Maximum spectral radius of outerplanar 3-uniform hypergraphs

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Abstract

In this paper, we study the maximum spectral radius of outerplanar 3-uniform hypergraphs. Given a hypergraph $\mathcal{H}$, the shadow of $\mathcal{H}$ is a graph G with $V(G) = V(\mathcal{H})$ and $E(G) = \{uv : uv \in h \text{ for some } h \in E(\mathcal{H})\}$. A graph is *outerplanar* if it can be embedded in the plane such that all its vertices lie on the outer face. A 3-uniform hypergraph $\mathcal{H}$ is called *outerplanar* if its shadow has an outerplanar embedding such that every hyperedge of $\mathcal{H}$ is the vertex set of an interior triangular face of the shadow. Cvetković and Rowlinson conjectured in 1990 that among all outerplanar graphs on n vertices, the graph $K_1 + P_{n-1}$ attains the maximum spectral radius. We show a hypergraph analogue of the Cvetković-Rowlinson conjecture. In particular, we show that for sufficiently large n, the n-vertex outerplanar 3-uniform hypergraph of maximum spectral radius is the unique 3-uniform hypergraph whose shadow is $K_1 + P_{n-1}$.

1 Introduction

A graph G is *planar* if it can be embedded in the plane, i.e., it can be drawn on the plane in such a way that edges intersect only at their endpoints. A graph is *outerplanar* if it can be embedded in the plane so that all vertices lie on the boundary of its outer face. The study of the spectral radius of (outer)planar graphs has a long history, dating back to Schwenk and Wilson [15]. Given a graph G, the *spectral radius* λ of G is the largest eigenvalue of the adjacency matrix of G. The spectral radius of planar graphs is useful in geography as a measure of the overall connectivity of a planar graph [1, 5]. It is therefore of interest to geographers to find the maximum spectral radius of a planar graph as a theoretical upper bound for the connectivity of networks. Boots and Royle [1], and independently Cao and Vince [2] conjectured that the extremal planar graph achieving the maximum spectral radius is $K_2 + P_{n-2}$ (see Figure 1). Hong [17] first showed that for an n-vertex plananr graph G, $\lambda(G) \leq \sqrt{5n-11}$. This was subsequently improved in a series of papers [2, 18, 8, 19, 6]. Guiduli and Hayes [9] showed in an unpublished preprint that the Boots-Royle-Cao-Vince conjecture is true for sufficiently large n. For outerplanar graphs, it is conjectured by Cvetković and Rowlinson [5] that among all outerplanar graph on n vertices, $K_1 + P_{n-1}$ attains the maximum spectral radius (see Figure 1). Partial progress has been made by Rowlinson [14], Cao and Vince [2], and Guiduli and Hayes [9]. Recently, Tait and Tobin [16] proved the Boots-Royle-Cao-Vince conjecture and the Cvetković-Rowlinson conjecture for large enough n. Lin and Ning [11] showed that the Cvetković-Rowlinson conjecture holds for all $n \geq 2$ except for $n = 6$.

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Figure 1: The graphs $K_1 + P_{n-1}$ (left) and $K_2 + P_{n-2}$ (right).

In this paper, we extend the investigations into the maximum spectral radius of outerplanar 3-uniform hypergraphs. Given a hypergraph $\mathcal{H}$, the *shadow* of $\mathcal{H}$, denoted by $\partial(\mathcal{H})$, is a 2-uniform graph G with $V(G) = V(\mathcal{H})$ and $E(G) = \{uv : uv \in h \text{ for some } h \in E(\mathcal{H})\}$.

We adopt Zykov's [20] definition of hypergraph planarity. In particular, a 3-uniform hypergraph $\mathcal{H}$ is called *planar* if $\partial(\mathcal{H})$ has a planar embedding so that every hyperedge of $\mathcal{H}$ is the vertex set of a triangular face of $\partial(\mathcal{H})$. A 3-uniform hypergraph $\mathcal{H}$ is called *outerplanar* if $\partial(\mathcal{H})$ has an outerplanar embedding such that every hyperedge of $\mathcal{H}$ is the vertex set of an interior triangular face of $\partial(\mathcal{H})$.

Now we define the spectral radius of an r-uniform hypergraph. Given positive integers r and n, an order r and dimension n *tensor* $\mathcal{A} = (a_{i_1 i_2 \cdots i_r})$ over $\mathbb{C}$ is a multidimensional array with all entries $a_{i_1 i_2 \cdots i_r} \in \mathbb{C}$ for all $i_1, i_2, \cdots, i_r \in [n] = \{1, 2, \ldots, n\}$. Given a column vector $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)^T \in \mathbb{C}^n$, $\mathcal{A}\boldsymbol{x}^{r-1}$ is defined to be a vector in $\mathbb{C}^n$ whose ith entry is

$$(\mathcal{A}\boldsymbol{x}^{r-1})_i = \sum_{i_2, \cdots, i_r = 1}^{n} a_{i i_2 \cdots i_r} x_{i_2} \cdots x_{i_r}.$$

In 2005, Qi [12] and Lim [10] independently proposed the definition of eigenvalues of a tensor. In particular, if there exists a number $\lambda \in \mathbb{C}$ and a nonzero vector $\boldsymbol{x} \in \mathbb{C}^n$ such that

$$\mathcal{A}\boldsymbol{x}^{r-1} = \lambda \boldsymbol{x}^{[r-1]}$$

where $\boldsymbol{x}^{[r-1]} = (x_1^{r-1}, x_2^{r-1}, \cdots, x_n^{r-1})^T$, then λ is called the *eigenvalue* of $\mathcal{A}$ and $\boldsymbol{x}$ is called an *eigenvector* of $\mathcal{A}$ corresponding to λ. The *spectral radius* of $\mathcal{A}$, denoted by $\lambda(\mathcal{A})$, is the maximum modulus of the eigenvalues of $\mathcal{A}$. It was shown in [13] that

$$\lambda(\mathcal{A}) = \max_{\substack{||\boldsymbol{x}||_r = 1 \\ \boldsymbol{x} \in \mathbb{R}_+^n}} \boldsymbol{x}^T \mathcal{A} \boldsymbol{x}^{r-1},$$

where $||\boldsymbol{x}||_r := (|x_1|^r + |x_2|^r + \cdots + |x_n|^r)^{1/r}$ and $\mathbb{R}_+$ is the set of nonnegative real numbers.

In 2012, Cooper and Dutle [4] defined the *adjacency tensor* of an r-uniform hypergraph. Given an r-uniform hypergraph $\mathcal{H}$ on n vertices, the adjacency tensor $\mathcal{A}(\mathcal{H})$ of $\mathcal{H}$ is defined as the order r dimension n tensor with entries $a_{i_1 i_2 \cdots i_r}$ such that

$$a_{i_1 i_2 \cdots i_r} = \begin{cases} \frac{1}{(r-1)!} & \text{if } \{i_1, i_2, \cdots i_r\} \in E(\mathcal{H}) \\ 0 & \text{otherwise.} \end{cases}$$

Let $\lambda(\mathcal{H})$ denote the spectral radius of $\mathcal{A}(\mathcal{H})$. Given an r-uniform hypergraph $\mathcal{H}$ and a vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, we can define a multi-linear function $P_{\mathcal{H}}(\boldsymbol{x}) : \mathbb{R}^n \to \mathbb{R}$ as follows:

$$P_H(\boldsymbol{x}) = r \sum_{\{i_1, i_2, \ldots, i_r\} \in E(\mathcal{H})} x_{i_1} x_{i_2} \cdots x_{i_r}.$$

Then the spectral radius of $\mathcal{H}$ can be also expressed as

$$\lambda(\mathcal{H}) := \max_{||\boldsymbol{x}||_r=1} P_{\mathcal{H}}(\boldsymbol{x}) = \max_{\boldsymbol{x}\in\mathbb{R}^n_+} \frac{P_{\mathcal{H}}(\boldsymbol{x})}{\|\boldsymbol{x}\|_r^r}.$$

The Perron-Frobenius theorem [3, 7] for nonnegative tensors implies that there is always a nonnegative vector $\boldsymbol{x}$ satisfying the maximum at right above. Any such $\boldsymbol{x}$ is called a *Perron-Frobenius eigenvector* of $\mathcal{A}(\mathcal{H})$ (corresponding to $\lambda(\mathcal{H})$). If $\mathcal{H}$ is connected then a Perron-Frobenius eigenvector is strictly positive and is unique up to scaling by a positive coefficient; moreover, the spectral radius $\lambda(\mathcal{H})$ is the unique eigenvalue with a strictly positive eigenvector. By definition, the spectral radius $\lambda(\mathcal{H})$ and its eigenvector $\boldsymbol{x} = (x_1, \cdots, x_n)$ also satisfy the following *eigenequation* for every x_i:

$$\lambda(H)x_i^{r-1} = \sum_{\{i,i_2,\ldots,i_r\}\in E(H)} x_{i_2}\cdots x_{i_r} \quad \text{for } x_i > 0.$$

Now we are ready to state our main theorem. We use $\mathcal{F}_n$ to denote the *fan hypergraph*, i.e., the unique 3-uniform hypergraph whose shadow is $K_1 + P_{n-1}$ (see Figure 1).

Theorem 1. *For large enough n, the n-vertex outerplanar 3-uniform hypergraph of maximum spectral radius is the fan hypergraph $\mathcal{F}_n$.*

The shadow of the extremal hypergraph attaining the maximum spectral radius among all outerplane 3-uniform hypergraphs is exactly the extremal graph attaining the maximum spectral radius among all outerplanar graphs. This motivates us to make the following analogous conjecture for planar 3-uniform hypergraphs:

Conjecture 1. *For large enough n, the n-vertex planar 3-uniform hypergraph graph $\mathcal{H}$ of maximum spectral radius is the unique maximal hypergraph whose shadow is $K_2 + P_{n-2}$.*

2 Proof of Theorem 1

Let $\mathcal{H}$ be an n-vertex outerplanar 3-uniform hypergraph of maximum spectral radius. Let G be the shadow of $\mathcal{H}$, i.e., $V(G) = V(\mathcal{H})$ and $E(G) = \{vu : \{v, u\} \subseteq h \text{ for some } h \in E(\mathcal{H})\}$. It follows by definition that G is outerplanar, and thus does not contain a $K_{2,3}$ minor or a K_4 minor. Observe that $\mathcal{H}$ must be edge-maximal (while maintaining the outerplanarity). Otherwise, we can obtain an outerplanar hypergraph $\mathcal{H}'$ such that $\mathcal{H} \subsetneq \mathcal{H}'$. It then follows from the Perron-Frobenius Theorem that $\mathcal{H}'$ attains a larger spectral radius than $\mathcal{H}$, giving us a contradiction. Now since $\mathcal{H}$ is edge-maximal, G must be a maximal outerplanar graph, with $2n-3$ edges. Then G is 2-connected, and has an outerplanar embedding, unique up to homeomorphisms of the plane, whose outer face is bounded by a Hamilton cycle. We always assume G has this outerplanar embedding. All interior faces of G are triangles, and every triangle of G is a facial triangle and a hyperedge of $\mathcal{H}$. The dual of G (excluding the outer face) is a tree, so the interior faces of G are connected together in a treelike fashion.

We use $N(v)$ to denote the set of neighbors of v in G, i.e., $N(v) = \{u : vu \in E(G)\}$ and $d(v)$ to denote the degree of v, i.e., $d(v) = |N(v)|$. We also use $d_F(v)$ to refer to degree in a subgraph F of G. The closed neighborhood of v, denoted by $N[v]$, is defined as $N[v] = N(v) \cup \{v\}$. More generally, we let $\text{dist}(u, v)$ denote the distance between u and v in G, and $N_k(v) = \{u \in V(G) : \text{dist}(v, u) = k\}$. Given an edge uw and vertex v define the *level of uw relative to v* to be $(\text{dist}(u, v) + \text{dist}(w, v))/2$, which is an integer or half-integer.

Let $\Gamma(v) = \{uw : \{v, u, w\} \in E(\mathcal{H})\}$ be the link of v in $\mathcal{H}$, and $d_{\mathcal{H}}(v) = |\Gamma(v)|$ be the degree of v in $\mathcal{H}$. The edges in $\Gamma(v)$ form an induced path in G whose ends are the neighbors of v on the outer cycle. For each edge e of G, $\Gamma^{-1}(e)$ is the set of vertices forming a triangle with e, and contains one vertex if e is an outer edge, and two vertices otherwise. We also use $\Sigma(v)$ to denote the set of edges incident with v in G. In our situation the edges in $\Sigma(v)$ and $\Gamma(v)$ are precisely the edges at levels $\frac{1}{2}$ and 1, respectively, relative to v.

Suppose we are given an edge uw and a vertex v not incident with uw. If uw is an outer edge, define $\Phi(uw, v)$ to be the empty graph. Otherwise, $G\backslash\{u, w\}$ has two components; define $\Phi(uw, v)$ to be the component not containing v, together with all edges joining that component to u or w. Loosely, $\Phi(uw, v)$ is the subgraph of G on the far side of uw from v.

Lemma 1. $\lambda(\mathcal{H}) \geq \sqrt[3]{4(n-1)}\left(1 - \frac{1}{n-1}\right)$.

Proof. Let $\mathcal{F}_n$ be the fan hypergraph on n vertices, i.e., the unique 3-uniform hypergraph on n vertices whose shadow is $K_1 + P_{n-1}$. Suppose w is the vertex that is adjacent to all the other vertices in $\partial(\mathcal{F}_n)$ and $v_1, v_2, \cdots, v_{n-1}$ are its neighbors. Clearly $\mathcal{F}_n$ is outerplanar. Consider the vector $\boldsymbol{x} \in \mathbb{R}^n$ with $x_w = 1/\sqrt[3]{3}$ and $x_{v_i} = \left(\frac{2}{3(n-1)}\right)^{1/3}$. Note that $\|\boldsymbol{x}\|_3 = 1$. It follows that

$$\lambda(\mathcal{H}) \geq \lambda(\mathcal{F}_n) \geq P_{\mathcal{F}_n}(\boldsymbol{x}) = 3(n-2) \cdot \frac{1}{\sqrt[3]{3}} \cdot \left(\frac{2}{3(n-1)}\right)^{2/3} = \sqrt[3]{4(n-1)}\left(1 - \frac{1}{n-1}\right).$$

□

Note that since $\mathcal{H}$ is connected, there exists an eigenvector corresponding to $\lambda(\mathcal{H})$ such that all its entries are strictly positive. In the rest of this section, for convenience we assume that this Perron-Frobenius eigenvector x of $\mathcal{H}$ is re-normalized so that the maximum eigenvector entry is 1. Let v_0 be the vertex with the maximum eigenvector entry, so that $x_{v_0} = 1$. We also define u_0 to be a vertex with the second largest eigenvector entry, i.e., $x_{u_0} = \max_{v \neq v_0} x_v$. We abbreviate $\lambda(\mathcal{H})$ to λ, and the eigenequation of $\mathcal{H}$ tells us that $\lambda x_v^2 = \sum_{uw \in \Gamma(v)} x_u x_w$ for every vertex v.

The following lemma says that $\mathcal{H}$ is very close to the fan hypergraph $\mathcal{F}_n$.

Lemma 2. *We have* $\lambda = (1 + o(1))\sqrt[3]{4n}$ *and* $d_G(v_0) \geq n - O(n^{2/3})$*. Moreover, for any other vertex* $u \neq v_0$*,* $x_u = O(n^{-1/3})$*.*

We first show a weaker version of Lemma 2. In particular, we show the following claim.

Claim 1. $d_G(v_0) \geq n - O(n^{5/6})$.

Proof of Claim 1. Let x and v_0 be as described above, so that $x_{v_0} = 1$. Let $d = d(v_0)$ and suppose that $\Gamma(v_0)$ forms the path $v_1 v_2 \ldots v_d$, where $v_1, v_2, \ldots, v_d$ are in clockwise order around v_0. Now by the eigenequation for x_{v_0},

$$\lambda = \lambda x_{v_0}^2 = \sum_{i=1}^{d-1} x_{v_i} x_{v_{i+1}} \leq \sum_{i=1}^{d} x_{v_i}^2,$$

using the fact $ab \leq (a^2 + b^2)/2$. Set $z = \sum_{i=1}^{d} x_{v_i}^2$. We have $\lambda \leq z$. It again follows from the eigenequation expansion that

$$\lambda z = \sum_{i=1}^{d} \lambda x_{v_i}^2 = \sum_{i=1}^{d} \sum_{vw \in \Gamma(v_i)} x_v x_w$$

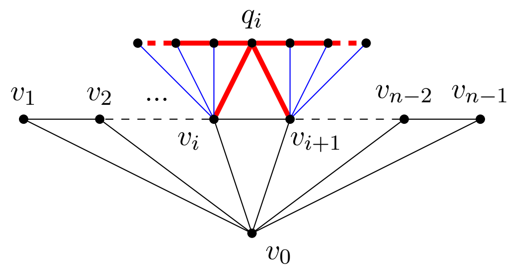

Figure 2: Neighborhood of v_0, and B_i

$$\leq 2\sum_{i=1}^{d} x_{v_0}x_{v_i} + \sum_{i=1}^{d} \sum_{vw\in\Gamma(v_i)\setminus\Sigma(v_0)} x_v x_w \tag{1}$$

Define B to be the subgraph consisting of the edges in $(\bigcup_{i=1}^{d}\Gamma(v_i))\setminus\Sigma(v_0)$ and their endvertices. The edges of B are at levels $1\frac{1}{2}$ and 2 relative to v_0. For $i \in [d-1]$ let $F_i = \Phi(v_iv_{i+1}, v_0)$ and $B_i = B\cap F_i$. Fig. 2 shows B_i, indicated in bold (and red, if color is visible). From (1), using $x_{v_0} = 1$ and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\lambda z &\leq 2\sum_{i=1}^{d} x_{v_i} + \sum_{vw\in E(B)} x_v x_w \\ &\leq 2\sqrt{dz} + \sum_{vw\in E(B)} x_v x_w. \end{aligned}\tag{2}$$

For ease of reference, set $R = \sum_{vw\in E(B)} x_v x_w$. Dividing both sides of inequality (2) by λ, we have $z - \frac{2\sqrt{dz}}{\lambda} \leq \frac{R}{\lambda}$. By completing the square, we have $\left(\sqrt{z} - \frac{\sqrt{d}}{\lambda}\right)^2 \leq \frac{R}{\lambda} + \frac{d}{\lambda^2}$. Rearranging the terms of the inequality, we obtain that

$$\begin{aligned} z &\leq \left(\frac{\sqrt{d}}{\lambda} + \sqrt{\frac{d}{\lambda^2} + \frac{R}{\lambda}}\right)^2 \\ &= \frac{4d}{\lambda^2} + \frac{2R}{\lambda} - \left(\sqrt{\frac{d}{\lambda^2} + \frac{R}{\lambda}} - \frac{\sqrt{d}}{\lambda}\right)^2. \end{aligned}\tag{3}$$

It follows that

$$\lambda^3 \leq \lambda^2 z \leq 4d + 2\lambda R - \left(\sqrt{d + R\lambda} - \sqrt{d}\right)^2. \tag{4}$$

By Lemma 1, we obtain that $\lambda^3 \geq 4n - 16$ when n is large enough.

Now we find a bound on $2\lambda R$. Using $ab \leq (a^2 + b^2)/2$ and then the eigenequations, twice, we have

$$\begin{aligned} 2\lambda R &= \sum_{vw\in E(B)} 2\lambda x_v x_w \\ &\leq \sum_{vw\in E(B)} \lambda(x_v^2 + x_w^2) = \sum_{u\in V(B)} d_B(u)\,\lambda x_u^2 \end{aligned}$$

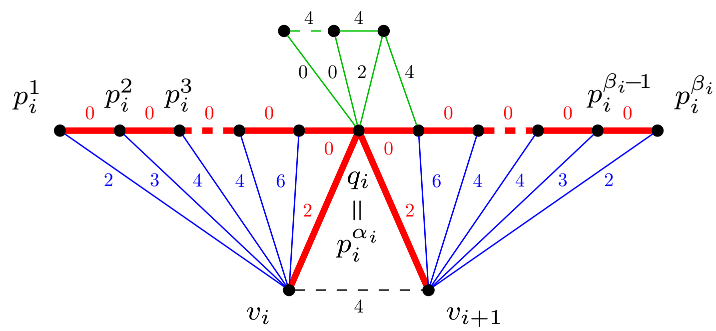

Figure 3: Structure of F_i and B_i, with edge coefficients

$$
\begin{aligned}
&= \sum_{u\in V(B)\cap N_1(v_0)} d_B(u)\,\lambda x_u^2 + \sum_{u\in V(B)\cap N_2(v_0)} d_B(u)\,\lambda x_u^2 \\
&\le 2\sum_{i=1}^{d} \lambda x_{v_i}^2 + \sum_{u\in V(B)\cap N_2(v_0)} d_B(u)\,\lambda x_u^2 \\
&= 2\lambda z + \sum_{u\in V(B)\cap N_2(v_0)} d_B(u) \sum_{vw\in\Gamma(u)} x_v x_w. \qquad (5)
\end{aligned}
$$

If $x_v x_w$ appears in the sum above then the level of vw relative to v_0 is between 1 and 3.

To investigate the sum in (5) we examine the structure of F_i and B_i more closely. If F_i is nonempty then it contains a common neighbor q_i of v_i and v_{i+1}. The vertices of $N_2(v_0)\cap F_i$ lie on a path $p_i^1p_i^2p_i^3\dots p_i^{\beta_i}$, with $q_i = p_i^{\alpha_i}$ for some α_i with $1\le\alpha_i\le\beta_i$. Here p_i^j is adjacent to v_i for $1\le j\le\alpha_i$, and to v_{i+1} for $\alpha_i\le j\le\beta_i$. The subgraph B_i contains the edges of this path and edges $v_iq_i, v_{i+1}q_i$. We let $F_i^j = \Phi(p_i^jp_i^{j+1}, v_0)$ be the part of F_i above $p_i^jp_i^{j+1}$ for $j\in[\beta_i-1]$. See Fig. 3, which illustrates part of F_i and the edge coefficients in the sum from (5) (v_iv_{i+1} is dashed because it is not an edge of F_i). For $u = p_i^j$ with $j\notin\{1,\alpha_i,\beta_i\}$ we have $d_B(u) = 2$; for $j\in\{1,\alpha_i,\beta_i\}$ the value of $d_B(u)$ depends on whether $\alpha_i = 1$ or $\alpha_i = \beta_i$ or both, but we always have $1\le d_B(u)\le 4$, and $d_B(u) > 2$ only if $j = \alpha_i$.

Each edge $v_iv_{i+1}\in\Gamma(v_0)$ occurs in the sum in (5) only as part of $\Gamma(q_i)$, when $q_i = p_i^{\alpha_i}$ exists, so the coefficient of $x_{v_i}x_{v_{i+1}}$ is at most 4. Thus, the contribution from $\Gamma(v_0)$, using the eigenequation for x_{v_0}, is

$$
S_0 = \sum_{u\in V(B)\cap N_2(v_0)} d_B(u) \sum_{vw\in\Gamma(u)\cap\Gamma(v_0)} x_vx_w \le 4\sum_{i=1}^{d-1} x_{v_i}x_{v_{i+1}} = 4\lambda x_{v_0}^2 = 4\lambda. \qquad (6)
$$

Assuming F_i is nonempty, the part of the sum in (5) coming from $vw\in E(F_i)$, $i\in[d-1]$, is

$$
S_i = \sum_{u\in V(B_i)\cap N_2(v_0)} d_B(u) \sum_{vw\in\Gamma(u)\setminus\Gamma(v_0)} x_vx_w
$$

We estimate S_i by computing the sum of the coefficients, i.e.,

$$
\widetilde{S}_i = \sum_{u\in V(B_i)\cap N_2(v_0)} d_B(u)\,|\Gamma(u)\setminus\Gamma(v_0)| = \sum_{j=1}^{\beta_i} d_B(p_i^j)\,|\Gamma(p_i^j)\setminus\Gamma(v_0)|.
$$

Let $d^-(p_i^j)$ be the degree of p_i^j in F_i^{j-1} (or 0 if $j=1$), and $d^+(p_i^j)$ be its degree in F_i^j (or 0 if $j=\beta_i$). For $j \notin \{1,\alpha_i,\beta_i\}$ we have

$$|\Gamma(p_i^j)\backslash\Gamma(v_0)| = |\Gamma(p_i^j)| = d(p_i^j)-1 = (d^-(p_i^j)+d^+(p_i^j)+3)-1 = d^-(p_i^j)+d^+(p_i^j)+2.$$

For $j=\alpha_i$, if $j \neq 1, \beta_i$ we have

$$|\Gamma(p_i^j)\backslash\Gamma(v_0)| = |\Gamma(p_i^j)|-1 = d(p_i^j)-2 = (d^-(p_i^j)+d^+(p_i^j)+4)-2 = d^-(p_i^j)+d^+(p_i^j)+2.$$

If $j=1$ we must reduce the above values by 1 since there is no edge $v_i p_i^{j-1}$, and similarly if $j=\beta_i$ we must reduce these values by 1 since there is no edge $v_{i+1}p_i^{j+1}$. These reductions are independent, and do not depend on whether $\alpha_i=1$ or $\alpha_i=\beta_i$ or both. Therefore,

$$\widetilde{S}_i = \sum_{j=1}^{\beta_i} d_B(p_i^j)\left(d^-(p_i^j)+d^+(p_i^j)+2\right) - d_B(p_i^1) - d_B(p_i^{\beta_i}).$$

We are going to compare $\widetilde{S}_i$ to $|E(F_i)|$. The number of edges in F_i at levels $1\frac{1}{2}$ and 2 relative to v_0 is just $2\beta_i$. The edges at higher levels belong to some F_i^j. If F_i^j is nonempty, then it contains $2d^+(p_i^j)$ edges of $\Sigma(p_i^j)\cup\Gamma(p_i^j)$, and $2d^-(p_i^{j+1})$ edges of $\Sigma(p_i^{j+1})\cup\Gamma(p_i^{j+1})$, but these two sets overlap in two edges. Thus, $|E(F_i^j)| \geq 2d^+(p_i^j)+2d^-(p_i^{j+1})-2$. Hence,

$$|E(F_i)| \geq 2\beta_i + \sum_{j:F_i^j\neq\emptyset}\left(2d^+(p_i^j)+2d^-(p_i^{j+1})-2\right) = 2\beta_i + \sum_{(j,\sigma):d^\sigma(p_i^j)>0}(2d^\sigma(p_i^j)-1)$$

where in the last sum $j \in [\beta_i]$ and $\sigma \in \{-,+\}$.

Therefore, when F_i is nonempty,

$$2|E(F_i)| - \widetilde{S}_i \geq \sum_{j=1}^{\beta_i}(4-2d_B(p_i^j)) + d_B(p_i^1) + d_B(p_i^{\beta_i}) + \sum_{(j,\sigma):d^\sigma(p_i^j)>0}\left((4-d_B(p_i^j))d^\sigma(p_i^j)-2\right). \quad (7)$$

In the first sum, only terms with $j \in \{1,\alpha_i,\beta_i\}$ can be nonzero. In the final sum, only terms with $d_B(p_i^j) > 2$, which requires $j=\alpha_i$, can be negative. (Fig. 3 shows a situation where we have a negative term in the final sum.) We consider several situations.

(i) If F_i is empty or $1=\alpha_i=\beta_i$ then $\widetilde{S}_i = 0$, so $\widetilde{S}_i \leq 2|E(F_i)|$.

(ii) Suppose that $1=\alpha_i<\beta_i$ or $1<\alpha_i=\beta_i$; these situations are symmetric so we may assume $1=\alpha_i<\beta_i$. Then $d_B(p_i^1)=3$ and $d_B(p_i^{\beta_i})=1$. The only possible negative term in the final sum of (7) is for $(j,\sigma)=(1,+)$, which is at least $(4-3)(1)-2=-1$. Hence $2|E(F_i)|-\widetilde{S}_i \geq (4-2(3))+(4-2(1))+3+1+(-1)=3$, and so $\widetilde{S}_i \leq 2|E(F_i)|$.

(iii) Suppose that $1<\alpha_i<\beta_i$. Then $d_B(p_i^1)=d_B(p_i^{\beta_i})=1$ and $d_B(p_i^{\alpha_i})=4$. We may have two negative terms in the final sum of (7), for $(j,\sigma)=(\alpha_i,\pm)$. Each negative term is equal to $(4-4)d^\sigma(p_i^j)-2=-2$. Therefore $2|E(F_i)|-\widetilde{S}_i \geq 2(4-2(1))+(4-2(4))+1+1+2(-2)=-2$, i.e., $\widetilde{S}_i \leq 2|E(F_i)|+2$.

In situations (i) and (ii), since each term $x_v x_w$ in S_i is at most $x_{u_0}^2$, we get $S_i \leq \widetilde{S}_i x_{u_0}^2 \leq 2|E(F_i)|x_{u_0}^2$. In situation (iii), $x_{v_i}x_{q_i}$ and $x_{v_{i+1}}x_{q_i}$ have coefficient at least 1 in S_i, so we have

$$S_i \leq x_{v_i}x_{q_i} + x_{v_{i+1}}x_{q_i} + (\widetilde{S}_i-2)x_{u_0}^2 \leq x_{v_i} + x_{v_{i+1}} + 2|E(F_i)|x_{u_0}^2.$$

Thus, in all cases $S_i \leq x_{v_i} + x_{v_{i+1}} + 2|E(F_i)|x_{u_0}^2$.

Hence, using our estimates above and the Cauchy-Schwarz inequality,

$$\begin{aligned}
2\lambda R &\le 2\lambda z + S_0 + \sum_{i=1}^{d-1} S_i \\
&\le 2\lambda z + 4\lambda + \sum_{i=1}^{d-1}\left(x_{v_i} + x_{v_{i+1}} + 2|E(F_i)|x_{u_0}^2\right) \\
&\le 2\lambda z + 4\lambda + 2\sum_{i=1}^{d} x_{v_i} + 2(|E(G)| - (2d-1))x_{u_0}^2 \\
&\le 2\lambda z + 4\lambda + 2\sqrt{dz} + (4n - 4d - 4)x_{u_0}^2. \qquad (8)
\end{aligned}$$

Substituting (8) into (4), it follows that when n is large enough,

$$4n - 16 \le \lambda^3 \le \lambda^2 z \le 4d + \left(2\lambda z + 4\lambda + 2\sqrt{dz} + 4n - 4d - 4\right) - \left(\sqrt{d + R\lambda} - \sqrt{d}\right)^2. \qquad (9)$$

Cancelling terms and rearranging the inequality, we obtain that

$$\left(\sqrt{d + R\lambda} - \sqrt{d}\right)^2 \le 2\lambda(z+2) + 2\sqrt{dz} + 12,$$

which can be written as

$$\frac{(\lambda R)^2}{\left(\sqrt{d + \lambda R} + \sqrt{d}\right)^2} \le 2\lambda(z+2) + 2\sqrt{dz} + 12. \qquad (10)$$

From here, we want to give an upper bound on λR. Note that from (9), we also have

$$\begin{aligned}
\lambda^2 z &\le 4d + (2\lambda z + 4\lambda + 2\sqrt{dz} + 4n - 4d - 4) \\
&\le 4n + 2\lambda z + 4\lambda + 2\sqrt{dz} \\
&\le 4n + 2\lambda z + 4\lambda + 2z\sqrt{d},
\end{aligned}$$

since $z \ge \lambda > 1$. Thus by the fact that $\lambda^3 \ge 4n - 16$, we obtain that

$$z \le \frac{4n + 4\lambda}{\lambda^2 - 2\lambda - 2\sqrt{d}} \le \frac{\lambda^3 + 16 + 4\lambda}{\lambda^2 - 2\lambda - \sqrt{\lambda^3 + 16}} \le (1 + o(1))\,\lambda. \qquad (11)$$

Since $\lambda^3 \le \lambda^2 z \le 4n + 2\lambda z + 4\lambda + 2\sqrt{dz}$, we also have

$$\begin{aligned}
4n &\ge \lambda^3 - 2\lambda z - 4\lambda - 2\sqrt{dz} \\
&\ge \lambda^3 - 2\lambda(1 + o(1))\lambda - 4\lambda - \sqrt{(\lambda^3 + 16)(1 + o(1))\lambda} \\
&\ge \lambda^3 - 3(1 + o(1))\lambda^2 - 4\lambda \\
&\ge (\lambda - (1 + o(1)))^3.
\end{aligned}$$

Thus, we have

$$\lambda \le \sqrt[3]{4n} + (1 + o(1)). \qquad (12)$$

Combining with Lemma 1, we get an asymptotic estimation of λ.

$$\lambda = (1 + o(1))\sqrt[3]{4n}.$$

Recall that $\lambda \le z$. Hence, using (11), we have $z = (1+o(1))\lambda = (1+o(1))\sqrt[3]{4n}$. Consequently we obtain from (8) that $\lambda R = O(n)$, which implies that $\left(\sqrt{d+\lambda R}+\sqrt{d}\right)^2 = O(n)$. Now it follows from (10) that

$$\lambda R = O\left(\sqrt{n\lambda z + n\sqrt{dz}}\right) = O\left(\sqrt{n\lambda^2 + n^{3/2}\lambda^{1/2}}\right) = O(n^{5/6}).$$

Substituting λR into (4) and using the fact that $\lambda^3 \ge 4n-16$, we obtain that

$$4n - 16 \le 4d + O(n^{5/6}),$$

which implies that $d \ge n - O(n^{5/6})$. This completes the proof of Claim 1. □

Proof of Lemma 2. In order to further improve the lower bound on d (as claimed in Lemma 2), we need to give a non-trivial upper bound on $x_{u_0}^2 = \max_{v \ne v_0} x_v^2$. We claim $x_{u_0} = O(n^{-1/3})$.

Let $d' = d_G(u_0)$ and $\{u_1, u_2, \cdots, u_{d'}\}$ be the neighbors of u_0 in G. Since G is outerplanar and has no $K_{2,3}$ subgraph, v_0 and u_0 have at most two common neighbors, so $d' \le n+2-d = O(n^{5/6})$. Most of the inequalities shown in Claim 1 hold in similar forms. However, we have to treat any terms that involve x_{v_0} separately from other terms, so our definitions of R' and B' will be slightly different.

By the eigenequation for x_{u_0}, allowing for the possibility that some u_i is v_0, and using $ab \le (a^2+b^2)/2$, we have

$$\lambda x_{u_0}^2 = \sum_{i=1}^{d'-1} x_{u_i}x_{u_{i+1}} \le 2x_{v_0}x_{u_0} + \sum_{u\in N(u_0)\setminus\{v_0\}} x_u^2 = 2x_{u_0} + z'$$

where we define $z' = \sum_{u\in N(u_0)\setminus\{v_0\}} x_u^2$. Let B' be the subgraph of G consisting of the edges in $(\bigcup_{u\in N(u_0)\setminus\{v_0\}} \Gamma(u))\setminus\Sigma(u_0)$ and their endvertices. Here B' is similar in structure to B, but B' is missing the edges in $\Gamma(u_i)$ if some u_i is v_0. In a similar way to (1) and (2), if we apply the eigenequations again and Cauchy-Schwartz, we have

$$\lambda z' \le 2x_{u_0} + 2x_{u_0}\sqrt{d'z'} + R',$$

where $R' = \sum_{vw\in E(B')} x_v x_w$.

It follows from the same logic as in (3) that

$$z' \le \frac{4d'x_{u_0}^2}{\lambda^2} + \frac{2(R'+2x_{u_0})}{\lambda} - \left(\sqrt{\frac{d'x_{u_0}^2}{\lambda^2} + \frac{R'+2x_{u_0}}{\lambda}} - \frac{\sqrt{d'}x_{u_0}}{\lambda}\right)^2.$$

Then

$$\begin{aligned}\lambda^2(z'+2x_{u_0}) \le& 4d'x_{u_0}^2 + 2\lambda(R'+2x_{u_0}) - \left(\sqrt{d'x_{u_0}^2 + \lambda(R'+2x_{u_0})} - \sqrt{d'}x_{u_0}\right)^2 + 2\lambda^2 x_{u_0} \\ \le& 4d'x_{u_0}^2 + 2\lambda R' + (2\lambda^2+4\lambda)x_{u_0}.\end{aligned}$$

Hence we have

$$(4n-16)x_{u_0}^2 \le \lambda^3 x_{u_0}^2 \le \lambda^2(z'+2x_{u_0}) \le 4d'x_{u_0}^2 + 2\lambda R' + (2\lambda^2+4\lambda)x_{u_0}. \tag{13}$$

We will use an inequality like (8) to bound $2\lambda R'$. Similarly to (5), we have

$$\begin{aligned}
2\lambda R' &\le \sum_{vw\in E(B')} \lambda(x_v^2+x_w^2) = \sum_{u\in V(B')} d_{B'}(u)\,\lambda x_u^2 \\
&= \sum_{u\in V(B')\cap N_1(u_0)} d_{B'}(u)\,\lambda x_u^2 + \sum_{u\in V(B)\cap N_2(u_0)} d_{B'}(u)\,\lambda x_u^2 \\
&\le 2\sum_{i=1}^{d'} \lambda x_{u_i}^2 + \sum_{u\in V(B)\cap N_2(u_0)} d_{B'}(u)\,\lambda x_u^2 \\
&\le 2\lambda\left(x_{v_0}^2 + \sum_{u\in N(u_0)\setminus\{v_0\}} x_u^2\right) + \sum_{u\in V(B')\cap N_2(u_0)} d_{B'}(u) \sum_{vw\in\Gamma(u)} x_v x_w. \\
&= 2\lambda(z'+1) + \sum_{u\in V(B')\cap N_2(u_0)} d_{B'}(u) \sum_{vw\in\Gamma(u)} x_v x_w. \qquad (14)
\end{aligned}$$

For any vertex v the terms containing x_v in the sum above are

$$\sum_{u\in N(v)\cap N_2(u_0)\cap V(B')} \; \sum_{w\in\Gamma^{-1}(uv)} d_{B'}(u) x_v x_w. \qquad (15)$$

For each u there are at most two vertices $w\in\Gamma^{-1}(uv)$.

We will break the sum of (14) into four parts: S_0' from $vw\in\Gamma(u_0)$, S_1' from $vw\notin\Gamma(u_0)$ but $vw\in\Gamma(v_0)$, S_2' from $vw\notin\Gamma(u_0)$ but $vw\in\Sigma(v_0)$, and S_3' from all remaining terms.

We can bound S_0' in a similar way to (6), as

$$S_0' = \sum_{u\in V(B')\cap N_2(u_0)} d_{B'}(u) \sum_{vw\in\Gamma(u)\cap\Gamma(u_0)} x_v x_w \le 4\lambda x_{u_0}^2.$$

For S_1', S_2', S_3' we use the following analysis. If $v\notin N[u_0]$ then (15) has at most two vertices u, both of which belong to the same subgraph $B_i' = B'\cap\Phi(u_iu_{i+1},u_0)$. Thus, one u has degree at most 4 in B', and the other has degree at most 2. Since there are at most two vertices w for each u, for $v\notin N[u_0]$ the sum of the coefficients of all terms x_vx_w is at most 12. Moreover, if we take any edge vw at level $1\frac{1}{2}$ or higher relative to u_0, then one endvertex v satisfies $v\notin N[u_0]$, and x_vx_w occurs in (15) with a total coefficient of at most 6.

Hence, using the eigenequation for x_{v_0}, we can bound S_1' as

$$S_1' = \sum_{u\in V(B')\cap N_2(u_0)} d_{B'}(u) \sum_{vw\in(\Gamma(u)\cap\Gamma(v_0))\setminus\Gamma(u_0)} x_v x_w \le 6\sum_{vw\in\Gamma(v_0)} x_v x_w = 6\lambda x_{v_0}^2 = 6\lambda.$$

Consider terms in the sum of (14) with $vw\notin\Gamma(u_0)$ and $vw\in\Sigma(v_0)$, i.e., $v=v_0$. If $v=v_0\notin N[u_0]$ then the total coefficient of x_vx_w is at most 12, as described above. If $v=v_0\in N[u_0]$ then $v_0=u_i$ for some i, and the only possible vertices u in (15) are $q_{i-1}'\in\Gamma^{-1}(u_{i-1}u_i)$ and $q_i'\in\Gamma^{-1}(u_iu_{i+1})$, which both have degree at most 2 in B', and for each such u there is only one w such that $vw\notin\Gamma(u_0)$. Thus, the total coefficient of x_vx_w is at most 4. In either case,

$$S_2' = \sum_{u\in V(B')\cap N_2(u_0)} d_{B'}(u) \sum_{vw\in(\Gamma(u)\cap\Sigma(v_0))\setminus\Gamma(u_0)} x_v x_w \le 12x_{v_0}x_{u_0} = O(x_{u_0}).$$

Finally, the terms in the sum of (14) with $vw\notin\Gamma(u_0)\cup\Gamma(v_0)\cup\Sigma(v_0)$ give

$$S_3' = \sum_{u\in V(B')\cap N_2(u_0)} d_{B'}(u) \sum_{vw\in\Gamma(u)\setminus(\Gamma(u_0)\cup\Gamma(v_0)\cup\Sigma(v_0))} x_v x_w$$

$$\leq 6|E(G)\backslash(\Gamma(v_0) \cup \Sigma(v_0))|x_{u_0}^2 = 6(2n-2d-2)x_{u_0}^2 = O(n^{5/6})x_{u_0}^2.$$

Therefore, using the fact that $\lambda = O(n^{1/3})$,

$$\begin{aligned} 2\lambda R' &\leq 2\lambda(z'+1) + S'_0 + S'_1 + S'_2 + S'_3 \\ &= 2\lambda(z'+1) + 4\lambda x_{u_0}^2 + 6\lambda + O(x_{u_0}) + O(n^{5/6})x_{u_0}^2 \\ &= 2\lambda z' + 8\lambda + O(x_{u_0}) + O(n^{5/6})x_{u_0}^2. \end{aligned} \tag{16}$$

Substituting (16) into (13), and using $d' = O(n^{5/6})$, we have

$$\begin{aligned} (4n-16)x_{u_0}^2 \leq \lambda^2(z' + 2x_{u_0}) &\leq 4d'x_{u_0}^2 + 2\lambda R' + (2\lambda^2 + 4\lambda)x_{u_0} \\ &\leq 4d'x_{u_0}^2 + \left(2\lambda z' + 8\lambda + O(x_{u_0}) + O(n^{5/6})x_{u_0}^2\right) + (2\lambda^2+4\lambda)x_{u_0} \\ &\leq 2\lambda z' + O(n^{5/6})x_{u_0}^2 + 8\lambda + (2\lambda^2 + 4\lambda + O(1))x_{u_0}. \end{aligned} \tag{17}$$

Rearranging the inequality in (17), we first obtain an upper bound on z':

$$z' \leq \frac{O(n^{5/6})x_{u_0}^2 + (4\lambda + O(1))x_{u_0} + 8\lambda}{\lambda^2 - 2\lambda} = O\left(n^{1/6}x_{u_0}^2 + \frac{4x_{u_0}}{\lambda} + \frac{1}{\lambda}\right).$$

Now using the upper bound on z' and (17), we have the following inequality:

$$\begin{aligned} (4n-16)x_{u_0}^2 &\leq 2\lambda z' + O(n^{5/6})x_{u_0}^2 + 8\lambda + (2\lambda^2 + 4\lambda + O(1))x_{u_0} \\ &= O\left(n^{5/6}x_{u_0}^2 + \lambda^2 x_{u_0} + \lambda\right). \end{aligned}$$

It follows from the fact that $\lambda = O(n^{1/3})$ that

$$x_{u_0} = O(n^{-1/3}).$$

Now using the bound $x_{u_0} = O(n^{-1/3})$ in (8), we obtain a better bound on $d = d_G(v_0)$ in Claim 1:

$$4n - 16 \leq \lambda^3 \leq 4d + 2\lambda z + 4\lambda + 2\sqrt{dz} + 4(n-d)\,O(n^{-2/3}),$$

which gives us

$$\begin{aligned} (4n-4d)(1 - O(n^{-2/3})) &\leq 16 + 2\lambda z + 4\lambda + 2\sqrt{dz} \\ &= O(1) + O(n^{2/3}) + O(n^{1/3}) + O(\sqrt{n \cdot n^{1/3}}) = O(n^{2/3}) \end{aligned}$$

and thus $d \geq n - O(n^{2/3})$. This completes the proof of Lemma 2. ☐

Lemma 3. *$d_{\mathcal{H}}(v_1) = 1$. Moreover, $x_{v_2} \geq x_{v_1}$.*

Proof. Assume for the sake of contradiction that $d_{\mathcal{H}}(v_1) \geq 2$. Recall that $\mathcal{H}$ is edge-maximal. It follows that there must exist a vertex $q_1 \neq v_0$ such that $\{v_1, v_2, q_1\}$ is a hyperedge, and q_1 is a vertex of $F_1 = \Phi(v_1v_2, v_0)$. Let F'_1 be F_1 but with v_2 renamed as v_0. Then $G' = G - (V(F_1) - \{v_1, v_2\}) \cup F'_1$ is outerplanar (we find the outerplanar embedding by flipping F'_1 over, i.e., reflecting it), and G' is the shadow of a 3-uniform hypergraph $\mathcal{H}'$ that can be obtained from $\mathcal{H}$ by replacing each hyperedge $\{v_2, u, w\}$ where $u, w \in V(F_1)$ by a hyperedge $\{v_0, u, w\}$. Suppose $\boldsymbol{x}$ is the Perron-Frobenius eigenvector of $\mathcal{H}$. Since $x_{v_0} > x_{v_2}$ by Lemma 2, it follows that

$$\sum_{\{i_1,i_2,i_3\} \in E(\mathcal{H}')} x_{i_1}x_{i_2}x_{i_3} - \sum_{\{i_1,i_2,i_3\} \in E(\mathcal{H})} x_{i_1}x_{i_2}x_{i_3} \geq x_{v_1}x_{q_1}(x_{v_0} - x_{v_2}) > 0.$$

This implies that $\lambda(\mathcal{H}') > \lambda(\mathcal{H})$, which contradicts $\mathcal{H}$ attaining the maximum spectral radius.

It remains to show that $x_{v_2} \geq x_{v_1}$. If $x_{v_2} < x_{v_1}$, then let $\boldsymbol{x'}$ be obtained from $\boldsymbol{x}$ by setting $x'_{v_1} = x_{v_2}$, $x'_{v_2} = x_{v_1}$ and keeping every other entry the same. Since $d_{\mathcal{H}}(v_1) = 1$, it follows that $P_{\mathcal{H}}(\boldsymbol{x'}) > P_{\mathcal{H}}(\boldsymbol{x})$, which contradicts $\boldsymbol{x}$ being the Perron-Frobenius eigenvector of $\mathcal{H}$. □

Now we are ready to show Theorem 1.

Proof of Theorem 1. Let $\mathcal{H}$ be an outerplanar 3-uniform hypergraph on n vertices with maximum spectral radius. Let G be the shadow of $\mathcal{H}$. Suppose the Perron–Frobenius eigenvector $\boldsymbol{x}$ of the adjacency tensor of $\mathcal{H}$ is normalized so that the maximum eigenvector entry is 1. Let v_0 be the vertex with the maximum eigenvector entry and $\{v_1, v_2, \cdots, v_d\}$ be the neighbors of v_0 in the clockwise order of the outerplanar drawing of G.

By Lemma 1, we have that $d(v_0) \geq n - O(n^{2/3})$ and for every other vertex $u \neq v_0$, $x_u = O(n^{-1/3})$. Now we claim that $x_{v_1} = \Omega(n^{-1/3})$. By Lemma 3, we have that $d_{\mathcal{H}}(v_1) = 1$, i.e., $\{v_1, v_2, v_0\}$ is the unique hyperedge containing v_1. It follows by Lemma 3 and the eigenequation for x_{v_1} that

$$\lambda x_{v_1}^2 = x_{v_0} x_{v_2} = x_{v_2} \geq x_{v_1}.$$

Together with (12), this implies that

$$x_{v_1} \geq \frac{1}{\lambda} = \Omega(n^{-1/3}).$$

Now we claim that for every vertex $u \in V(G)\backslash\{v_0\}$, u is a neighbor of v_0 in G. Suppose not. The hyperedges incident with v_0 form a path in the dual of G and there must be a hyperedge $\{w, s, t\}$ that is a leaf of the dual tree (excluding the outer face) but not an end of this path. Then $w, s, t \neq v_0$ and one of these vertices, say w, has degree 2 in G and degree 1 in $\mathcal{H}$. Now similarly to Lemma 3, consider the hypergraph $\mathcal{H}'$ obtained from $\mathcal{H}$ by by removing the hyperedge $\{w, s, t\}$ and adding the hyperedge $\{w, v_0, v_1\}$. It follows that

$$\sum_{\{i_1,i_2,i_3\} \in E(\mathcal{H}')} x_{i_1} x_{i_2} x_{i_3} - \sum_{\{i_1,i_2,i_3\} \in E(\mathcal{H})} x_{i_1} x_{i_2} x_{i_3} \geq x_w x_{v_0} x_{v_1} - x_w x_s x_t.$$

Note that $x_s x_t = O(n^{-2/3})$ while $x_{v_0} x_{v_1} = \Omega(n^{-1/3})$. It follows that $x_w x_{v_0} x_{v_1} > x_w x_s x_t$, which implies that $\lambda(\mathcal{H}') > \lambda(\mathcal{H})$, contradicting $\mathcal{H}$ being the extremal hypergraph of maximum spectral radius. Hence every vertex $u \in V(G)\backslash\{v_0\}$ is a neighbor of v_0 in G.

Again by the fact that $\mathcal{H}$ attains the maximum spectral radius, it follows that $\mathcal{H}$ is the unique 3-uniform hypergraph $\mathcal{F}_n$ with $K_1 + P_{n-1}$ as it shadow. □

References

[1] B. N. Boots and G. F. Royle, A conjecture on the maximum value of the principal eigenvalue of a planar graph, *Geograph. Anal.* **23** (1991), 276-282.

[2] D. Cao and A. Vince, Spectral radius of a planar graph, *Linear Algebra Appl.* **187** (1993), 251-257.

[3] K. Chang, K. Pearson, T. Zhang, Perron–Frobenius theorem for nonegative tensors, *Commun. Math. Sci.* **6** (2008), 507-520.

[4] J. Cooper and A. Dutle, Spectra of uniform hypergraphs, *Linear Algebra Appl.* **436** (2012), 3268–3299.

[5] D. Cvetković and P. Rowlinson, The largest eigenvalue of a graph: A survey, *Linear and Multilinear Algebra* **28** (1990), 3-33.

[6] M. N. Ellingham and X. Zha, The spectral radius of graphs on surfaces, *J. Combin. Theory Ser. B* **78(1)** (2000), 45-56.

[7] S. Friedland, A. Gaubert, L. Han, Perron–Frobenius theorems for nonnegative multilinear forms and extensions, *Linear Algebra Appl.* **438** (2013), 738–749.

[8] B. D. Guiduli, "Spectral Extrema for Graphs", Ph.D. thesis, University of Chicago, 1996.

[9] B. D. Guiduli and T. P. Hayes, On the spectral radius of graphs of given genus, preprint, Institut IMAG, Grenoble, France, and University of Chicago, 1998.

[10] L-H. Lim, Singular values and eigenvalues of tensors: a variational approach. In: Proceedings of the IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing CAMSAP'05. Vol. 1; Puerto Vallarta, Mexico: 2005. p. 129–132.

[11] H. Lin and B. Ning, A complete solution to the Cvetković–Rowlinson conjecture, *J. Graph Theory* **97(3)** (2021), 441–450.

[12] L. Qi, Eigenvalues of a real supersymmetric tensor, *J. Symb Comput.* **40** (2005), 1302–1324.

[13] L. Qi, Symmetric nonnegative tensors and copositive tensors, *Linear Algebra Appl.* **439** (2013), 228-238.

[14] P. Rowlinson, On the index of certain outerplanar graphs, *Ars Combin.* **29C** (1990), 221-225.

[15] A. J. Schwenk and R. J. Wilson, On the eigenvalues of a graph, in "Selected Topics in Graph Theory" (L. W. Beineke and R. J. Wilson, Eds.), Chap. 11, pp. 307-336, Academic Press, London, 1978.

[16] M. Tait and J. Tobin, Three conjectures in extremal spectral graph theory, *J. Combin. Theory Ser. B* **126** (2017), 137–161.

[17] Hong Yuan, A bound on the spectral radius of graphs, *Linear Algebra Appl.* **108** (1988), 135-139.

[18] Hong Yuan, On the spectral radius and the genus of graphs, *J. Combin. Theory Ser. B* **65** (1995), 262-268.

[19] Hong Yuan, Upper bounds of the spectral radius of graphs in terms of genus, *J. Combin. Theory Ser. B* **74** (1998), 153-159.

[20] A. A. Zykov, Hypergraphs, *Uspeki Mat. Nauk* **6** (1974), 89–154 (in Russian).