Maximum spectral radius of outerplanar 3-uniform hypergraphs

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Abstract

In this paper, we study the maximum spectral radius of outerplanar 3-uniform hypergraphs. Given a hypergraph \( H \), the shadow of \( H \) is a graph \( G \) with \( V(G) = V(H) \) and \( E(G) = \{uv : uv \in h \text{ for some } h \in E(H)\} \). A graph is outerplanar if it can be embedded in the plane such that all its vertices lie on the outer face. A 3-uniform hypergraph \( H \) is called outerplanar if its shadow has an outerplanar embedding such that every hyperedge of \( H \) is the vertex set of an interior triangular face of the shadow. Cvetković and Rowlinson [Linear and Multilinear Algebra, 1990] conjectured that among all outerplanar graphs on \( n \) vertices, the graph \( K_1 + P_{n-1} \) attains the maximum spectral radius. We show a hypergraph analogue of the Cvetković-Rowlinson conjecture. In particular, we show that for sufficiently large \( n \), the \( n \)-vertex outerplanar 3-uniform hypergraph of maximum spectral radius is the unique 3-uniform hypergraph whose shadow is \( K_1 + P_{n-1} \).

1 Introduction

A graph \( G \) is planar if it can be embedded in the plane, i.e., it can be drawn on the plane in such a way that edges intersect only at their endpoints. A graph is outerplanar if it can be embedded in the plane such that all vertices lie on the boundary of its outer face. The study of the spectral radius of (outer)planar graphs has a long history, dating back to Schwenk and Wilson [15]. Given a graph \( G \), the spectral radius \( \lambda(G) \) is the largest eigenvalue of the adjacency matrix of \( G \). The spectral radius of planar graphs is useful in geography as a measure of the overall connectivity of a planar graph [1, 5]. It is therefore of interest to geographers to find the maximum spectral radius of a planar graph as a theoretical upper bound for the connectivity of networks. Boots and Royle [1], and independently Cao and Vince [2] conjectured that the extremal planar graph achieving the maximum spectral radius is \( K_2 + P_{n-2} \) (see Figure 1). Hong [17] first showed that for an \( n \)-vertex planar graph \( G \), \( \lambda(G) \leq \sqrt{5m - 11} \). This was subsequently improved in a series of papers [2, 18, 8, 19, 6]. Guiduli and Hayes [9] showed in an unpublished preprint that the Boots-Royle-Cao-Vince conjecture is true for sufficiently large \( n \). For outerplanar graphs, it is conjectured by Cvetković and Rowlinson [5] that among all outerplanar graphs on \( n \) vertices, \( K_1 + P_{n-1} \) attains the maximum spectral radius (see Figure 1). Partial progress has been made by Rowlinson [14], Cao and Vince [2], and Guiduli and Hayes [9]. Recently, Tait and Tobin [16] proved the Boots-Royle-Cao-Vince conjecture and the Cvetković-Rowlinson conjecture for large enough \( n \). Lin and Ning [11] showed that the Cvetković-Rowlinson conjecture holds for all \( n \geq 17 \).

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In this paper, we extend the investigations into the maximum spectral radius of outerplanar 3-uniform hypergraphs. Given a hypergraph \( \mathcal{H} \), the shadow of \( \mathcal{H} \), denoted by \( \partial(\mathcal{H}) \), is a 2-uniform graph \( G \) with \( V(G) = V(\mathcal{H}) \) and \( E(G) = \{ uv : uv \in h \text{ for some } h \in E(\mathcal{H}) \} \).

We adopt Zykov’s [20] definition of hypergraph planarity. In particular, a 3-uniform hypergraph \( \mathcal{H} \) is called planar if \( \partial(\mathcal{H}) \) has a planar embedding so that every hyperedge of \( \mathcal{H} \) is the vertex set of a triangular face of \( \partial(\mathcal{H}) \). A 3-uniform hypergraph \( \mathcal{H} \) is called outerplanar if \( \partial(\mathcal{H}) \) has an outerplanar embedding such that every hyperedge of \( \mathcal{H} \) is the vertex set of an interior triangular face of \( \partial(\mathcal{H}) \).

Now we define the spectral radius of an \( r \)-uniform hypergraph. Given positive integers \( r \) and \( n \), an order \( r \) and dimension \( n \) tensor \( \mathbf{A} = (a_{i_1i_2...i_r}) \) over \( \mathbb{C} \) is a multidimensional array with all entries \( a_{i_1i_2...i_r} \in \mathbb{C} \) for all \( i_1, i_2, \ldots, i_r \in [n] \). Given a column vector \( \mathbf{x} = (x_1, x_2, \ldots, x_n)^T \in \mathbb{C}^n \), \( \mathbf{A} \mathbf{x}^{r-1} \) is defined to be a vector in \( \mathbb{C}^n \) whose \( i \)th entry is

\[
(\mathbf{A} \mathbf{x}^{r-1})_i = \sum_{i_2, \ldots, i_r=1}^{n} a_{i_1i_2...i_r} x_{i_2} \cdots x_{i_r}.
\]

In 2005, Qi [12] and Lim [10] independently proposed the definition of eigenvalues of a tensor. In particular, if there exists a number \( \lambda \in \mathbb{C} \) and a nonzero vector \( \mathbf{x} \in \mathbb{C}^n \) such that

\[
\mathbf{A} \mathbf{x}^{r-1} = \lambda \mathbf{x}^{r-1}
\]

where \( \mathbf{x}^{[r-1]} = (x_1^{r-1}, x_2^{r-1}, \ldots, x_n^{r-1})^T \), then \( \lambda \) is called the eigenvalue of \( \mathbf{A} \) and \( \mathbf{x} \) is called an eigenvector of \( \mathbf{A} \) corresponding to \( \lambda \). The spectral radius of \( \mathbf{A} \), denoted by \( \lambda(\mathbf{A}) \), is the maximum modulus of the eigenvalues of \( \mathbf{A} \). It was shown in [13] that

\[
\lambda(\mathbf{A}) = \max_{||\mathbf{x}||_r=1} \mathbf{x}^T \mathbf{A} \mathbf{x}^{r-1},
\]

where \( ||\mathbf{x}||_r := (|x_1|^r + |x_2|^r + \cdots + |x_n|^r)^{1/r} \) and \( \mathbb{R}_+ \) is the set of nonnegative real numbers.

In 2012, Cooper and Dutle [4] defined the adjacency tensor of an \( r \)-uniform hypergraph. Given an \( r \)-uniform hypergraph \( \mathcal{H} \) on \( n \) vertices, the adjacency tensor \( \mathbf{A}(\mathcal{H}) \) of \( \mathcal{H} \) is defined as the order \( r \) dimension \( n \) tensor with entries \( a_{i_1i_2...i_r} \) such that

\[
a_{i_1i_2...i_r} = \begin{cases} 
\frac{1}{(r-1)!} & \text{if } \{i_1, i_2, \ldots, i_r\} \in E(\mathcal{H}) \\
0 & \text{otherwise.}
\end{cases}
\]

Let \( \lambda(\mathcal{H}) \) denote the spectral radius of \( \mathbf{A}(\mathcal{H}) \). Given an \( r \)-uniform hypergraph \( \mathcal{H} \) and a vector \( \mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \), we can define a multi-linear function \( P_{\mathcal{H}}(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R} \) as follows:

\[
P_{\mathcal{H}}(\mathbf{x}) = r \sum_{\{i_1, i_2, \ldots, i_r\} \in E(\mathcal{H})} x_{i_1} x_{i_2} \cdots x_{i_r}.
\]
Then the spectral radius of $\mathcal{H}$ can be also expressed as

$$\lambda(\mathcal{H}) := \max_{\|x\|_r = 1} P_\mathcal{H}(x) = \max_{x \in \mathbb{R}_+^n} \frac{P_\mathcal{H}(x)}{\|x\|_r}. \quad (1)$$

The Perron-Frobenius theorem \cite{3, 7} for nonnegative tensors implies that there is always a nonnegative vector $x$ satisfying the maximum at right above. Any such $x$ is called a Perron-Frobenius eigenvector of $\mathcal{A}(\mathcal{H})$ (corresponding to $\lambda(\mathcal{H})$). If $\mathcal{H}$ is connected then a Perron-Frobenius eigenvector is strictly positive and is unique up to scaling by a positive coefficient; moreover, the spectral radius $\lambda(\mathcal{H})$ is the unique eigenvalue with a strictly positive eigenvector. By definition, the spectral radius $\lambda(\mathcal{H})$ and its eigenvector $x = (x_1, \cdots, x_n)$ also satisfy the following eigenequation on every $x_i$:

$$\lambda(H)x_i^{r-1} = \sum_{\{i, j_2, \cdots, j_r\} \in E(\mathcal{H})} x_{j_2} \cdots x_{j_r} \text{ for } x_i > 0. \quad (2)$$

Now we are ready to state our main theorem. We use $\mathcal{F}_n$ to denote the fan hypergraph, i.e., the unique 3-uniform hypergraph whose shadow is $K_1 + P_{n-1}$ (see Figure 1).

**Theorem 1.** For large enough $n$, the $n$-vertex outerplanar 3-uniform hypergraph of maximum spectral radius is the fan hypergraph $\mathcal{F}_n$.

The shadow of the extremal hypergraph attaining the maximum spectral radius among all outerplane 3-uniform hypergraphs is exactly the extremal graph attaining the maximum spectral radius among all outplanar graphs. This motivates us to make the following analogous conjecture for planar 3-uniform hypergraphs:

**Conjecture 1.** For large enough $n$, the $n$-vertex planar 3-uniform hypergraph $\mathcal{H}$ of maximum spectral radius is the unique maximal hypergraph whose shadow is $K_2 + P_{n-2}$.

## 2 Proof of Theorem 1

Given a graph $G$ and $v \in V(G)$, we use $N_G(v)$ to denote the set of neighbors of $v$, i.e., $N_G(v) = \{u : vu \in E(G)\}$. The closed neighborhood of $v$, denoted by $N_G[v]$, is defined as $N_G[v] = N_G(v) \cup \{v\}$. Given a 3-uniform hypergraph $\mathcal{H}$ and $v \in V(\mathcal{H})$, we define $\Gamma_\mathcal{H}(v) = \{uw : vuw \in E(\mathcal{H})\}$. Moreover, set $d_G(v) = |N_G(v)|$ and $d_\mathcal{H}(v) = |\Gamma_\mathcal{H}(v)|$. In all the definitions above, we may ignore the subscript if the underlying (hyper)graph is clear from the context.

Let $\mathcal{H}$ be an $n$-vertex outerplanar 3-uniform hypergraph of maximum spectral radius. Throughout this section, let $G$ be the shadow of $\mathcal{H}$, i.e., $V(G) = V(\mathcal{H})$ and $E(G) = \{vu : \{v, u\} \subseteq h \text{ for some } h \in E(\mathcal{H})\}$. It follows by definition that $G$ is outerplanar, thus does not contain a $K_{2,3}$ minor. Observe that $\mathcal{H}$ must be edge-maximal (while maintaining the outerplanarity). Otherwise, we can obtain an outerplanar hypergraph $\mathcal{H}'$ such as $\mathcal{H} \subseteq \mathcal{H}'$. It then follows from the Perron-Frobenius Theorem that $\mathcal{H}'$ attains a larger spectral radius than $\mathcal{H}$, giving us a contradiction. Now since $\mathcal{H}$ is edge-maximal, $G$ must be a maximal outerplanar graph. It follows that $G$ must be 2-connected, thus does not contain a cut vertex. All interior faces of $G$ are triangles, and $G$ has $2n - 3$ edges. The dual of $G$ (excluding the outer face) is a tree, so the interior faces of $G$ are connected together in a treelike fashion.

**Lemma 1.** $\lambda(\mathcal{H}) \geq \sqrt[4]{4(n-1)} \left(1 - \frac{1}{n-1}\right)$. 

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Proof. Let $F_n$ be the fan hypergraph on $n$ vertices, i.e., the unique 3-uniform hypergraph on $n$ vertices whose shadow is $K_1 + P_{n-1}$. Suppose $w$ is the vertex that is adjacent to all the other vertices in $\partial(F_n)$ and $v_1, v_2, \ldots, v_{n-1}$ are its neighbors. Clearly $F_n$ is outerplanar. Consider the vector $x \in \mathbb{R}^n$ with $x_w = 1/\sqrt[3]{3}$ and $x_{v_i} = \left(\frac{2}{3(n-1)}\right)^{1/3}$. Note that $\|x\|_3 = 1$. It follows that

$$\lambda(H) \geq \lambda(F_n) \geq P_{F_n}(x) = 3(n-2) \cdot \frac{1}{\sqrt[3]{3}} \cdot \left(\frac{2}{3(n-1)}\right)^{2/3} = \sqrt{4(n-1)} \left(1 - \frac{1}{n-1}\right).$$

Note that since $H$ is connected, there exists an eigenvector corresponding to $\lambda(H)$ such that all its entries are strictly positive. In the rest of this section, for convenience we assume that the Perron-Frobenius eigenvector of $H$ is re-normalized so that the maximum eigenvector entry is 1.

Let $v_0$ be the vertex with the maximum eigenvector entry, i.e., $x_{v_0} = 1$. The following lemma says that $H$ is very close to the fan hypergraph $F_n$.

Lemma 2. We have $\lambda = (1 + o(1))\sqrt[3]{3n}$ and $d_G(v_0) > n - O(n^{2/3})$. Moreover, for any other vertex $u \neq v_0$, $x_u = O(n^{-1/3})$.

We first show a weaker version of Lemma 2. In particular, we show the following claim.

Claim 1. $d_G(v_0) > n - O(n^{5/6})$.

Proof of Claim 1. Recall that $x_{v_0} = 1$ where $v_0$ is the vertex with the maximum eigenvector entry of the Perron-Frobenius eigenvector of $H$. Let $d = d_G(v_0)$. Let \{\(v_1, v_2, \ldots, v_d\)\} be the neighbors of $v_0$ in the clockwise order of some outerplanar drawing of $G$. Observe that we can relabel them in such a way that \{\(v_i, v_{i+1}, v_0\)\} $E(H)$ for each $i \in [d-1]$. This is because if for some $j \neq d$ such that \(v_j, v_{j+1}, v_0\) $E(H)$, then we can add the hyperedge \(v_j, v_{j+1}, v_0\) to $H$ and obtain an outerplanar hypergraph with larger spectral radius. Now by the eigenequation on $v_0$, we have

$$\lambda = \lambda x_{v_0}^2 = \sum_{i=1}^{d-1} x_{v_i} x_{v_{i+1}} \leq \sum_{i=1}^{d} x_{v_i}^2,$$

using the fact $ab \leq \frac{a^2 + b^2}{2}$. Set $z = \sum_{i=1}^{d} x_{v_i}^2$. We have $\lambda \leq z$. It again follows from the eigenequation expansion that

$$\lambda z \leq \sum_{i=1}^{d} \lambda x_{v_i}^2 \leq 2x_{v_0} \sum_{i=1}^{d} x_{v_i} + \sum_{i=1}^{d} \sum_{v,w \in E(v_i)} x_v x_w$$

$$= 2 \sum_{i=1}^{d} x_{v_i} + \sum_{i=1}^{d} \sum_{v,w \in E(v_i), v \neq v_0} x_v x_w.$$
Figure 2: Neighborhood of $v_0$

\[
\leq 2\sqrt{dz} + \sum_{i=1}^{d} \sum_{\substack{vw \in \Gamma(v_i) \\ v,w \neq v_0}} x_v x_w,
\]

where the last inequality is by the Cauchy-Schwarz inequality.

For ease of reference, set $\displaystyle R = \sum_{i=1}^{d} \sum_{\substack{vw \in \Gamma(v_i) \\ v,w \neq v_0}} x_v x_w$. In Figure 2, all the edges $vw \in E(G)$ corresponding to the summands $x_v x_w$ in $R$ are colored red. Dividing both sides of the inequality above by $\lambda$, we then have $\displaystyle z - \frac{2\sqrt{dz}}{\lambda} \leq \frac{R}{\lambda}$. By completing the square, we have $\left(\sqrt{z} - \frac{\sqrt{d}}{\lambda}\right)^2 \leq \frac{R}{\lambda} + \frac{d}{\lambda^2}$. Rearranging the terms of the inequality, we obtain that

\[
z \leq \left(\frac{\sqrt{d}}{\lambda} + \sqrt{\frac{d}{\lambda^2} + \frac{R}{\lambda}}\right)^2 = \frac{4d}{\lambda^2} + \frac{2R}{\lambda} - \left(\sqrt{\frac{d}{\lambda^2} + \frac{R}{\lambda}} - \sqrt{d}\right)^2.
\]

It follows that

\[
\lambda^3 \leq \lambda^2 z \leq 4d + 2\lambda R - \left(\sqrt{d + R\lambda} - \sqrt{d}\right)^2.
\]

By Lemma 1, we obtain that $\lambda^3 \geq 4n - 16$ when $n$ is large enough. Let’s now give a bound on $2\lambda R$. Observe that since $G$ is an outerplanar graph, the neighborhood around an edge $v_i v_{i+1}$ is a subgraph of the structure shown in Figure 2. The edges $vw$ for which $x_v x_w$ appears in the summands of $R$ are colored red. Let $E_r$ be the collection of these red edges. Again using the fact that $2ab \leq a^2 + b^2$, we replace all $2x_v x_w$ in $R$ by $x_v^2 + x_w^2$. We then use the eigenvalue on $x_v$ and $x_w$ to expand $\lambda(x_v^2 + x_w^2)$.

To make the analysis easier, we partition the vertices into three classes and pay attention to their multiplicity in the summation. Note that we only need to consider the vertices that are the endpoints of red edges. The first class of vertices (denoted by $V_1$) are the ones that are adjacent to $v_0$. It’s easy to see that

\[
\sum_{h \in E_r} \sum_{u \in V_1 \cap h} x_u^2 \leq 2 \sum_{i=1}^{d} x_{v_i}^2.
\]

Hence we have

\[
\lambda \sum_{h \in E_r} \sum_{u \in V_1 \cap h} x_u^2 \leq 2 \sum_{i=1}^{d} \lambda x_{v_i}^2 = 2\lambda z.
\]
The next class of vertices (denoted by $V_2$) consists of the ones that form a hyperedge with two adjacent neighbors of $v_0$ (labeled as $q$ in Figure 2). The set of the remaining vertices are denoted by $V_3$. Now by the eigenequations on these vertices, we have

$$
\lambda \sum_{h \in E_r} \sum_{u \in V_2 \cap h} x_u^2 + \sum_{h \in E_r} \sum_{u \in V_3 \cap h} x_u^2 = \sum_{h \in E_r} \sum_{u \notin N(v_0)} \sum_{vw \in \Gamma(u)} x_v x_w.
$$

(7)

Let $E'$ be the set of edges $vw$ in $G$ for which $x_v x_w$ appears as summands in the summation above. Note that none of the edges in $E'$ contain $v_0$. For edges $vw \in E'$, we need to count the multiplicity of $x_v x_w$ in the summation above. For edges $vw$ in $E'$ such that $vwv_0 \in E(H)$, it’s easy to see that $x_v x_w$ has multiplicity at most 4 since these terms come from the eigenequation expansion on some vertex of $V_2$, which is incident to at most 4 red edges. Moreover, by the eigenequation on $x_{v_{i-1}}$, we have

$$
\sum_{h \in E_r} \sum_{u \in h} \sum_{vw \in \Gamma(v_{i-1}) \cap \Gamma(u)} x_v x_w \leq 4 \lambda x_{v_{i-1}}^2 = 4 \lambda.
$$

Figure 3: Neighborhood around edges $v_{i-1}v_i$.

Next we analyze the average number of times that edges in $E' \setminus \Gamma(v_0)$ appear in the summands of (7). We do this by first considering the the structure of the neighborhood around each vertex $v_i \in N_G(v_0)$. Observe that since $G$ is outerplanar and 2-connected, the neighborhood of each $v_i$ must intersect the neighborhood of either $v_{i-1}$ or $v_{i+1}$ (besides at $v_0$). Moreover, WLOG, if $N(v_i) \setminus \{v_0\}$ intersects $N(v_{i-1}) \setminus \{v_0\}$, they intersect exactly at one point since $G$ is outerplanar. Thus, each vertex in $N_G(v_i) \setminus \{v_0\}$ must lie in a subgraph of the structure in Figure 3 (or a similar structure around the edge $v_i v_{i+1}$, but not both). The multiplicities of the edges in each structure, i.e. the edges of $E' \setminus \Gamma(v_0)$ that is either incident to some $v_i$ or forms a hyperedge with some $v_i$ are labelled in Figure 3. It is easy to compute that the average multiplicity of such edges is at most 2.

Figure 4: Average multiplicity of edges in $E' \setminus \Gamma(v_0)$

For the edges of $E' \setminus \Gamma(v_0)$ that is not incident to some $v_i$ or forms some hyperedge with some $v_i$ (colored in green in Figure 4 and will be referred as green edges), we analyze their multiplicities similarly. In particular, similar to before, the neighborhood around each vertex in $N(v_i) \setminus N[v_0]$ has similar structure as the neighborhood around the vertices in $N(v_0)$. Observe that the average multiplicity of these green edges is at most 2 except in the case shown in Figure 4 when there is
a vertex $a$ that forms a hyperedge with a red edge in $E_r$ (e.g., $qb \in E_r$) with $q \in V_2$. However observe that the average multiplicity of the green edges is at most 2 if we can subtract 2 from the total multiplicities for each appearance of the hyperedge $qab$ (which is the reason that the average multiplicity is above 2). Moreover, notice there are at most 2d number of such hyperedges $qab$ since $|V_2| \leq d$. To solve this, for each appearance of $qab$, we will count one term of $x_qx_{v_1}$ and $x_qx_{v_1-1}$ separately and use the fact that $x_qx_{v_1} \leq x_{v_1}x_{v_1} \leq x_{v_1}$ (similar for $x_{v_1-1}$). Now, since the average multiplicity of the non-green edges in $E' \setminus \Gamma(v_0)$ is at most 2, this implies that the average multiplicity of all edges in $E' \setminus \Gamma(v_0)$ is at most 2 if we count some occurrences of $x_qx_{v_1}$ separately. In particular, we have

$$
\sum_{h \in E_r} \sum_{u \in h} \sum_{v \in (u) \setminus \Gamma(v_0)} x_qx_w \leq 2|E' \setminus \Gamma(v_0)| \max_{x_v, v \neq v_0} x_v^2 + 4 \sum_{i \in [d]} x_{v_i}. \tag{8}
$$

Observe that $|E' \setminus \Gamma(v_0)| \leq E(G) - (2d - 1) \leq 2n - 2d - 2$ since $G$ is outerplanar. It follows from (7) that

$$
\lambda \sum_{h \in E_r} \sum_{u \in h} x_u^2 + \sum_{h \in E_r} \sum_{u \in V_2 \setminus h} x_u^2 = \sum_{h \in E_r} \sum_{u \in h} \sum_{v \in \Gamma(u)} x_u x_w \\
\leq \sum_{h \in E_r} \sum_{u \in h} \sum_{v \in \Gamma(v_0) \setminus \Gamma(u)} x_u x_w + \sum_{h \in E_r} \sum_{u \in h} \sum_{v \in \Gamma(u) \setminus \Gamma(v_0)} x_u x_w \\
\leq 4\lambda x_{v_0}^2 + 4 \sum_{i \in [d]} x_{v_i} + 2(E(G) - (2d - 1)) \max_{x_v, v \neq v_0} x_v^2 \\
\leq 4\lambda + 4\sqrt{dz} + (4n - 4d - 4 \max_{x_v, v \neq v_0} x_v^2.
$$

Hence in summary, by (6) and the inequality above, we have

$$
2\lambda R = 2\lambda \sum_{i=1}^{d} \sum_{v, w \in \Gamma(v_i)} x_v x_w \\
\leq \lambda \sum_{h \in E_r} \sum_{u \in V_2 \setminus h} x_u^2 + \lambda \sum_{h \in E_r} \sum_{u \in V_2 \setminus h} x_u^2 + \lambda \sum_{h \in E_r} \sum_{u \in V_3 \setminus h} x_u^2 \\
\leq 2\lambda z + 4\lambda + 4\sqrt{dz} + (4n - 4d - 4 \max_{x_v, v \neq v_0} x_v^2. \tag{9}
$$

Substitute (9) into (5), it follows that when $n$ is large enough,

$$
4n - 16 \leq \lambda^3 \leq \lambda^2 z \leq 4d + \left(2\lambda z + 4\lambda + 4\sqrt{dz} + 4n - 4d - 4\right) - \left(\sqrt{d + R\lambda} - \sqrt{d}\right)^2. \tag{10}
$$

Cancelling terms and rearranging the inequality, we obtain that

$$
\left(\sqrt{d + R\lambda} - \sqrt{d}\right)^2 \leq 2\lambda(z + 2) + 4\sqrt{dz} + 12,
$$
which can be written as
\[
\frac{(\lambda R)^2}{(\sqrt{d + \lambda R} + \sqrt{d})^2} \leq 2\lambda(z + 2) + 4\sqrt{dz} + 12. \tag{11}
\]

From here, we want to give an upper bound on \(\lambda R\). Note that from (10), we also have
\[
\lambda^2 z \leq 4d + (2\lambda z + 4\lambda + 4\sqrt{dz} + 4n - 4d - 4)
\leq 4n + 2\lambda z + 4\lambda + 4\sqrt{dz}
\leq 4n + 2\lambda z + 4\lambda + 4\sqrt{d},
\]
since \(z \geq \lambda > 1\). Thus by the fact that \(\lambda^3 \geq 4n - 16\), we obtain that
\[
z \leq \frac{4n + 4\lambda}{\lambda^3 - 2\lambda - 4\sqrt{d}} \leq \frac{\lambda^3 + 16 + 4\lambda}{\lambda^3 - 2\lambda - 2\sqrt{\lambda^3 + 16}} \leq (1 + o(1)) \lambda.
\]

Since \(\lambda^3 \leq \lambda^2 z \leq 4n + 2\lambda z + 4\lambda + 4\sqrt{dz}\), we also have
\[
4n \geq \lambda^3 - 2\lambda z - 4\lambda - 4\sqrt{dz}
\geq \lambda^3 - 2\lambda(1 + o(1))\lambda - 4\lambda - 2\sqrt{(\lambda^3 + 16)(1 + o(1))}\lambda
\geq \lambda^3 - 4(1 + o(1))\lambda^2 - 4\lambda
\geq \left(\lambda - (1 + o(1))\frac{2}{3}\sqrt{3}\right)^3.
\]

Thus, we have
\[
\lambda \leq \sqrt{4n} + (1 + o(1))\frac{2}{3}\sqrt{3}. \tag{12}
\]

Combining with Lemma 1, we get asymptotic estimation of \(\lambda\).
\[
\lambda = (1 + o(1))\sqrt{4n}. \tag{13}
\]

Recall that \(\lambda \leq z\). Hence we have \(z = (1 + o(1))\lambda = (1 + o(1))\sqrt{4n}\). Consequently we obtain from (9) that \(\lambda R = O(n)\), which implies that \(\left(\sqrt{d + \lambda R} + \sqrt{d}\right)^2 = O(n)\). Now it follows from (11) that
\[
\lambda R = O\left(\sqrt{n\lambda z + n\sqrt{dz}}\right) = O\left(\sqrt{n\lambda^2 + n^{3/2}\lambda^{1/2}}\right) = O(n^{5/6}).
\]
Substitute \(\lambda R\) into (5) and use the fact that \(\lambda^3 \geq 4n - 16\), we obtain that
\[
4n - 16 \leq 4d + O(n^{5/6}),
\]
which implies that \(d \geq n - O(n^{5/6})\). This completes the proof of Claim 1.

In order to further improve the lower bound of \(d\) (as claimed in Lemma 2), we need to give a non-trivial upper bound on \(\max_{v \neq v_0} x_v^2\). Let \(u_0\) be a vertex attaining the second largest Perron-Frobenius eigenvector entry of the adjacency tensor of \(H\). We claim \(x_{u_0} = O(n^{-1/3})\).

Let \(d' = d_G(u_0)\) and \(\{u_1, u_2, \ldots, u_{d'}\}\) be the neighbors of \(u_0\) in \(G\). Moreover, let \(\Delta' = \max_{w \neq v_0} d_G(w)\). Note that since \(d_G(v_0) \geq n - O(n^{5/6})\), it follows that \(d' \leq \Delta' = O(n^{5/6})\). Otherwise by pigeonhole principle \(G\) has a \(K_{2,3}\), which contradicts that \(G\) is outerplanar.
Most of the inequalities shown in Claim 1 hold in similar forms. In particular, by the eigenequation expansion on $x_{u_0}$, we have

$$\lambda x_{u_0}^2 = \sum_{i=1}^{d'-1} x_{u_i} x_{u_{i+1}} \leq 2x_{v_0} x_{u_0} + \sum_{u \in N_G(u_0), u \neq v_0} x_u^2.$$ \hspace{1cm}

Let $z' = \sum_{u \in N_G(u_0), u \neq v_0} x_u^2$. Similar to (3), if we apply the eigenequations on $z'$, we have

$$\lambda z' \leq 2x_{u_0} + 2x_{u_0} \sqrt{d'z'} + R',$$

where

$$R' = \sum_{u \in N_G(u_0) \setminus \{v_0\}} \sum_{v, w \neq u_0} x_v x_w \leq \sum_{u \in N_G(u_0) \setminus \{v_0\}} \sum_{v, w \neq u_0} \frac{x_v^2 + x_w^2}{2}.$$

It follows from the same logic in (4) that

$$z' \leq \left( \frac{\sqrt{d'x_{u_0}}}{\lambda} + \frac{\sqrt{d'x_{u_0}^2}}{\lambda^2} + \frac{R' + 2x_{u_0}}{\lambda} \right)^2 \leq \frac{4d'x_{u_0}^2}{\lambda^2} + \frac{2(R' + 2x_{u_0})}{\lambda} - \left( \frac{\sqrt{d'x_{u_0}^2}}{\lambda^2} + \frac{R' + 2x_{u_0}}{\lambda} - \frac{\sqrt{d'x_{u_0}}}{\lambda} \right)^2.$$

Then it follows that

$$\lambda^2(z' + 2x_{u_0}) \leq 4d'x_{u_0}^2 + 2\lambda(R' + 2x_{u_0}) - \left( \frac{\sqrt{d'x_{u_0}^2}}{\lambda^2} + \frac{R' + 2x_{u_0}}{\lambda} - \frac{\sqrt{d'x_{u_0}}}{\lambda} \right)^2 + 2\lambda^2 x_{u_0}$$

$$\leq 4d'x_{u_0}^2 + 2\lambda R' + (2\lambda^2 + 4\lambda)x_{u_0}.$$ \hspace{1cm}

Hence we have

$$(4n - 16)x_{u_0}^2 \leq \lambda^2 x_{u_0}^2 \leq \lambda^2(z' + 2x_{u_0}) \leq 4d'x_{u_0}^2 + 2\lambda R' + (2\lambda^2 + 4\lambda)x_{u_0}. \hspace{1cm} (14)$$

We will use an inequality similar to (9) to bound $2\lambda R'$. Let $E(R') = \{vw \in \Gamma(u) : v, w \neq u_0, u \in N_G(u_0) \setminus \{v_0\}\}$. Similar to before, we have

$$2\lambda R' \leq \sum_{vw \in E(R')} \lambda x_v^2 + \lambda x_w^2 \leq 2\lambda \left( x_{v_0}^2 + \sum_{u \in N_G(u_0), u \neq v_0} x_u^2 \right) + 4 \sum_{i=1}^{d-1} x_{u_i} x_{u_{i+1}} + \sum_{h \in E(R')} \sum_{w \in h} \sum_{pq \in \Gamma(w)} x_p x_q \leq 2\lambda (z' + 1) + 4\lambda x_{u_0}^2 + \sum_{h \in E(R')} \sum_{w \in h} \sum_{pq \in \Gamma(w)} x_p x_q. \hspace{1cm} (15)$$

We bound $x_p x_q$ by $x_{u_0}^2$ if neither $p$ nor $q$ is equal to $v_0$; else by $x_{u_0}$. So again it’s important to bound the multiplicities of the terms $x_p x_q$ in the summation above. For convenience, let $E''$ be the collection of edges $pq \in E(G)$ with $x_p x_q$ appearing in the summation above.
It’s easy to see from Figure 4 that due to outerplanarity of G the multiplicity of each \( pq \in E'' \) is at most 6. Thus by eigenequation on \( v_0 \), we can bound the sum of all \( x_p x_q \) (including multiplicities) for which \( p, q \) form a hyperedge together with \( v_0 \):

\[
\sum_{h \in E(R')} \sum_{w \in h \cap N(u)} \sum_{pq \in \Gamma(w) \cap \Gamma(v_0)} x_p x_q \leq 6\lambda v_0^2 = 6\lambda.
\]

Now we claim that there at most \( O(1) \) terms \( x_p x_q \) (including multiplicities) containing \( x_{v_0} \) in the triple summations in (15). Indeed, if \( v_0 \notin N(u) \), then observe that \( vw \notin \Gamma(v_0) \) for all edges \( vw \in E(R') \). In this case due to the outerplanarity of \( G \) there is no edge \( pq \in E'' \) for which the term \( x_p x_q \) contains \( x_{v_0} \). Otherwise, \( v_0 \notin N(u) \). It follows from the outerplanarity of \( G \) again that there are at most two vertices \( w \) such as \( w \in h \) for some \( h \in E(R') \) and \( v_0 \in \Gamma(w) \). Moreover, each \( w \) is incident to at most 4 edges in \( E(R') \). Hence there are at most 8 terms \( x_p x_q \) (including multiplicities) containing \( x_{v_0} \) in the sums in (15). We bound each such term \( x_p x_q \) by \( x_{u_0} x_{v_0} = x_{u_0} \).

As a result, there are at most \( (E(G) - (2d(v_0) - 1) + O(1)) = O(n^{5/6}) \) edges in \( E'' \) that is not incident to \( v_0 \) and not in \( \Gamma(v_0) \). For such edges \( pq \), we bound \( x_p x_q \) by \( x_{u_0}^2 \). It follows from (15) that

\[
2\lambda R' \leq 2\lambda (z' + 1) + 6\lambda + O(x_{u_0}) + O(n^{5/6}) x_{u_0}^2
\]

\[
\leq 2\lambda z' + 8\lambda + O(x_{u_0}) + O(n^{5/6}) x_{u_0}^2,
\]

(16)

Substituting (16) into (14), we have

\[
(4n - 16)x_{u_0}^2 \leq \lambda^2(z' + 2x_{u_0}) \leq 4d^2 x_{u_0}^2 + 2\lambda R' + (2\lambda^2 + 4\lambda)x_{u_0}
\]

\[
\leq 4d^2 x_{u_0}^2 + \left(2\lambda z' + 8\lambda + O(x_{u_0}) + O(n^{5/6}) x_{u_0}^2 \right) + (2\lambda^2 + 4\lambda)x_{u_0}
\]

\[
\leq 2\lambda z' + O(n^{5/6}) x_{u_0}^2 + 8\lambda + (2\lambda^2 + 4\lambda + O(1)) x_{u_0}.
\]

(17)

Rearranging the inequality in (17), we first obtain an upper bound on \( z' \):

\[
z' \leq \frac{O(n^{5/6}) x_{u_0}^2 + (4\lambda + O(1)) x_{u_0} + 8\lambda}{\lambda^2 - 2\lambda} = O\left( n^{1/6} x_{u_0}^2 + \frac{4x_{u_0}}{\lambda} + \frac{1}{\lambda} \right).
\]

Now using the upper bound on \( z' \) and (17), we have the following inequality:

\[
(4n - 16)x_{u_0}^2 \leq 2\lambda z' + O(n^{5/6}) x_{u_0}^2 + 8\lambda + (2\lambda^2 + 4\lambda + O(1)) x_{u_0}
\]

\[
= O\left( n^{5/6} x_{u_0}^2 + \lambda^2 x_{u_0} + \lambda \right).
\]

It follows from the fact that \( \lambda = O(n^{1/3}) \) that

\[
x_{u_0} = O(n^{-1/3}).
\]

Now use the bound \( x_{u_0} = O(n^{-1/3}) \) in (9), we obtain a better bound of \( d = d_G(v_0) \) in Claim 1:

\[
4n - 16 \leq \lambda^2 \leq 4d + 2\lambda z + 4\lambda + 4\sqrt{dz} + \left( 4(n - d) \cdot O((n^{-1/3})^2) \right),
\]

(18)

which gives us

\[
d \geq n - O(n^{2/3}).
\]

This completes the proof of Lemma 1.
Lemma 3. $d_{\mathcal{H}}(v_1) = 1$. Moreover, $x_{v_2} \geq x_{v_1}$.

Proof. Assume for the sake of contradiction that $d_{\mathcal{H}}(v_1) \geq 2$. Recall that $\mathcal{H}$ is edge-maximal. It follows that there must exist some vertex $t \neq v_0$ such that $\{v_1, v_2, t\}$ is a hyperedge. Because $G = \partial(\mathcal{H})$ is maximally outerplanar, there is a unique component $S$ of $G - \{v_1, v_2\}$ containing $t$ and not $v_0$. Let $S'$ be the subgraph of $G$ consisting of $S$ and all edges joining $S$ to $\{v_1, v_2\}$. Let $S''$ be $S'$ but with $v_2$ renamed as $v_0$. Then $G' = (G - V(S)) \cup S''$ is outerplanar (we can attach $S''$ to $G - V(S)$ at $v_0$ and $v_1$ after flipping it over, i.e., reflecting it), and $G'$ is the shadow of a 3-uniform hypergraph $\mathcal{H}'$ that can be obtained from $\mathcal{H}$ by replacing each hyperedge $\{v_2, u, w\}$ where $u, w \in V(S')$ by a hyperedge $\{v_0, u, w\}$. Suppose $x$ is the Perron-Frobenius eigenvector of $\mathcal{H}$. Then it follows that

$$\sum_{\{i_1, i_2, i_3\} \in E(\mathcal{H}')} x_{i_1}x_{i_2}x_{i_3} - \sum_{\{i_1, i_2, i_3\} \in E(\mathcal{H})} x_{i_1}x_{i_2}x_{i_3} \geq x_{v_1}x_t(x_{v_0} - x_{v_2}) > 0.$$  

This implies that $\lambda(\mathcal{H}') > \lambda(\mathcal{H})$, which contradicts that $\mathcal{H}$ attains the maximum spectral radius.

It remains to show that $x_{v_2} \geq x_{v_1}$. If $x_{v_2} < x_{v_1}$, then let $x'$ be obtained from $x$ by setting $x_{v_1}' = x_{v_2}$, $x_{v_2}' = x_{v_1}$ and every other entry the same. Since $d_{\mathcal{H}}(v_1) = 1$, it follows that $P_\mathcal{H}(x') > P_\mathcal{H}(x)$, which contradicts that $x$ is the Perron-Frobenius eigenvector of $\mathcal{H}$.

Now we are ready to show Theorem 1.

Proof of Theorem 1. Let $\mathcal{H}$ be an outerplanar 3-uniform hypergraph on $n$ vertices with maximum spectral radius. Let $G$ be the shadow of $\mathcal{H}$. Suppose the Perron–Frobenius eigenvector $x$ of the adjacency tensor of $\mathcal{H}$ is normalized so that the maximum eigenvector entry is 1. Let $v_0$ be the vertex with the maximum eigenvector entry and $\{v_1, v_2, \cdots, v_d\}$ be the neighbors of $v_0$ in the clockwise order of the planar drawing of $G$.

By Lemma 1, we have that $d(v_0) \geq n - O(n^{2/3})$ and for every other vertex $u \neq v_0$, $x_u = O(n^{-1/3})$. Now we claim that $x_{v_1} = \Omega(n^{-1/3})$. By Lemma 3, we have that $d_{\mathcal{H}}(v_1) = 1$, i.e., $v_1v_2v_0$ is the unique hyperedge containing $v_1$. It follows by Lemma 3 and the eigenequation on $v_1$ that

$$\lambda x_{v_1}^2 = x_{v_0}x_{v_2} = x_{v_2} \geq x_{v_1}.$$  

Together with (12), this implies that

$$x_{v_1} \geq \frac{1}{\lambda} = \Omega(n^{-1/3}).$$  

Now we claim that for every vertex $u \in V(G) \setminus \{v_0\}$, $u$ is a neighbor of $v_0$ in $G$. Suppose not, then it follows from the outerplanarity of $G$ that there exists some vertex $w$ not adjacent to $v_0$ such that $w$ is contained in a unique hyperedge $\{w, s, t\}$ ($s, t \neq v_0$). Now similar to Lemma 3, consider the hypergraph $\mathcal{H}'$ obtained from $\mathcal{H}$ by removing the hyperedge $\{w, s, t\}$ and adding the hyperedge $\{w, v_0, v_1\}$. It follows that

$$\sum_{\{i_1, i_2, i_3\} \in E(\mathcal{H}')} x_{i_1}x_{i_2}x_{i_3} - \sum_{\{i_1, i_2, i_3\} \in E(\mathcal{H})} x_{i_1}x_{i_2}x_{i_3} \geq x_wx_{v_0}x_{v_1} - x_{w}x_{s}x_{t}.$$  

Note that $x_wx_t = O(n^{-2/3})$ while $x_{v_0}x_{v_1} = \Omega(n^{-1/3})$. It follows that $x_wx_{v_0}x_{v_1} > x_{w}x_{s}x_{t}$, which implies that $\lambda(\mathcal{H}') > \lambda(\mathcal{H})$, contradicting that $\mathcal{H}$ is the extremal hypergraph of maximum spectral radius. Hence by contradiction, every vertex $u \in V(G) \setminus \{v_0\}$ is a neighbor of $v_0$ in $G$. Again by the fact that $\mathcal{H}$ attains the maximum spectral radius, it follows that $\mathcal{H}$ is the unique 3-uniform hypergraph with $K_1 + P_{n-1}$ as it shadow.
References


