'_{jl}$ to show that we retain 2-connectedness when we add F_{jl}. This completes the proof. ∎

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