Maximum spectral radius of outerplanar 3-uniform hypergraphs

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Abstract

In this paper, we study the maximum spectral radius of outerplanar 3-uniform hypergraphs. Given a hypergraph $H$, the shadow of $H$ is a graph $G$ with $V(G) = V(H)$ and $E(G) = \{uv : uv \in h \text{ for some } h \in E(H)\}$. A graph is outerplanar if it can be embedded in the plane such that all its vertices lie on the outer face. A 3-uniform hypergraph $H$ is called outerplanar if its shadow has an outerplanar embedding such that every hyperedge of $H$ is the vertex set of an interior triangular face of the shadow. Cvetković and Rowlinson conjectured in 1990 that among all outerplanar graphs on $n$ vertices, the graph $K_1 + P_{n-1}$ attains the maximum spectral radius. We show a hypergraph analogue of the Cvetković-Rowlinson conjecture. In particular, we show that for sufficiently large $n$, the $n$-vertex outerplanar 3-uniform hypergraph of maximum spectral radius is the unique 3-uniform hypergraph whose shadow is $K_1 + P_{n-1}$.

1 Introduction

A graph $G$ is planar if it can be embedded in the plane, i.e., it can be drawn on the plane in such a way that edges intersect only at their endpoints. A graph is outerplanar if it can be embedded in the plane so that all vertices lie on the boundary of its outer face. The study of the spectral radius of (outer)planar graphs has a long history, dating back to Schwenk and Wilson [15]. Given a graph $G$, the spectral radius $\lambda$ of $G$ is the largest eigenvalue of the adjacency matrix of $G$. The spectral radius of planar graphs is useful in geography as a measure of the overall connectivity of a planar graph [1, 5]. It is therefore of interest to geographers to find the maximum spectral radius of a planar graph as a theoretical upper bound for the connectivity of networks. Boots and Royle [1], and independently Cao and Vince [2] conjectured that the extremal planar graph achieving the maximum spectral radius is $K_2 + P_{n-2}$ (see Figure 1). Hong [17] first showed that for an $n$-vertex planar graph $G$, $\lambda(G) \leq \sqrt{5n - 11}$. This was subsequently improved in a series of papers [2, 18, 8, 19, 6]. Guiduli and Hayes [9] showed in an unpublished preprint that the Boots-Royle-Cao-Vince conjecture is true for sufficiently large $n$. For outerplanar graphs, it is conjectured by Cvetković and Rowlinson [5] that among all outerplanar graph on $n$ vertices, $K_1 + P_{n-1}$ attains the maximum spectral radius (see Figure 1). Partial progress has been made by Rowlinson [14], Cao and Vince [2], and Guiduli and Hayes [9]. Recently, Tait and Tobin [16] proved the Boots-Royle-Cao-Vince conjecture and the Cvetković-Rowlinson conjecture for large enough $n$. Lin and Ning [11] showed that the Cvetković-Rowlinson conjecture holds for all $n \geq 2$ except for $n = 6$.

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In this paper, we extend the investigations into the maximum spectral radius of outerplanar 3-uniform hypergraphs. Given a hypergraph \( \mathcal{H} \), the \textit{shadow} of \( \mathcal{H} \), denoted by \( \partial(\mathcal{H}) \), is a 2-uniform graph \( G \) with \( V(G) = V(\mathcal{H}) \) and \( E(G) = \{ uv : uv \in h \text{ for some } h \in E(\mathcal{H}) \} \).

We adopt Zykov’s [20] definition of hypergraph planarity. In particular, a 3-uniform hypergraph \( \mathcal{H} \) is called \textit{planar} if \( \partial(\mathcal{H}) \) has a planar embedding so that every hyperedge of \( \mathcal{H} \) is the vertex set of a triangular face of \( \partial(\mathcal{H}) \). A 3-uniform hypergraph \( \mathcal{H} \) is called \textit{outerplanar} if \( \partial(\mathcal{H}) \) has an outerplanar embedding such that every hyperedge of \( \mathcal{H} \) is the vertex set of an interior triangular face of \( \partial(\mathcal{H}) \).

Now we define the spectral radius of an \( r \)-uniform hypergraph. Given positive integers \( r \) and \( n \), an order \( r \) and dimension \( n \) tensor \( \mathbf{A} = (a_{i_1i_2\ldots i_r}) \) over \( \mathbb{C} \) is a multidimensional array with all entries \( a_{i_1i_2\ldots i_r} \in \mathbb{C} \) for all \( i_1, i_2, \ldots, i_r \in [n] = \{1, 2, \ldots, n\} \). Given a column vector \( \mathbf{x} = (x_1, x_2, \ldots, x_n)^T \in \mathbb{C}^n \), \( \mathbf{A}\mathbf{x}^{r-1} \) is defined to be a vector in \( \mathbb{C}^n \) whose \( i \)-th entry is

\[
(\mathbf{A}\mathbf{x}^{r-1})_i = \sum_{i_2, \ldots, i_r=1}^n a_{i_1i_2\ldots i_r}x_{i_2}\ldots x_{i_r}.
\]

In 2005, Qi [12] and Lim [10] independently proposed the definition of eigenvalues of a tensor. In particular, if there exists a number \( \lambda \in \mathbb{C} \) and a nonzero vector \( \mathbf{x} \in \mathbb{C}^n \) such that

\[
\mathbf{A}\mathbf{x}^{r-1} = \lambda \mathbf{x}^{[r-1]}
\]

where \( \mathbf{x}^{[r-1]} = (x_1^{r-1}, x_2^{r-1}, \ldots, x_n^{r-1})^T \), then \( \lambda \) is called the \textit{eigenvalue} of \( \mathbf{A} \) and \( \mathbf{x} \) is called an \textit{eigenvector} of \( \mathbf{A} \) corresponding to \( \lambda \). The \textit{spectral radius} of \( \mathbf{A} \), denoted by \( \lambda(\mathbf{A}) \), is the maximum modulus of the eigenvalues of \( \mathbf{A} \). It was shown in [13] that

\[
\lambda(\mathbf{A}) = \max_{\mathbf{x} \in \mathbb{R}_+^n} \frac{\mathbf{x}^T \mathbf{A}\mathbf{x}^{r-1}}{||\mathbf{x}||_r^{r-1}}
\]

where \( ||\mathbf{x}||_r := (|x_1|^r + |x_2|^r + \cdots + |x_n|^r)^{1/r} \) and \( \mathbb{R}_+ \) is the set of nonnegative real numbers.

In 2012, Cooper and Dutle [4] defined the \textit{adjacency tensor} of an \( r \)-uniform hypergraph. Given an \( r \)-uniform hypergraph \( \mathcal{H} \) on \( n \) vertices, the adjacency tensor \( \mathcal{A}(\mathcal{H}) \) of \( \mathcal{H} \) is defined as the order \( r \) dimension \( n \) tensor with entries \( a_{i_1i_2\ldots i_r} \) such that

\[
a_{i_1i_2\ldots i_r} = \begin{cases} 
\frac{1}{(r-1)!} & \text{if } \{i_1, i_2, \ldots, i_r\} \in E(\mathcal{H}) \\
0 & \text{otherwise}.
\end{cases}
\]

Let \( \lambda(\mathcal{H}) \) denote the spectral radius of \( \mathcal{A}(\mathcal{H}) \). Given an \( r \)-uniform hypergraph \( \mathcal{H} \) and a vector \( \mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \), we can define a multi-linear function \( P_{\mathcal{H}}(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R} \) as follows:

\[
P_{\mathcal{H}}(\mathbf{x}) = r \sum_{\{i_1, i_2, \ldots, i_r\} \in E(\mathcal{H})} x_{i_1}x_{i_2}\ldots x_{i_r}.
\]
Then the spectral radius of $\mathcal{H}$ can be also expressed as

$$\lambda(\mathcal{H}) := \max_{\|x\|_r = 1} P_{\mathcal{H}}(x) = \max_{x \in \mathbb{R}^n_+} \frac{P_{\mathcal{H}}(x)}{\|x\|_r^r}.$$ 

The Perron-Frobenius theorem [3, 7] for nonnegative tensors implies that there is always a nonnegative vector $x$ satisfying the maximum at right above. Any such $x$ is called a Perron-Frobenius eigenvector of $A(\mathcal{H})$ (corresponding to $\lambda(\mathcal{H})$). If $\mathcal{H}$ is connected then a Perron-Frobenius eigenvector is strictly positive and is unique up to scaling by a positive coefficient; moreover, the spectral radius $\lambda(\mathcal{H})$ is the unique eigenvalue with a strictly positive eigenvector. By definition, the spectral radius $\lambda(\mathcal{H})$ and its eigenvector $x = (x_1, \ldots, x_n)$ also satisfy the following eigenequation for every $x_i$:

$$\lambda(H)x_i^{r-1} = \sum_{\{i, j_2, \ldots, j_r\} \in E(\mathcal{H})} x_{j_2} \cdots x_{j_r} \text{ for } x_i > 0.$$ 

Now we are ready to state our main theorem. We use $\mathcal{F}_n$ to denote the fan hypergraph, i.e., the unique 3-uniform hypergraph whose shadow is $K_1 + P_{n-1}$ (see Figure 1).

**Theorem 1.** For large enough $n$, the $n$-vertex outerplanar 3-uniform hypergraph of maximum spectral radius is the fan hypergraph $\mathcal{F}_n$.

The shadow of the extremal hypergraph attaining the maximum spectral radius among all outerplane 3-uniform hypergraphs is exactly the extremal graph attaining the maximum spectral radius among all outerplanar graphs. This motivates us to make the following analogous conjecture for planar 3-uniform hypergraphs:

**Conjecture 1.** For large enough $n$, the $n$-vertex planar 3-uniform hypergraph graph $\mathcal{H}$ of maximum spectral radius is the unique maximal hypergraph whose shadow is $K_2 + P_{n-2}$.

### 2 Proof of Theorem 1

Let $\mathcal{H}$ be an $n$-vertex outerplanar 3-uniform hypergraph of maximum spectral radius. Let $G$ be the shadow of $\mathcal{H}$, i.e., $V(G) = V(\mathcal{H})$ and $E(G) = \{vu : \{v, u\} \subseteq h \text{ for some } h \in E(\mathcal{H})\}$. It follows by definition that $G$ is outerplanar, and thus does not contain a $K_{2,3}$ minor or a $K_4$ minor. Observe that $\mathcal{H}$ must be edge-maximal (while maintaining the outerplanarity). Otherwise, we can obtain an outerplanar hypergraph $\mathcal{H}'$ such that $\mathcal{H} \subseteq \mathcal{H}'$. It then follows from the Perron-Frobenius Theorem that $\mathcal{H}'$ attains a larger spectral radius than $\mathcal{H}$, giving us a contradiction. Now since $\mathcal{H}$ is edge-maximal, $G$ must be a maximal outerplanar graph, with $2n - 3$ edges. Then $G$ is 2-connected, and has an outerplanar embedding, unique up to homeomorphisms of the plane, whose outer face is bounded by a Hamilton cycle. We always assume $G$ has this outerplanar embedding. All interior faces of $G$ are triangles, and every triangle of $G$ is a facial triangle and a hyperedge of $\mathcal{H}$. The dual of $G$ (excluding the outer face) is a tree, so the interior faces of $G$ are connected together in a treelike fashion.

We use $N(v)$ to denote the set of neighbors of $v$ in $G$, i.e., $N(v) = \{u : vu \in E(G)\}$ and $d(v)$ to denote the degree of $v$, i.e., $d(v) = |N(v)|$. We also use $d_F(v)$ to refer to degree in a subgraph $F$ of $G$. The closed neighborhood of $v$, denoted by $\overline{N}[v]$, is defined as $\overline{N}[v] = N(v) \cup \{v\}$. More generally, we let $\text{dist}(u,v)$ denote the distance between $u$ and $v$ in $G$, and $N_k(v) = \{u \in V(G) : \text{dist}(v,u) = k\}$. Given an edge $uv$ and vertex $v$ define the level of $uv$ relative to $v$ to be $(\text{dist}(u,v) + \text{dist}(w,v))/2$, which is an integer or half-integer.
Let $\Gamma(v) = \{uw : \{v, u, w\} \in E(H)\}$ be the link of $v$ in $H$, and $d_H(v) = |\Gamma(v)|$ be the degree of $v$ in $H$. The edges in $\Gamma(v)$ form an induced path in $G$ whose ends are the neighbors of $v$ on the outer cycle. For each edge $e$ of $G$, $\Gamma^{-1}(e)$ is the set of vertices forming a triangle with $e$, and contains one vertex if $e$ is an outer edge, and two vertices otherwise. We also use $\Sigma(v)$ to denote the set of edges incident with $v$ in $G$. In our situation the edges in $\Sigma(v)$ and $\Gamma(v)$ are precisely the edges at levels $\frac{1}{2}$ and 1, respectively, relative to $v$.

Suppose we are given an edge $uw$ and a vertex $v$ not incident with $uw$. If $uw$ is an outer edge, define $\Phi(uw, v)$ to be the empty graph. Otherwise, $G\setminus \{u, w\}$ has two components; define $\Phi(uw, v)$ to be the component not containing $v$, together with all edges joining that component to $u$ or $w$. Loosely, $\Phi(uw, v)$ is the subgraph of $G$ on the far side of $uw$ from $v$.

**Lemma 1.** $\lambda(H) \geq \sqrt[3]{4(n-1)} \left(1 - \frac{1}{n-1}\right)$.

**Proof.** Let $F_n$ be the fan hypergraph on $n$ vertices, i.e., the unique 3-uniform hypergraph on $n$ vertices whose shadow is $K_1 + P_{n-1}$. Suppose $w$ is the vertex that is adjacent to all the other vertices in $\partial(F_n)$ and $v_1, v_2, \ldots, v_{n-1}$ are its neighbors. Clearly $F_n$ is outerplanar. Consider the vector $x \in \mathbb{R}^n$ with $x_w = 1/\sqrt[3]{3}$ and $x_{v_i} = \left(\frac{2}{3(n-1)}\right)^{1/3}$. Note that $\|x\|_3 = 1$. It follows that

$$\lambda(H) \geq \lambda(F_n) \geq P_{F_n}(x) = 3(n-2) \cdot \frac{1}{\sqrt[3]{3}} \cdot \left(\frac{2}{3(n-1)}\right)^{2/3} = \sqrt[3]{4(n-1)} \left(1 - \frac{1}{n-1}\right).$$

\[\Box\]

Note that since $H$ is connected, there exists an eigenvector corresponding to $\lambda(H)$ such that all its entries are strictly positive. In the rest of this section, for convenience we assume that this Perron-Frobenius eigenvector $x$ of $H$ is re-normalized so that the maximum eigenvector entry is 1. Let $v_0$ be the vertex with the maximum eigenvector entry, so that $x_{v_0} = 1$. We also define $u_0$ to be a vertex with the second largest eigenvector entry, i.e., $x_{u_0} = \max_{v \neq v_0} x_v$. We abbreviate $\lambda(H)$ to $\lambda$, and the eigenequation of $H$ tells us that $\lambda x_v^2 = \sum_{uw \in \Gamma(v)} x_u x_w$ for every vertex $v$.

The following lemma says that $H$ is very close to the fan hypergraph $F_n$.

**Lemma 2.** We have $\lambda = (1 + o(1)) \sqrt[3]{4n}$ and $d_G(v_0) \geq n - O(n^{2/3})$. Moreover, for any other vertex $u \neq v_0$, $x_u = O(n^{-1/3})$.

We first show a weaker version of Lemma 2. In particular, we show the following claim.

**Claim 1.** $d_G(v_0) \geq n - O(n^{5/6})$.

**Proof of Claim 1.** Let $x$ and $v_0$ be as described above, so that $x_{v_0} = 1$. Let $d = d(v_0)$ and suppose that $\Gamma(v_0)$ forms the path $v_1v_2\ldots v_d$, where $v_1, v_2, \ldots, v_d$ are in clockwise order around $v_0$. Now by the eigenequation for $x_{v_0}$,

$$\lambda = \lambda x_{v_0}^2 = \sum_{i=1}^{d-1} x_{v_i} x_{v_{i+1}} \leq \sum_{i=1}^{d} x_{v_i}^2,$$

using the fact $ab \leq (a^2 + b^2)/2$. Set $z = \sum_{i=1}^{d} x_{v_i}^2$. We have $\lambda \leq z$. It again follows from the eigenequation expansion that

$$\lambda z = \sum_{i=1}^{d} \lambda x_{v_i}^2 = \sum_{i=1}^{d} \sum_{vw \in \Gamma(v_i)} x_v x_w$$

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Define $B$ to be the subgraph consisting of the edges in $\bigcup_{i=1}^{d} \Gamma(v_i) \setminus \Sigma(v_0)$ and their endvertices. The edges of $B$ are at levels 1 and 2 relative to $v_0$. For $i \in [d-1]$ let $F_i = \Phi(v_i, v_{i+1}, v_0)$ and $B_i = B \cap F_i$. Fig. 2 shows $B_i$, indicated in bold (and red, if color is visible). From (1), using $x_{v_0} = 1$ and the Cauchy-Schwarz inequality, we have

$$\lambda z \leq 2 \left( \sum_{i=1}^{d} x_{v_i} + \sum_{vw \in E(B)} x_v x_w \right) \leq 2 \sqrt{dz} + \sum_{vw \in E(B)} x_v x_w. \quad (2)$$

For ease of reference, set $R = \sum_{vw \in E(B)} x_v x_w$. Dividing both sides of inequality (2) by $\lambda$, we have

$$z - 2 \sqrt{d} \leq \frac{R}{\lambda} \leq \frac{R}{\lambda} + \frac{d}{\lambda^2}. \quad (3)$$

Rearranging the terms of the inequality, we obtain that

$$z \leq \left( \sqrt{\frac{d}{\lambda^2} + \frac{R}{\lambda}} \right)^2 = \frac{4d}{\lambda^2} + \frac{2R}{\lambda} \left( \sqrt{\frac{d}{\lambda^2} + \frac{R}{\lambda}} - \sqrt{d} \right)^2. \quad (3)$$

It follows that

$$\lambda^3 \leq \lambda^2 z \leq 4d + 2\lambda R - \left( \sqrt{\frac{d}{\lambda^2} + \frac{R}{\lambda}} - \sqrt{d} \right)^2. \quad (4)$$

By Lemma 1, we obtain that $\lambda^3 \geq 4n - 16$ when $n$ is large enough.

Now we find a bound on $2\lambda R$. Using $ab \leq (a^2 + b^2)/2$ and then the eigenequations, twice, we have

$$2\lambda R = \sum_{vw \in E(B)} 2\lambda x_v x_w \leq \sum_{vw \in E(B)} \lambda(x_v^2 + x_w^2) = \sum_{u \in V(B)} d_B(u) \lambda x_u^2.$$
If $x$ is the coefficient of $x$, if it is not an edge of $v$, then it contains a common neighbor $p$

We estimate $S_i v_i$, $v_{i+1}$.

Each edge $F_i$ appears in the sum above then the level of $w$ relative to $v_0$ is between 1 and 3.

To investigate the sum in (5) we examine the structure of $F_i$ and $B_i$ more closely. If $F_i$ is nonempty then it contains a common neighbor $q_i$ of $v_i$ and $v_{i+1}$. The vertices of $N_2(v_0) \cap F_i$ lie on a path $p_i^1 p_i^2 \ldots p_i^j$, with $q_i = p_{i}^{\alpha_i}$ for some $\alpha_i$ with $1 \leq \alpha_i \leq \beta_i$. Here $p_i^j$ is adjacent to $v_i$ for $1 \leq j \leq \alpha_i$, and to $v_{i+1}$ for $\alpha_i \leq j \leq \beta_i$. The subgraph $B_i$ contains the edges of this path and edges $v_i q_i$, $v_{i+1} q_i$. We let $F_i' = \Phi(p_i^j p_i^{j+1}, v_0)$ be the part of $F_i$ above $p_i^j p_i^{j+1}$ for $j \in [\beta_i - 1]$. See Fig. 3, which illustrates part of $F_i$ and the edge coefficients in the sum from (5) ($v_i v_{i+1}$ is dashed because it is not an edge of $F_i$). For $u = p_i^j$ with $j \notin \{1, \alpha_i, \beta_i\}$ we have $d_B(u) = 2$; for $j \in \{1, \alpha_i, \beta_i\}$ the value of $d_B(u)$ depends on whether $\alpha_i = 1$ or $\beta_i = \beta_i$ or both, but we always have $1 \leq d_B(u) \leq 4$, and $d_B(u) > 2$ only if $j = \alpha_i$.

Each edge $v_i v_{i+1} \in \Gamma(v_0)$ occurs in the sum in (5) only as part of $\Gamma(q_i)$, when $q_i = p_i^{\alpha_i}$ exists, so the coefficient of $x_{v_i} x_{v_{i+1}}$ is at most 4. Thus, the contribution from $\Gamma(v_0)$, using the eigenequation for $x_{v_0}$, is

$$S_0 = \sum_{u \in V(B) \cap N_2(v_0)} d_B(u) \sum_{vw \in \Gamma(u) \cap \Gamma(v_0)} x_v x_w \leq 4 \sum_{i=1}^{d-1} x_{v_i} x_{v_{i+1}} = 4 \lambda x_{v_0}^2 = 4 \lambda.$$  

Assuming $F_i$ is nonempty, the part of the sum in (5) coming from $vw \in E(F_i)$, $i \in [d - 1]$, is

$$S_i = \sum_{u \in V(B_i) \cap N_2(v_0)} d_B(u) \sum_{vw \in \Gamma(u) \cap \Gamma(v_0)} x_v x_w.$$

We estimate $S_i$ by computing the sum of the coefficients, i.e.,

$$\tilde{S}_i = \sum_{u \in V(B_i) \cap N_2(v_0)} d_B(u) |\Gamma(u) \cap \Gamma(v_0)| \sum_{j=1}^{\beta_i} d_B(p_i^j) |\Gamma(p_i^j) \cap \Gamma(v_0)|.$$
Let $d^-(p^j_i)$ be the degree of $p^j_i$ in $F^j_{i-1}$ (or 0 if $j = 1$), and $d^+(p^j_i)$ be its degree in $F^j_i$ (or 0 if $j = \beta_i$). For $j \notin \{1, \alpha_i, \beta_i\}$ we have

$$|\Gamma(p^j_i) \setminus \Gamma(v_0)| = |\Gamma(p^j_i)| = d(p^j_i) - 1 = (d^-(p^j_i) + d^+(p^j_i) + 3) - 1 = d^-(p^j_i) + d^+(p^j_i) + 2.$$  

For $j = \alpha_i$, if $j \neq 1, \beta_i$ we have

$$|\Gamma(p^j_i) \setminus \Gamma(v_0)| = |\Gamma(p^j_i)| - 1 = d(p^j_i) - 2 = (d^-(p^j_i) + d^+(p^j_i) + 4) - 2 = d^-(p^j_i) + d^+(p^j_i) + 2.$$  

If $j = 1$ we must reduce the above values by 1 since there is no edge $vvp^j_{i-1}$, and similarly if $j = \beta_i$ we must reduce these values by 1 since there is no edge $v_{i+1}p^j_{i+1}$. These reductions are independent, and do not depend on whether $\alpha_i = 1$ or $\alpha_i = \beta_i$ or both. Therefore,

$$\tilde{S}_i = \sum_{j=1}^{\beta_i} d_B(p^j_i)\left(d^-(p^j_i) + d^+(p^j_i) + 2\right) - d_B(p^j_i) - d_B(p^j_i).$$

We are going to compare $\tilde{S}_i$ to $|E(F_i)|$. The number of edges in $F_i$ at levels $1\frac{1}{2}$ and 2 relative to $v_0$ is just $2\beta_i$. The edges at higher levels belong to some $F^j_i$. If $F^j_i$ is nonempty, then it contains $2d^+(p^j_i)$ edges of $\Sigma(p^j_i) \cup \Gamma(p^j_i)$, and $2d^-(p^j_i+1)$ edges of $\Sigma(p^j_i+1) \cup \Gamma(p^j_i+1)$, but these two sets overlap in two edges. Thus, $|E(F^j_i)| \geq 2d^+(p^j_i) + 2d^-(p^j_i+1) - 2$. Hence,

$$|E(F^j_i)| \geq 2\beta_i + \sum_{j:F^j_i \neq \emptyset} \left(2d^+(p^j_i) + 2d^-(p^j_i+1) - 2\right) = 2\beta_i + \sum_{(j,\sigma):d^\sigma(p^j_i) > 0} \left(2d^\sigma(p^j_i) - 1\right)$$

where in the last sum $j \in [\beta_i]$ and $\sigma \in \{-, +\}$.

Therefore, when $F_i$ is nonempty,

$$2|E(F^j_i)| - \tilde{S}_i \geq \sum_{j=1}^{\beta_i} \left(4 - 2d_B(p^j_i)\right) + d_B(p^j_i) + d_B(p^j_i) + \sum_{(j,\sigma):d^\sigma(p^j_i) > 0} \left(4 - d_B(p^j_i)\right) d^\sigma(p^j_i) - 2). \quad (7)$$

In the first sum, only terms with $j \in \{1, \alpha_i, \beta_i\}$ can be nonzero. In the final sum, only terms with $d_B(p^j_i) > 2$, which requires $j = \alpha_i$, can be negative. (Fig. 3 shows a situation where we have a negative term in the final sum.) We consider several situations.

(i) If $F_i$ is empty or $1 = \alpha_i = \beta_i$ then $\tilde{S}_i = 0$, so $\tilde{S}_i \leq 2|E(F^j_i)|$.

(ii) Suppose that $1 = \alpha_i < \beta_i$ or $1 < \alpha_i = \beta_i$; these situations are symmetric so we may assume $1 = \alpha_i < \beta_i$. Then $d_B(p^1_i) = 3$ and $d_B(p^\beta_i) = 1$. The only possible negative term in the final sum of (7) is for $(j,\sigma) = (1, +)$, which is at least $4 - 3(1) - 2 = -1$. Hence

$$2|E(F^j_i)| - \tilde{S}_i \geq 4 - 2(3) + 4 - 2(1) + 3 + 1 + (-1) = 3,$

and so $\tilde{S}_i \leq 2|E(F^j_i)|$.

(iii) Suppose that $1 < \alpha_i < \beta_i$. Then $d_B(p^1_i) = d_B(p^\alpha_i) = 1$ and $d_B(p^\beta_i) = 4$. We may have two negative terms in the final sum of (7), for $(j,\sigma) = (\alpha_i, \pm)$. Each negative term is equal to $(4 - 4)d^\sigma(p^\alpha_i) - 2 = -2$. Therefore $2|E(F^j_i)| - \tilde{S}_i \geq 2(4 - 2(1)) - 2 = -2$, i.e., $\tilde{S}_i \leq 2|E(F^j_i)|$.

In situations (i) and (ii), since each term $x_vx_w$ in $S_i$ is at most $x^2 u_0$, we get $S_i \leq \tilde{S}_i x^2 u_0 \leq 2|E(F^j_i)| x^2 u_0$. In situation (iii), $x_v x_q_i$ and $x_{v_i+1} x_q_i$ have coefficient at least 1 in $S_i$, so we have

$$S_i \leq x_v x_q_i + x_{v_i+1} x_q_i + (\tilde{S}_i - 2)x^2 u_0 \leq x_v + x_{v_i+1} + 2|E(F^j_i)| x^2 u_0.$$  

Thus, in all cases $S_i \leq x_v + x_{v_i+1} + 2|E(F^j_i)| x^2 u_0$. 

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Hence, using our estimates above and the Cauchy-Schwarz inequality,

\[
2\lambda R \leq 2\lambda z + S_0 + \sum_{i=1}^{d-1} S_i \\
\leq 2\lambda z + 4\lambda + \sum_{i=1}^{d-1} (x_{v_i} + x_{v_{i+1}} + 2|E(F_i)|x_{u_0}^2) \\
\leq 2\lambda z + 4\lambda + 2\sum_{i=1}^{d} x_{v_i} + 2(|E(G)| - (2d - 1))x_{u_0}^2 \\
\leq 2\lambda z + 4\lambda + 2\sqrt{dz} + (4n - 4d - 4)x_{u_0}^2. \tag{8}
\]

Substituting (8) into (4), it follows that when \(n\) is large enough,

\[
4n - 16 \leq \lambda^3 \leq 4d + \left(2\lambda z + 4\lambda + 2\sqrt{dz} + 4n - 4d - 4\right) - \left(\sqrt{d + R\lambda} - \sqrt{d}\right)^2. \tag{9}
\]

Cancelling terms and rearranging the inequality, we obtain that

\[
\left(\frac{\sqrt{d + R\lambda} - \sqrt{d}}{\lambda R}\right)^2 \leq 2\lambda(z + 2) + 2\sqrt{dz} + 12,
\]

which can be written as

\[
\frac{(\lambda R)^2}{(\sqrt{d + \lambda R} + \sqrt{d})^2} \leq 2\lambda(z + 2) + 2\sqrt{dz} + 12. \tag{10}
\]

From here, we want to give an upper bound on \(\lambda R\). Note that from (9), we also have

\[
\lambda^2 z \leq 4d + (2\lambda z + 4\lambda + 2\sqrt{dz} + 4n - 4d - 4) \\
\leq 4n + 2\lambda z + 4\lambda + 2\sqrt{dz} \\
\leq 4n + 2\lambda z + 4\lambda + 2\lambda\sqrt{d},
\]

since \(z \geq \lambda > 1\). Thus by the fact that \(\lambda^3 \geq 4n - 16\), we obtain that

\[
z \leq \frac{4n + 4\lambda}{\lambda^2 - 2\lambda - 2\sqrt{d}} \leq \frac{\lambda^3 + 16 + 4\lambda}{\lambda^2 - 2\lambda - \sqrt{\lambda^3 + 16}} \leq (1 + o(1))\lambda. \tag{11}
\]

Since \(\lambda^3 \leq \lambda^2 z \leq 4n + 2\lambda z + 4\lambda + 2\sqrt{dz}\), we also have

\[
4n \geq \lambda^3 - 2\lambda z - 4\lambda - 2\sqrt{dz} \\
\geq \lambda^3 - 2\lambda(1 + o(1))\lambda - 4\lambda - \sqrt{(\lambda^3 + 16)(1 + o(1))\lambda} \\
\geq \lambda^3 - 3(1 + o(1))\lambda^2 - 4\lambda \\
\geq (\lambda - (1 + o(1)))^3.
\]

Thus, we have

\[
\lambda \leq \sqrt[3]{4n} + (1 + o(1)). \tag{12}
\]

Combining with Lemma 1, we get an asymptotic estimation of \(\lambda\).

\[
\lambda = (1 + o(1))\sqrt[3]{4n}.
\]
Recall that \( \lambda \leq \zeta \). Hence, using (11), we have \( \zeta = (1 + o(1))\lambda = (1 + o(1))\sqrt{d}n \). Consequently we obtain from (8) that \( \lambda R = O(n) \), which implies that \( \left( \sqrt{d + \lambda R} + \sqrt{d} \right)^2 = O(n) \). Now it follows from (10) that

\[
\lambda R = O \left( \sqrt{n\lambda z + n\sqrt{d}z} \right) = O \left( \sqrt{n\lambda^2 + n^{3/2}\lambda^{1/2}} \right) = O(n^{5/6}).
\]

Substituting \( \lambda R \) into (4) and using the fact that \( \lambda^3 \geq 4n - 16 \), we obtain that

\[
4n - 16 \leq 4d + O(n^{5/6}),
\]

which implies that \( d \geq n - O(n^{5/6}) \). This completes the proof of Claim 1.

**Proof of Lemma 2.** In order to further improve the lower bound on \( d \) (as claimed in Lemma 2), we need to give a non-trivial upper bound on \( x_{u_0}^2 = \max_{v \neq u_0} x_v^2 \). We claim \( x_{u_0} = O(n^{-1/3}) \).

Let \( d' = d_G(u_0) \) and \( \{u_1, u_2, \ldots, u_{d'}\} \) be the neighbors of \( u_0 \) in \( G \). Since \( G \) is outerplanar and has no \( K_{2,3} \) subgraph, \( v_0 \) and \( u_0 \) have at most two common neighbors, so \( d' \leq n + 2 - d = O(n^{5/6}) \). Most of the inequalities shown in Claim 1 hold in similar forms. However, we have to treat any terms that involve \( x_{v_0} \) separately from other terms, so our definitions of \( R' \) and \( B' \) will be slightly different.

By the eigenequation for \( x_{u_0} \), allowing for the possibility that some \( u_i \) is \( v_0 \), and using \( ab \leq \frac{a^2 + b^2}{2} \), we have

\[
\lambda x_{u_0}^2 = \sum_{i=1}^{d'} x_{u_i} x_{u_{i+1}} \leq 2x_{v_0} x_{u_0} + \sum_{u \in N(u_0) \setminus \{v_0\}} x_u^2 = 2x_{u_0} + z'
\]

where we define \( z' = \sum_{u \in N(u_0) \setminus \{v_0\}} x_u^2 \). Let \( B' \) be the subgraph of \( G \) consisting of the edges in \( (\bigcup_{u \in N(u_0) \setminus \{v_0\}} \Gamma(u)) \setminus \Sigma(u_0) \) and their endvertices. Here \( B' \) is similar in structure to \( B \), but \( B' \) is missing the edges in \( \Gamma(u_i) \) if some \( u_i \) is \( v_0 \). In a similar way to (1) and (2), if we apply the eigenequations again and Cauchy-Schwartz, we have

\[
\lambda z' \leq 2x_{u_0} + 2x_{u_0} \sqrt{d' z'} + R',
\]

where \( R' = \sum_{u \in E(B')} x_v x_w \).

It follows from the same logic as in (3) that

\[
z' \leq \frac{4d' x_{u_0}^2}{\lambda^2} + \frac{2(R' + 2x_{u_0})}{\lambda} - \left( \frac{\sqrt{d' x_{u_0}^2}}{\lambda^2} + \frac{R' + 2x_{u_0}}{\lambda} - \sqrt{d' x_{u_0}} \right)^2.
\]

Then

\[
\lambda^2 (z' + 2x_{u_0}) \leq 4d' x_{u_0}^2 + 2\lambda (R' + 2x_{u_0}) - \left( \sqrt{d' x_{u_0}^2} + \lambda (R' + 2x_{u_0}) - \sqrt{d' x_{u_0}} \right)^2 + 2\lambda^2 x_{u_0}
\]

\[
\leq 4d' x_{u_0}^2 + 2\lambda R' + (2\lambda^2 + 4\lambda)x_{u_0}.
\]

Hence we have

\[
(4n - 16)x_{u_0}^2 \leq \lambda^2 x_{u_0}^2 \leq \lambda^2 (z' + 2x_{u_0}) \leq 4d' x_{u_0}^2 + 2\lambda R' + (2\lambda^2 + 4\lambda)x_{u_0}, \tag{13}
\]

\[
(4n - 16)x_{u_0}^2 \leq \lambda^2 x_{u_0}^2 \leq \lambda^2 (z' + 2x_{u_0}) \leq 4d' x_{u_0}^2 + 2\lambda R' + (2\lambda^2 + 4\lambda)x_{u_0},
\]
We will use an inequality like (8) to bound $2\lambda R'$. Similarly to (5), we have

$$2\lambda R' \leq \sum_{v \in V(B')} \lambda (x_v^2 + x_w^2) = \sum_{u \in V(B')} d_{B'}(u) \lambda x_u^2$$

$$= \sum_{u \in V(B') \cap N_1(u_0)} d_{B'}(u) \lambda x_u^2 + \sum_{u \in V(B') \cap N_2(u_0)} d_{B'}(u) \lambda x_u^2$$

$$\leq 2 \sum_{i=1}^{d'} \lambda x_{u_i}^2 + \sum_{u \in V(B') \cap N_2(u_0)} d_{B'}(u) \lambda x_u^2$$

$$\leq 2\lambda \left( x_{v_0}^2 + \sum_{u \in N(u_0) \setminus \{v_0\}} x_u^2 \right) + \sum_{u \in V(B') \cap N_2(u_0)} d_{B'}(u) \sum_{vw \in \Gamma(u)} x_v x_w.$$  \hspace{1cm} (14)

For any vertex $v$ the terms containing $x_v$ in the sum above are

$$\sum_{u \in N(v) \cap N_2(u_0) \cap V(B')} d_{B'}(u) x_v x_w.$$  \hspace{1cm} (15)

For each $u$ there are at most two vertices $w \in \Gamma^{-1}(uv)$.

We will break the sum of (14) into four parts: $S'_0$ from $vw \in \Gamma(u_0)$, $S'_1$ from $vw \notin \Gamma(u_0)$ but $vw \in \Gamma(v_0)$, $S'_2$ from $vw \notin \Gamma(u_0)$ but $vw \in \Sigma(v_0)$, and $S'_3$ from all remaining terms.

We can bound $S'_0$ in a similar way to (6), as

$$S'_0 = \sum_{u \in V(B') \cap N_2(u_0)} d_{B'}(u) \sum_{vw \in \Gamma(u) \cap \Gamma(u_0)} x_v x_w \leq 4\lambda x_{u_0}^2.$$  

For $S'_1, S'_2, S'_3$ we use the following analysis. If $v \notin N[u_0]$ then (15) has at most two vertices $u$, both of which belong to the same subgraph $B'_i = B' \cap \Phi(u_iu_{i+1}, u_0)$. Thus, one $u$ has degree at most 4 in $B'$, and the other has degree at most 2. Since there are at most two vertices $w$ for each $u$, for $v \notin N[u_0]$ the sum of the coefficients of all terms $x_v x_w$ is at most 12. Moreover, if we take any edge $vw$ at level $1\frac{1}{2}$ or higher relative to $u_0$, then one endvertex $v$ satisfies $v \notin N[u_0]$, and $x_v x_w$ occurs in (15) with a total coefficient of at most 6.

Hence, using the eigenequation for $x_{v_0}$, we can bound $S'_1$ as

$$S'_1 = \sum_{u \in V(B') \cap N_2(u_0)} d_{B'}(u) \sum_{vw \in \Gamma(u) \cap \Gamma(u_0)} x_v x_w \leq 6 \sum_{vw \in \Gamma(u_0)} x_v x_w = 6\lambda x_{u_0}^2 = 6\lambda.$$  

Consider terms in the sum of (14) with $vw \notin \Gamma(u_0)$ and $vw \in \Sigma(v_0)$, i.e., $v = v_0$. If $v = v_0 \notin N[u_0]$ then the total coefficient of $x_v x_w$ is at most 12, as described above. If $v = v_0 \in N[u_0]$ then $v_0 = u_i$ for some $i$, and the only possible vertices $u$ in (15) are $q'_{i-1} \in \Gamma^{-1}(u_{i-1}u_i)$ and $q'_i \in \Gamma^{-1}(u_iu_{i+1})$, which both have degree at most 2 in $B'$, and for each such $u$ there is only one $w$ such that $vw \notin \Gamma(u_0)$. Thus, the total coefficient of $x_v x_w$ is at most 4. In either case,

$$S'_2 = \sum_{u \in V(B') \cap N_2(u_0)} d_{B'}(u) \sum_{vw \in \Gamma(u) \setminus \Gamma(u_0)} x_v x_w \leq 12 x_{v_0} x_{u_0} = O(x_{u_0}).$$  

Finally, the terms in the sum of (14) with $vw \notin \Gamma(u_0) \cup \Gamma(v_0) \cup \Sigma(v_0)$ give

$$S'_3 = \sum_{u \in V(B') \cap N_2(u_0)} d_{B'}(u) \sum_{vw \in \Gamma(u) \setminus \Gamma(u_0) \cup \Gamma(v_0) \cup \Sigma(v_0)) x_v x_w.$$
\[ \leq 6|E(G)\setminus (\Gamma(v_0) \cup \Sigma(v_0))|x_{u_0}^2 = 6(2n - 2d) x_{u_0}^2 = O(n^{5/6})x_{u_0}^2. \]

Therefore, using the fact that \( \lambda = O(n^{1/3}) \),

\[ 2\lambda R' \leq 2\lambda(z' + 1) + S'_0 + S'_1 + S'_2 + S'_3 \]
\[ = 2\lambda(z' + 1) + 4\lambda x_{u_0}^2 + 6\lambda + O(x_{u_0}) + O(n^{5/6})x_{u_0}^2 \]
\[ = 2\lambda z' + 8\lambda + O(x_{u_0}) + O(n^{5/6})x_{u_0}^2. \]

Substituting (16) into (13), and using \( d' = O(n^{5/6}) \), we have

\[ (4n - 16)x_{u_0}^2 \leq \lambda^2(z' + 2x_{u_0}) \leq 4d'x_{u_0}^2 + 2\lambda R' + (2\lambda^2 + 4\lambda)x_{u_0} \]
\[ \leq 4d'x_{u_0}^2 + \left( 2\lambda z' + 8\lambda + O(x_{u_0}) + O(n^{5/6})x_{u_0}^2 \right) + (2\lambda^2 + 4\lambda)x_{u_0} \]
\[ \leq 2\lambda z' + O(n^{5/6})x_{u_0}^2 + 8\lambda + (2\lambda^2 + 4\lambda + O(1))x_{u_0}. \]

Rearranging the inequality in (17), we first obtain an upper bound on \( z' \):

\[ z' \leq \frac{O(n^{5/6})x_{u_0} + (4\lambda + O(1))x_{u_0} + 8\lambda}{\lambda^2 - 2\lambda} = O\left(n^{1/6}x_{u_0} + \frac{4x_{u_0}}{\lambda} + \frac{1}{\lambda}\right). \]

Now using the upper bound on \( z' \) and (17), we have the following inequality:

\[ (4n - 16)x_{u_0}^2 \leq 2\lambda z' + O(n^{5/6})x_{u_0}^2 + 8\lambda + (2\lambda^2 + 4\lambda + O(1))x_{u_0} \]
\[ = O\left(n^{5/6}x_{u_0}^2 + \lambda^2x_{u_0} + \lambda\right). \]

It follows from the fact that \( \lambda = O(n^{1/3}) \) that

\[ x_{u_0} = O(n^{-1/3}). \]

Now using the bound \( x_{u_0} = O(n^{-1/3}) \) in (8), we obtain a better bound on \( d = d_G(v_0) \) in Claim 1:

\[ 4n - 16 \leq \lambda^3 \leq 4d + 2\lambda z + 4\lambda + 2\sqrt{d\lambda} + 4(n - d)O(n^{-2/3}), \]

which gives us

\[ (4n - 4d)(1 - O(n^{-2/3})) \leq 16 + 2\lambda z + 4\lambda + 2\sqrt{d\lambda} \]
\[ = O(1) + O(n^{2/3}) + O(n^{1/3}) + O(\sqrt{n \cdot n^{1/3}}) = O(n^{2/3}) \]

and thus \( d \geq n - O(n^{2/3}) \). This completes the proof of Lemma 2. \( \square \)

**Lemma 3.** \( d_H(v_1) = 1 \). Moreover, \( x_{v_2} \geq x_{v_1} \).

**Proof.** Assume for the sake of contradiction that \( d_H(v_1) \geq 2 \). Recall that \( H \) is edge-maximal. It follows that there must exist a vertex \( q_1 \neq v_0 \) such that \( \{v_1, v_2, q_1\} \) is a hyperedge, and \( q_1 \) is a vertex of \( F_1 = \Phi(v_1 v_2, v_0) \). Let \( F_1' \) be \( F_1 \) but with \( v_2 \) renamed as \( v_0 \). Then \( G' = G - (V(F_1) - \{v_1, v_2\}) \cup F_1' \) is outerplanar (we find the outerplanar embedding by flipping \( F_1 \) over, i.e., reflecting it), and \( G' \) is the shadow of a 3-uniform hypergraph \( H' \) that can be obtained from \( H \) by replacing each hyperedge \( \{v_2, u, w\} \) where \( u, w \in V(F_1) \) by a hyperedge \( \{v_0, u, w\} \). Suppose \( x \) is the Perron-Frobenius eigenvector of \( H \). Since \( x_{v_0} > x_{v_2} \) by Lemma 2, it follows that

\[ \sum_{\{i_1, i_2, i_3\} \in E(H')} x_{i_1}x_{i_2}x_{i_3} - \sum_{\{i_1, i_2, i_3\} \in E(H)} x_{i_1}x_{i_2}x_{i_3} \geq x_{v_1}x_{q_1}(x_{v_0} - x_{v_2}) > 0. \]
This implies that \( \lambda(H') > \lambda(H) \), which contradicts \( H \) attaining the maximum spectral radius.

It remains to show that \( x_{v_2} \geq x_{v_1} \). If \( x_{v_2} < x_{v_1} \), then let \( x' \) be obtained from \( x \) by setting \( x'_v = x_{v_2}, \ x'_v = x_{v_1} \) and keeping every other entry the same. Since \( d_H(v_1) = 1 \), it follows that \( P_H(x') > P_H(x) \), which contradicts \( x \) being the Perron-Frobenius eigenvector of \( H \). \( \square \)

Now we are ready to show Theorem 1.

**Proof of Theorem 1.** Let \( H \) be an outerplanar 3-uniform hypergraph on \( n \) vertices with maximum spectral radius. Let \( G \) be the shadow of \( H \). Suppose the Perron–Frobenius eigenvector \( x \) of the adjacency tensor of \( H \) is normalized so that the maximum eigenvector entry is 1. Let \( v_0 \) be the vertex with the maximum eigenvector entry and \( \{v_1, v_2, \ldots, v_d\} \) be the neighbors of \( v_0 \) in the clockwise order of the outerplanar drawing of \( G \).

By Lemma 1, we have that \( d(v_0) \geq n - O(n^{2/3}) \) and for every other vertex \( u \neq v_0, x_u = O(n^{-1/3}) \).

Now we claim that \( x_{v_1} = \Omega(n^{-1/3}) \). By Lemma 3, we have that \( d_H(v_1) = 1 \), i.e., \( \{v_1, v_2, v_0\} \) is the unique hyperedge containing \( v_1 \). It follows by Lemma 3 and the eigenequation for \( x_{v_1} \) that

\[
\lambda x_{v_1}^2 = x_{v_0}x_{v_2} = x_{v_2} \geq x_{v_1}.
\]

Together with (12), this implies that

\[
x_{v_1} \geq \frac{1}{\lambda} = \Omega(n^{-1/3}).
\]

Now we claim that for every vertex \( u \in V(G) \setminus \{v_0\} \), \( u \) is a neighbor of \( v_0 \) in \( G \). Suppose not. The hyperedges incident with \( v_0 \) form a path in the dual of \( G \) and there must be a hyperedge \( \{w, s, t\} \) that is a leaf of the dual tree (excluding the outer face) but not an end of this path. Then \( w, s, t \neq v_0 \) and one of these vertices, say \( w \), has degree 2 in \( G \) and degree 1 in \( H \). Now similarly to Lemma 3, consider the hypergraph \( H' \) obtained from \( H \) by by removing the hyperedge \( \{w, s, t\} \) and adding the hyperedge \( \{w, v_0, v_1\} \). It follows that

\[
\sum_{\{i_1, i_2, i_3\} \in E(H')} x_{i_1}x_{i_2}x_{i_3} - \sum_{\{i_1, i_2, i_3\} \in E(H)} x_{i_1}x_{i_2}x_{i_3} \geq x_wx_{v_0}x_{v_1} - x_wx_sx_t.
\]

Note that \( x_ax_t = O(n^{-2/3}) \) while \( x_{v_0}x_{v_1} = \Omega(n^{-1/3}) \). It follows that \( x_wx_{v_0}x_{v_1} > x_wx_sx_t \), which implies that \( \lambda(H') > \lambda(H) \), contradicting \( H \) being the extremal hypergraph of maximum spectral radius. Hence every vertex \( u \in V(G) \setminus \{v_0\} \) is a neighbor of \( v_0 \) in \( G \).

Again by the fact that \( H \) attains the maximum spectral radius, it follows that \( H \) is the unique 3-uniform hypergraph \( F_n \) with \( K_1 + P_{n-1} \) as it shadow. \( \square \)

**References**


