Separating cycles in doubly toroidal embeddings

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Abstract

We show that every 4-representative graph embedding in the double torus contains a noncontractible cycle which separates the surface into two pieces.

1. Introduction

If a graph is embedded in a surface of genus (orientable or nonorientable) at least 2, then it may have a *noncontractible separating cycle* (*NSC*), a cycle in the graph which is noncontractible and separates the surface into two pieces. Sufficient conditions for the existence of an NSC are of interest because they may provide a way to prove results about graph embeddings by induction on genus.

Several conditions of this kind have been proved or proposed. Many of them involve the *representativity*, or *facewidth*, of a graph embedding, which is the smallest number of points in which any noncontractible closed curve in the surface intersects the graph. The representativity of the embedding Ψ is denoted $\rho(\Psi)$, and Ψ is *k-representative* if $\rho(\Psi) \geq k$.

In what follows, a 'suitable' surface is one of genus (orientable or nonorientable) at least 2. Barnette, at a meeting in Tacoma, Wash. in the mid-1980's, conjectured that every triangulation of a suitable orientable surface has an NSC. Triangulations are a subset of the 3-representative embeddings. In a workshop in Vermont, one of us (Zha) [**7**] conjectured more generally that every 3-representative embedding in a suitable (orientable or nonorientable) surface has an NSC. The representativity condition here would be best possible, as Zha and Zhao [**8**] have given examples

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of embeddings with representativity 2 and no NSC. Robertson and Thomas [**5**] proved that every 3-representative embedding in the Klein bottle has an NSC. Richter and Vitray [**4**] proved that every 11-representative embedding in any suitable surface has an NSC. Zha and Zhao [**8**] reduced the needed representativity condition to 6-representative for orientable surfaces and 5-representative for nonorientable surfaces. Brunet, Mohar, and Richter [**1**] proved that a graph embedding of representativity w in a suitable orientable surface contains $\lfloor (w-9)/8 \rfloor$ disjoint and pairwise homotopic NSCs. Thomassen (see [**3**]) conjectured that given a triangulation of a surface of genus $g \geq 2$ and a number h, $1 \leq h \leq g-1$, there must be an NSC Γ such that the two surfaces separated by Γ have genus h and $g-h$, respectively. Mohar [**3**] conjectured that the same is true for any 3-representative embedding.

In this paper we tackle the simplest suitable orientable surface, the double torus. We show that 4-representative graph embeddings in the double torus have an NSC. This improves on the best previous condition ($\rho \geq 6$) but does not achieve the goal of Zha's conjecture ($\rho \geq 3$).

We believe that an argument similar to the one in this paper can be used to verify Barnette's conjecture (on triangulations) in the case of the double torus. However, the argument would be long and tedious, involving examination of many cases. It might even be possible to verify Zha's conjecture (for $\rho \geq 3$) in the case of the double torus with this approach, but in practice any such proof is likely to be unmanageably lengthy.

2. Punctured tori

In this section we introduce our basic definitions and then prove some properties of punctured tori that we need later.

A *circle* in a surface is a simple closed curve; an *arc* is a simple non-closed curve, including its endpoints. A single point is not considered to be an arc. If a graph is embedded in a surface, each cycle of the graph is embedded as a circle, and each nontrivial (not a single vertex) path as an arc. If a, b are sections of an oriented arc or oriented circle Q, then aQb denotes the part of Q from the last point of a to the first point of b, inclusive. Q^{-1} denotes Q traversed in the opposite direction. A *section* of an arc or circle on a surface is a subarc or single point contained in the arc or circle. A *segment* of a path or cycle in a graph is a subpath, which may consist of just a single vertex. The number of components of a set S in a surface Σ is denoted $\|S\|$.

For convenience, all topological sets we deal with are considered closed unless we explicitly indicate otherwise. Arcs include their endpoints, and disks, faces, and surfaces with boundary include their boundaries. When a surface Σ is separated into two parts by a circle Γ, each of the parts is assumed to contain Γ.

T denotes the torus S_1. We assume we are using a fixed orientation of T. Any contractible circle in T then has a natural clockwise orientation. The following facts are well known.

Lemma 2.1.

(i) *Two disjoint noncontractible circles in the torus are homotopic, and together they separate the torus into two cylinders.*

(ii) *Two circles in the torus are disjoint under homotopy if and only if they are homotopic.* ∎

Suppose Σ_0 is a surface with one boundary circle, Γ. Let Σ be the surface (without boundary) obtained by pasting a disk D along Γ. Suppose P is an arc that joins two distinct points of Γ in Σ_0, with $P^\circ \cap \Gamma = \emptyset$. The endpoints of P divide Γ into two subarcs Γ_1, Γ_2, and the two circles $P \cup \Gamma_1$, $P \cup \Gamma_2$ are homotopic in Σ. If these circles are noncontractible then P will be called an *essential* arc (or path, if appropriate).

Now assume that $\Sigma_0 = T_0$ is a punctured torus and $\Sigma = T$ the torus. Suppose P and P' are disjoint essential arcs (so their four endpoints are all distinct). We say P and P' are *parallel* if the endpoints of P are not separated on Γ by the endpoints of P'. In this case we may label the four endpoints in order along Γ as x, y, x', y' with P from x to y, P' from x' to y'. By Lemma 2.1 (1), the disjoint homotopic circles $P \cup x\Gamma y$ and $P' \cup x'\Gamma y'$ separate T into disjoint cylinders C, C'. Let C' be the cylinder containing D°, then C' is further separated by $y\Gamma x' \cup y'\Gamma x$ into D° and a disk S bounded by $P \cup y\Gamma x' \cup P' \cup y'\Gamma x$: we call S a *strip* with *ends* $y\Gamma x'$ and $y'\Gamma x$. Thus, $\Gamma \cup P \cup P'$ separates the torus T into C, S, and D, and $P \cup P'$ separates the punctured torus $T_0 = T \backslash D^\circ$ into C and S. Call $\{C, S\}$ the *cylinder-strip partition* of T_0 induced by P and P', denoted $CS(P, P')$. This is illustrated on the left of Figure 2.1. In T_0 it is easy to recognize the cylinder and the strip because $\Gamma \cup P \cup P'$ provides two disjoint boundary circles for C and one boundary circle for S.

The following two obvious properties of $CS(P, P')$ will be used frequently, and we refer to them as $CS(P, P')$(i) and $CS(P, P')$(ii), respectively.

Lemma 2.2 [Cylinder-Strip]. *Given a punctured torus T_0, let P and P' be parallel disjoint essential arcs. Let P'' be an essential arc disjoint from P and P'. Then*

(i) *Both ends of P'' must lie in the cylinder, or both ends must lie in the strip, of $CS(P, P')$.*

(ii) *If both ends of P'' lie in the strip, then they lie at opposite ends.* ∎

When dealing with essential arcs in a given punctured torus T_0, with boundary Γ, we have found it useful to represent Γ as a circle and the essential arcs as chords of the circle. This is

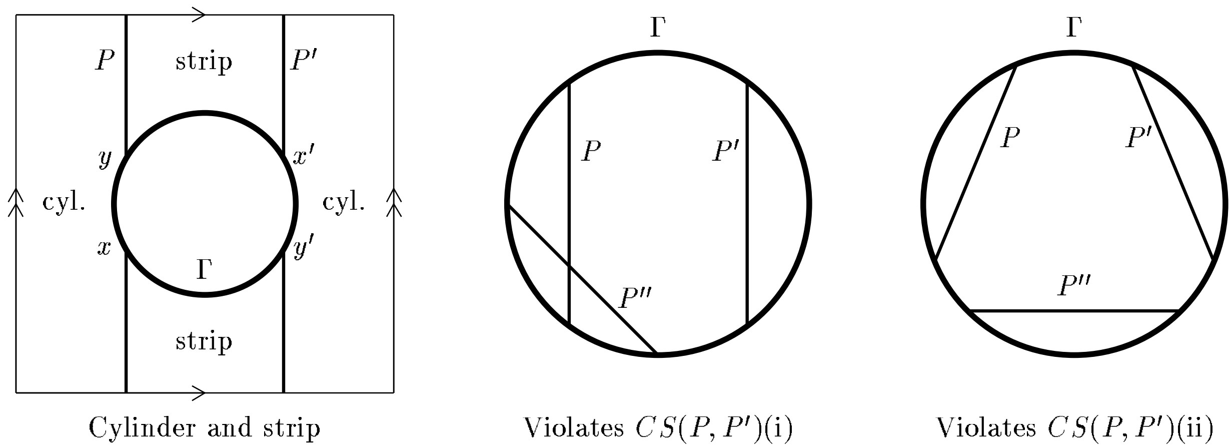

Figure 2.1

illustrated by the centre and right parts of Figure 2.1, which show essential arcs P, P', P'' that violate the Cylinder-Strip Lemma.

Suppose Γ is the single boundary circle of a (closed) surface Σ_0. Let D be a closed disk in Σ_0 and suppose that $\Gamma \cap D = \Gamma \cap \partial D$ consists of a finite number of components. We say that Γ and D intersect *essentially* if every arc in D joining two distinct components of $\Gamma \cap D$ is essential.

Lemma 2.3. *Suppose Γ is the boundary circle of a punctured torus and Γ intersects a disk D essentially. Let $L = \partial D$, oriented clockwise. Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ be the components of $\Gamma \cap D = \Gamma \cap L$ in clockwise order around L, where $\Gamma_i = x_i L y_i$ for each i.*

(i) *If $k = 2$ then (y_1, x_1, y_2, x_2) occur in that clockwise order along Γ;*

(ii) *If $k = 3$ then $(y_1, x_1, y_2, x_2, y_3, x_3)$ occur in that clockwise order along Γ;*

(iii) *If $k = 4$ then $(y_1, x_1, y_2, x_2, y_3, x_3, y_4, x_4)$ occur in that clockwise order along Γ.*

(iv) $k \leq 4$.

Proof. By expanding D slightly if necessary, we may assume that $x_i \neq y_i$ for all i. If we add a disk along Γ, L and Γ are both contractible and have natural clockwise orientations, which must oppose each other where they meet. Thus, y_i is followed on Γ by x_i, and x_i must be followed by some y_j; it cannot be followed by x_j.

(i) When $k = 2$ the given order is the only possible one.

(ii) Suppose $k = 3$. There are only two possible clockwise orders along Γ. If the order is $(y_1, x_1, y_3, x_3, y_2, x_2)$ then the essential arc $y_3 L x_1$ has both ends at the same end of the strip of $CS(y_1 L x_2, y_2 L x_3)$, contradicting (ii) of the Cylinder-Strip Lemma.

(iii) By shifting D slightly we may apply (ii) separately to both collections $\Gamma_1, \Gamma_2, \Gamma_3$ and $\Gamma_1, \Gamma_3, \Gamma_4$, giving the required order.

(iv) If $k \geq 5$, then by shifting D slightly we may assume that $k = 5$. By similar reasoning to (iii), the clockwise order must be $(y_1, x_1, y_2, x_2, y_3, x_3, y_4, x_4, y_5, x_5)$. Now the essential arc $y_4 L x_5$ has one end in the cylinder and the other end in the strip of $CS(y_5 L x_1, y_2 L x_3)$, contradicting (i) of the Cylinder-Strip Lemma. ∎

Now we consider the double torus S_2. Note that any contractible circle in S_2 has a natural clockwise orientation, but noncontractible circles must be given an orientation.

Suppose we have an oriented noncontractible separating circle Γ of the double torus S_2. It separates S_2 into two punctured tori A_0, B_0 (which we take to be closed, each including Γ). When convenient we complete A_0 with a disk A^* to a torus A, and B_0 with a disk B^* to a torus B. If A, B both inherit the orientation of S_2, Γ will be clockwise in one, which we assume to be A, and anticlockwise in the other, B. In other words, Γ goes clockwise around A^*, so A^* is to the right of Γ in A, and A_0 is to the left of Γ in both A and S_2. Similarly, B_0 is to the right of Γ.

We discuss some of the ways Γ can pass through a given closed disk D. Let $L = \partial D$. Suppose $\Gamma \cap D$ has finitely many components, including (but not limited to) $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ with the following properties:

(1) Each $\Gamma_i \cap L$ has at most two components, $1 \leq i \leq 4$.

(2) There is an arc P in D with ends on L and $P^\circ \subset D^\circ$, such that $P \cap \Gamma = \{x_1, x_2, x_3, x_4\}$, where x_1, x_2, x_3, x_4 are in that order along P, each x_i belongs to Γ_i, and Γ_2 and Γ_3 cross (not just intersect) P at x_2 and x_3, respectively.

Assume that $x_1 P x_2 \subset A_0$. For each i, let ia denote the first component of $\Gamma_i \cap L$ (following Γ_i along Γ) and ib the last. Then by the fact that Γ is separating, and using the orientations of A_0 and B_0, the components $1a, 2b, 3a, 4b, 4a, 3b, 2a, 1b$ occur in that clockwise order along L (so $2a \neq 2b$, $3a \neq 3b$, but possibly $1a = 1b$ or $4a = 4b$).

There are six possible cyclic orders in which the components $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ (or $1, 2, 3, 4$ for short) can occur along Γ; they occur in pairs which are equivalent up to reversal of Γ. If we know that Γ intersects the (closures of the) components of $D \backslash \Gamma$ essentially, then for some of these orders we can place restrictions on where additional components of $\Gamma \cap D$ can be.

Lemma 2.4. *Suppose Γ is a noncontractible separating circle of S_2 with a disk D and components $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ of $\Gamma \cap D$ as described above. Suppose further that Γ intersects the closure of every component of $D \backslash \Gamma$ essentially (in A_0 or B_0, as appropriate).*

(i) *If the components occur along* Γ *in the order* (1432) *then* $\Gamma \cap (2aL1b)^\circ = \Gamma \cap (4aL3b)^\circ = \emptyset$.

(ii) *If the components occur along* Γ *in the order* (1342) *then* $\Gamma \cap (2aL1b)^\circ = \Gamma \cap (3aL4b)^\circ = \emptyset$.

Proof. Suppose in case (i) that $\Gamma \cap (2aL1b)^\circ \neq \emptyset$. Let Γ_5 (or 5 for short) be the component of $\Gamma \cap (2aL1b)^\circ$ closest to $1b$ on L. By Lemma 2.3 (2), $5 \subset (2b\Gamma 1a)^\circ$. But then $5L1b$ violates $CS(1aL2b, 3aL4b)$(i). (Note: we assume that essential arcs can be shifted slightly if necessary so that we can apply the Cylinder-Strip Lemma. Here this is necessary if $\Gamma_1 = 1a = 1b$ is a single point.) Thus, $\Gamma \cap (2aL1b)^\circ = \emptyset$. The rest of the proof is similar. ∎

3. Construction of noncontractible separating circles

In this section we describe two methods for constructing noncontractible separating circles. Our first method constructs a new noncontractible separating circle from an old one.

Theorem 3.1. *Let* Σ *be a surface with an oriented noncontractible separating circle* Γ *separating* Σ *into two (closed) components* A_0 *and* B_0*. Suppose there are sections* $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ *of* Γ *in that order along* Γ*, and arcs* P_{12}, P_{34} *in* A_0 *and* Q_{23}, Q_{41} *in* B_0 *such that*

(i) $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ *are disjoint;*

(ii) $P_{12}^\circ, P_{34}^\circ, Q_{23}^\circ, Q_{41}^\circ$ *are disjoint from each other and from* Γ*;*

(iii) P_{12} *has ends* a_1, a_2*,* P_{34} *has ends* a_3, a_4*,* Q_{23} *has ends* b_2, b_3 *and* Q_{41} *has ends* b_4, b_1*, where* a_i *and* b_i *are the two ends of each* Γ_i *(not necessarily in order along* Γ*);*

(iv) $P_{12} \cup P_{34}$ *separates* A_0 *into a component* A_1 *with boundary* $P_{12} \cup a_2\Gamma a_3 \cup P_{34} \cup a_4\Gamma a_1$ *(one circle) and a component* A_2 *with boundary* $(a_1\Gamma a_2 \cup P_{12}) \cup (a_3\Gamma a_4 \cup P_{34})$ *(two circles), while* $Q_{23} \cup Q_{41}$ *similarly separates* B_0 *into a component* B_1 *with boundary* $Q_{23} \cup b_3\Gamma b_4 \cup Q_{41} \cup b_1\Gamma b_2$ *(one circle) and a component* B_2 *with boundary* $(b_2\Gamma b_3 \cup Q_{23}) \cup (b_4\Gamma b_1 \cup Q_{41})$ *(two circles).*

Then

$$\Gamma' = \Gamma_1 \cup P_{12} \cup \Gamma_2 \cup Q_{23} \cup \Gamma_3 \cup P_{34} \cup \Gamma_4 \cup Q_{41}$$

is also a noncontractible separating circle in Σ*, separating* $A_1 \cup B_2$ *from* $A_2 \cup B_1$*.*

Proof. The conditions of the theorem clearly guarantee that Γ' separates $A_1 \cup B_2$ from $A_2 \cup B_1$. We must show that Γ' is noncontractible, or, equivalently, that neither $A_1 \cup B_2$ nor $A_2 \cup B_1$ is homeomorphic to a disk. Since A_1 has one boundary circle, it is homeomorphic to a disk with handles and/or crosscaps attached. Since B_2 has two boundary circles, it is homeomorphic to a cylinder with handles and/or crosscaps attached. If A_1 is just a disk and B_2 is just a cylinder, the way they are attached along the segments of Γ between Γ_2 and Γ_3 and between Γ_4 and Γ_1 means that the result would be homeomorphic to a punctured torus. More generally, the result

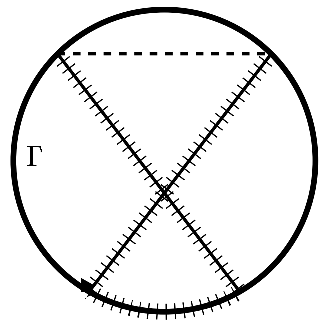

Figure 3.1: Equivalent to an essential arc

is homeomorphic to a punctured torus with handles and/or crosscaps added, which is not a disk. Similarly, $A_2 \cup B_1$ is not a disk. Therefore Γ is noncontractible, as required. ∎

We now apply this to the double torus, with weaker versions of conditions (ii) and (iv), and stating condition (iv) in a way specific to the double torus.

Corollary 3.2. *Suppose Γ is a noncontractible separating circle in the double torus S_2, separating S_2 into two (closed) punctured tori A_0 and B_0. Suppose there are sections $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ of Γ in that order along Γ, and arcs P_{12}, P_{34} in A_0 and Q_{23}, Q_{41} in B_0 such that*

(i) $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ *are disjoint;*

(ii) $P_{12}^\circ, P_{34}^\circ, Q_{23}^\circ, Q_{41}^\circ$ *are disjoint from each other and from* $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$;

(iii) P_{12} *has ends* a_1, a_2, P_{34} *has ends* a_3, a_4, Q_{23} *has ends* b_2, b_3 *and* Q_{41} *has ends* b_4, b_1, *where* a_i *and* b_i *are the two ends of each* Γ_i *(not necessarily in order along* Γ*);*

(iv) P_{12}, P_{34} *are homotopic with endpoints fixed in* A_0 *to a pair of parallel essential arcs, and* Q_{23}, Q_{41} *are homotopic with endpoints fixed in* B_0 *to a pair of parallel essential arcs.*

Then

$$\Gamma' = \Gamma_1 \cup P_{12} \cup \Gamma_2 \cup Q_{23} \cup \Gamma_3 \cup P_{34} \cup \Gamma_4 \cup Q_{41}$$

is also a noncontractible separating circle in S_2.

Proof. Conditions (ii) and (iv) mean that by shifting Γ slightly we can make P_{12}, P_{34} and Q_{23}, Q_{41} into pairs of parallel essential arcs. Then each of the (slightly shifted) punctured tori is separated into a cylinder and a strip. Now apply Theorem 3.1. ∎

A set of arcs $P_{12}, P_{34}, Q_{23}, Q_{41}$ satisfying Corollary 3.2 is called an *orthogonal arrangement of parallel arcs*, or OP for short; we refer to it as $OP(P_{12}, P_{34}; Q_{23}, Q_{41})$.

Note in particular that $P_{12}, P_{34}, Q_{23}, Q_{41}$ are *not* required to have interiors disjoint from Γ, just from the sections $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$. The most common (but not the only) case of this is illustrated in Figure 3.1. When forming an OP the two hatched essential arcs joined by a hatched segment of Γ may be considered equivalent to the single dashed essential arc, as long as the hatched segment of Γ does not intersect $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$.

4. Critical embeddings

Our approach to proving the existence of noncontractible separating cycles (NSCs) is to examine embeddings which are very close to having an NSC.

Lemma 4.1. *Let Σ be a suitable surface, and $k \geq 3$. Suppose there is a k-representative embedding in Σ that does not have an NSC. Then there exists a k-representative embedding Ψ of a simple 2-connected graph in Σ that does not have an NSC, with a face f containing nonadjacent vertices x, y, so that when the edge xy is inserted across the face f, $\Psi^+ = \Psi \cup xy$ has an NSC.*

We call Ψ a *critical embedding* with *critical edge xy*.

Proof. Let Ψ_0 be a k-representative embedding of a graph G_0 in Σ with no NSC. Since $k \geq 3$ and since multiple edges bounding a face can be reduced to a single edge without affecting the existence of an NSC, we may assume that G_0 is simple. Moreover, by reducing to the 'essential 2-component' (see [**6**, Section 7]) we may assume that G_0 is 2-connected.

Define an *augmentation* of an embedding to be either (1) the addition of an edge across a face between two vertices nonadjacent on that face, or (2) if every face is a triangle (bounded by a 3-cycle), then in some face (uvw) subdivide one edge vw with a new vertex x, then add the edge ux. Neither (1) nor (2) decreases the representativity. In a sequence of augmentations, any augmentation following one of type (2) must be of type (1).

If we apply a sequence of augmentations to Ψ_0, each embedding is k-representative with a graph that is simple and 2-connected. Moreover, by applying a sequence of augmentations to Ψ_0 we can increase its representativity arbitrarily. First we complete Ψ_0 to a triangulation using type (1) augmentations. It is well known that in a triangulation the representativity equals the length of a shortest noncontractible cycle. Given an edge $e = vw$ on a shortest noncontractible cycle, belonging to two triangles (uvw) and (twv), we can apply four augmentations of type (2), (1), (2), (1) with the effect of deleting vw, adding two new vertices x_1, x_2, and adding paths vx_1w, vx_2w, ux_1x_2t. This destroys all shortest noncontractible cycles through e without creating any new shortest noncontractible cycles. After destroying all shortest noncontractible cycles in this way the representativity must increase by at least one; then we can repeat the process.

Therefore, it is possible to apply a sequence of augmentations to Ψ_0 to raise the representativity to at least 6, at which point an NSC exists (see Section 1). Let Ψ be the embedding in the sequence before the first augmentation that creates an NSC. Type (2) augmentations cannot create an NSC if one does not already exist. So, that augmentation is of type (1), and the result follows. ∎

5. Main theorem

In this section, we show that every embedding in the double torus with representativity at least 4 contains an NSC. We begin with a standard result on 4-representative graphs.

Lemma 5.1. *Let f be a face of Ψ, and let F be the union of f and all faces that share at least one vertex with f.*

(i) *The face f is a disk D_1 with boundary cycle L_1, and there is a disk $D_2 \supset D_1$ with boundary cycle L_2, such that $F \subseteq D_2$ and $L_2 = \partial D_2 \subseteq \partial F$.*

(ii) *Any path P in D_2 with both ends on L_2 must be a segment of L_2 or must contain a vertex of L_1.*

Proof. (i) is a special case, for representativity at least 4, of a standard result. The facts that $F \subseteq D_2$ and $L_2 \subseteq \partial F$ are not part of the usual statement, but follow from the standard proof. See [**2**, Prop. 5] or [**3**, Prop. 3.7].

(ii) If (ii) fails there would be a path P in D_2 internally disjoint from L_2 joining two vertices of L_2 and not intersecting L_1. Labelling the ends a, b of P appropriately, $P \cup bL_2a$ would separate $(aL_2b)^\circ$ from L_1. But this contradicts the fact that since $L_2 \subseteq \partial F$, every point of L_2 has an arc joining it to L_1 that does not intersect the graph except at its endpoints. ∎

Now we state and prove our main result.

Theorem 5.2. *Every 4-representative embedding on the double torus contains a noncontractible separating cycle.*

Proof. Suppose the theorem is false. By Theorem 4 there is a critical 4-representative embedding Ψ (of a simple 2-connected graph) with no NSC, while $\Psi^+ = \Psi \cup xy$ has an NSC. Suppose that xy is added across the face f. Let D_1, D_2, L_1, L_2 be as provided by Lemma 5.1 for f.

Every NSC in Ψ^+ must contain the edge xy. Of all NSCs in Ψ^+, let Γ be one that minimizes $\|\Gamma \cap D_2\|$, and, subject to this, also minimizes $\|\Gamma \cap D_1\|$. Then each component of $\Gamma \cap D_2$ contains

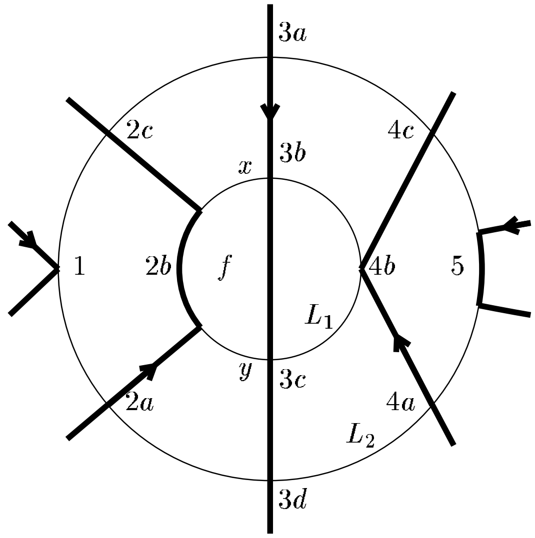

Figure 5.1: Some parts of Γ in D_2

at most one component of $\Gamma \cap D_1$, and (using Lemma 5.1(ii)) at most two components of $\Gamma \cap L_2$. The components of $\Gamma \cup D_2$ will be denoted Γ_i, for $i = 1, 2, \ldots$. We often abbreviate Γ_i to i. (In later parts of the proof we also use components Γ'_i, abbreviated i', where $i = 1, 2, \ldots$.) We represent subsegments of component i as ij where j is a letter, e.g. $3b$ is a subsegment of $3 = \Gamma_3$.

The minimality assumption further guarantees us that any arc in D_2 that joins different components Γ_i, Γ_j of $\Gamma \cap D_2$ and is otherwise disjoint from Γ is essential. If it were not essential, then we could replace one of the segments $i\Gamma j$ or $j\Gamma i$ with a segment of L_2, reducing $\|\Gamma \cap D_2\|$. This is true even if the segment of L_2 we wish to use intersects other components of $\Gamma \cap D_2$, because those other components must also be part of the segment of Γ we are replacing.

Let Γ_3 be the component of $\Gamma \cap D_2$ that contains xy. Then $\Gamma_3 \cup L$ has four components which we name $3a, 3b, 3c, 3d$ in order on Γ, with $3a, 3d \subset L_2$ and $3b, 3c \subset L_1$. We may assume that $x \in 3b$ and $y \in 3c$. For ease of description, we assume D_2 is drawn as a circular disk and Γ_3 passes downwards through D_2, with $3a$ containing its top point and $3d$ containing its bottom point. Other than Γ_3, no component of $\Gamma \cap D_2$ contains more than one component of $\Gamma \cap L_1$.

In fact, any other component Γ_i of $\Gamma \cap D_2$ is of one of two types. If $\|\Gamma_i \cap L\| = 1$, i is a segment of L_2. Otherwise, $\|\Gamma_i \cap L\| = 3$ and i includes two segments ia, ic of L_2 and one segment ib of L_1, with ia, ib, ic in that order along Γ.

Since Ψ is critical, Γ intersects both $(3bL_1 3c)^\circ$ and $(3cL_1 3b)^\circ$, otherwise we could reroute Γ to avoid (the interior of) $3b\Gamma 3c = xy$. Moreover, each component of $\Gamma \cap D_2$ which intersects $(3cL_1 3b)^\circ$

cannot be rerouted via $3dL_23a$, or we could reduce $\|\Gamma \cap D_1\|$. Let Γ_2 denote any such component, with $2a, 2c \subset L_2$ and $2b \subset L_1$. Note that Γ_2 passes upwards through D_2, so that the half of f to the right of Γ_3 is also to the right of Γ_2. Since Γ_2 cannot be rerouted there is at least one component of $\Gamma \cap D_2$ that intersects $(2aL_22c)^\circ$. Let Γ_1 denote any such component, which must be a segment of L_2 and pass downwards through D_2. In a similar way we can find Γ_4 passing upwards through D_2, intersecting $(3aL_23d)^\circ$ at $4a$ and $4c$ and intersecting $(3bL_13c)^\circ$ at $4b$. Then we must also have $\Gamma_5 = 5$ passing downwards through D_2 and contained in $(4cL_24a)^\circ$. The situation is illustrated in Figure 5.1. Note that in general we do not know whether a given component of $\Gamma \cap L$ is trivial or nontrivial.

Let A_0 denote the part of S_2 to the left of Γ, and B_0 the part to the right; A_0 and B_0 are punctured tori. For $i \geq 1$, let A_i denote the unique component of $A_0 \cap D_2$ to which Γ_i belongs; define B_i similarly. (If $\Gamma_i \subset L_2$, one of A_i or B_i will be just Γ_i itself.) We know that $A_1 = A_2$, $B_2 = B_3$, $A_3 = A_4$ and $B_4 = B_5$.

We will frequently use orthogonal arrangements of parallel paths to construct a new NSC Γ' in Ψ^+. In the notation $OP(P, P'; Q, Q')$, P, P' will be the paths in A_0 and Q, Q' those in B_0. There are two common ways in which this provides a contradiction. First, Γ' may avoid the edge xy, and so be an NSC for Ψ: we indicate this by $AOP(P, P'; Q, Q')$. Second, $\Gamma' \cap D_2$ may have fewer components than $\Gamma \cap D_2$: we indicate this by $COP(P, P'; Q, Q')$.

Suppose P, P', Q, Q' all lie in D_2. When we form Γ' from Γ we delete the interiors of four nontrivial segments of Γ, say S_1, S_2, S_3, S_4, and then add the interiors of P, P', Q, Q'. Each end of each S_j lies in some component of $\Gamma \cap D_2$. Suppose each S_j intersects s_j components of $\Gamma \cap D_2$, then $s_j \geq 1$. When we delete S_j°, the number of components in D_2 changes by $2 - s_j$. When we add $P^\circ, P'^\circ, Q^\circ, Q'^\circ$ the number of components in D_2 changes by -4. Thus, $\|\Gamma' \cap D_2\| = \|\Gamma \cap D_2\| + 4 - s_1 - s_2 - s_3 - s_4$. This analysis is valid even when the interiors of P, P', Q, Q' intersect components of $\Gamma \cap D_2$. If we do not have $s_1 = s_2 = s_3 = s_4 = 1$ then we have $COP(P, P'; Q, Q')$. In particular, let $OOP[i](P, P'; Q, Q')$ denote the situation in which some component i of $\Gamma \cap D_2$ contains an odd number of the eight endpoints of P, P', Q, Q' (counted with multiplicity). Then $s_j > 1$ for some j, so this is a special case of $COP(P, P'; Q, Q')$.

Now we break into cases according to the order of $1, 2, 3, 4, 5$ along Γ. For any distinct components $i_1, i_2, \ldots, i_k$ of $\Gamma \cap D_2$, we say that Γ *has* $(i_1 i_2 \ldots i_k)$ if $i_1, i_2, \ldots, i_k$ occur in that order along Γ.

(A) Suppose Γ has (1432). By Lemma 2.4, $\Gamma \cap (2aL_21)^\circ = \Gamma \cap (4aL_23d)^\circ = \emptyset$. Note that by Lemma

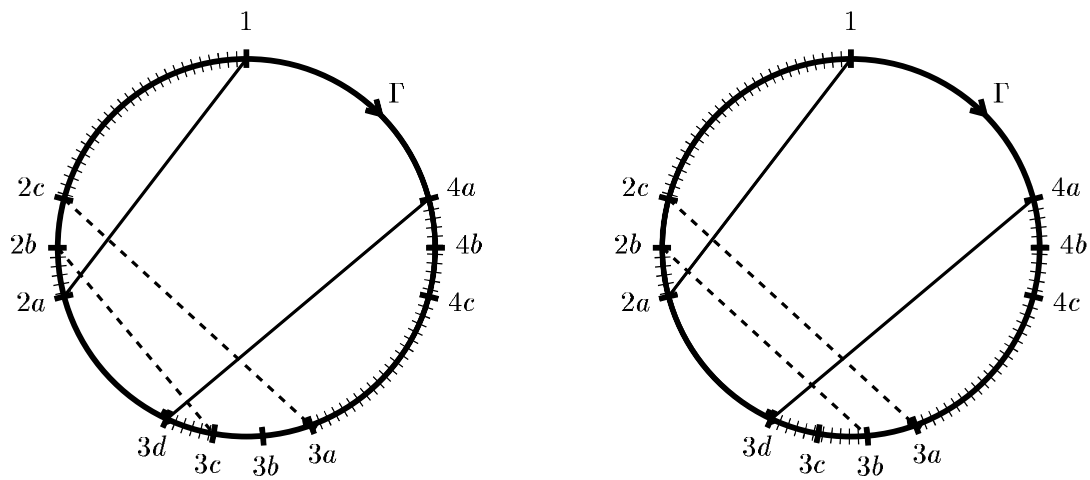

Figure 5.2: Some cases from (A)

2.3, $\Gamma \cap (2cL_2 3a)^\circ \subset (3\Gamma 2)^\circ$.

To help the reader begin following our arguments, Figure 5.2 shows how we construct new NSCs using Corollary 3.2 in the first two cases here. The solid chords are essential paths in A_0, the dashed chords are essential paths in B_0, and the edges of Γ used by the new NSC are hatched.

First, suppose that $\Gamma \cap (3cL_1 2b)^\circ = \emptyset$. Let $P = 2c(\partial B_3)3a$ (P is just $2cL_2 3a$ if $\Gamma \cap (2cL_2 3a)^\circ = \emptyset$). Since $\Gamma \cap (2cL_2 3a)^\circ$ is contained in $(3\Gamma 2)^\circ$, we have $P^\circ \cap \Gamma \subset (3\Gamma 2)^\circ$. Now we have $AOP(2aL_2 1, 4aL_2 3d; 3cL_1 2b, P)$. This is illustrated on the left of Figure 5.2. Note that $(3\Gamma 2)^\circ$ is not hatched, showing that this part of Γ may be used by P if necessary.

Second, suppose that $\Gamma \cap (2bL_1 3b)^\circ = \emptyset$. Since $\Gamma \cap (2cL_2 3a)^\circ \subset (3\Gamma 2)^\circ$, we have $OOP[1](2aL_2 1, 4aL_2 3d; 2cL_2 3a, 2bL_1 3b)$. This is illustrated on the right of Figure 5.2. While this appears very similar to the left of Figure 5.2, we obtain different types of contradiction in these two cases.

Finally, we may suppose that there exist $\Gamma_6 = 6$ that intersects $(2bL_1 3b)^\circ$ and $\Gamma_7 = 7$ that intersects $(3cL_1 2b)^\circ$. By Lemma 2.3, $2, 3, 6, 7$ are the only components of $\Gamma \cap D_2$ intersecting B_3, and Γ has (3627). If Γ has (271) then we have $OOP[1](2aL_2 1, 4aL_2 3d; 2bL_1 6b, 3cL_1 7b)$. If Γ has (173) then we have $OOP[1](2aL_2 1, 4aL_2 3d; 6bL_1 3b, 7bL_1 2b)$.

By symmetry we may also exclude the cases where Γ has (1234), (5234) or (5432).

(B) Suppose Γ has (1342). By Lemma 2.4, $\Gamma \cap (2aL_2 1)^\circ = \Gamma \cap (3aL_2 4c)^\circ = \emptyset$.

Suppose $\Gamma \cap (2cL_2 3a)^\circ = \emptyset$, then we have $OOP[1](2aL_2 1, 3aL_2 4c; 2cL_2 3a, 2bL_1 3b)$. Therefore, we may assume $\Gamma \cap (2cL_2 3a)^\circ \neq \emptyset$, and, similarly, $\Gamma \cap (3dL_2 2a)^\circ \neq \emptyset$. Let $\Gamma_6 = 6$ intersect $(2cL_2 3a)^\circ$, and $\Gamma_7 = 7$ intersect $(3dL_2 2a)^\circ$. By Lemma 2.3, $2, 3, 6, 7$ are the only components of

$\Gamma \cap D_2$ intersecting B_3 and Γ has (3627). Then we have $OOP[1](2aL_21, 3aL_24c; 2cL_26, 3dL_27)$.

By symmetry we may also exclude the cases where Γ has (1243), (5324) or (5423).

(C) Now we know that Γ must have either (1324) or (1423), and either (5243) or (5342). So the overall order must be (13524) or (14253). These cases are symmetric, so let us assume the order is (14253). Given this order, there is a symmetry that reverses Γ and swaps A_0 and B_0. For our standard picture of D_2, this amounts to rotating D_2 by 180° and reversing Γ.

Consider the components of $\Gamma \cap D_2$ that intersect $(2aL_22c)^\circ$. Each such component lies in $(3\Gamma 4)^\circ$, otherwise we could choose that component as 1 and have case (A) or (B). Let 1 be the first and $1'$ the last such component along $3\Gamma 4$ (possibly $1 = 1'$). By Lemma 2.3 there are at most three such components, and 1 is the first, and $1'$ the last, along $2aL_22c$. Thus, $\Gamma \cap (2aL_21)^\circ = \Gamma \cap (1'L_22c)^\circ = \emptyset$. If there are three distinct components $1, 1^*, 1'$ in order along Γ, then $3aL_24c$ violates $CS(1L_21^*, 1'L_22c)$(i) in A_0. Therefore, there are at most two such components. Similarly, at most two components of $\Gamma \cap D_2$ intersect $(4cL_24a)^\circ$, they lie in $(2\Gamma 3)^\circ$, and if 5 is the first and $5'$ the last along $2\Gamma 3$ (possibly $5 = 5'$), then $\Gamma \cap (4cL_25')^\circ = \Gamma \cap (5L_24a)^\circ = \emptyset$.

If $\Gamma \cap (2cL_23a)^\circ \neq \emptyset$, we denote the component of $\Gamma \cap D_2$ closest to $3a$ by 6, and that closest to $2c$ by $6'$ (possibly $6 = 6'$). If $\Gamma \cap (3dL_22a)^\circ \neq \emptyset$, we denote the component of $\Gamma \cap D_2$ closest to $2a$ by 7, and that closest to $3d$ by $7'$ (possibly $7 = 7'$). By Lemma 2.3, Γ has $(366'277')$, suitably modified to identify components that are the same and delete components that do not exist. Similarly, if $\Gamma \cap (3aL_24c)^\circ \neq \emptyset$, we denote the component of $\Gamma \cap D_2$ closest to $3a$ by 8, and that closest to $4c$ by $8'$ (possibly $8 = 8'$). If $\Gamma \cap (4aL_23d)^\circ \neq \emptyset$, we denote the component of $\Gamma \cap D_2$ closest to $4a$ by 9, and that closest to $3d$ by $9'$ (possibly $9 = 9'$). By Lemma 2.3, Γ has $(388'499')$, suitably modified. Note that $6, 6', 7, 7', 8, 8', 9, 9'$ may or may not intersect L_1.

Claim 1. At least one of 7 and 9 exists.

Proof. If not, we have the $OP(3bL_14b, 3aL_24c; 3cL_12b, 3dL_22a)$ which produces Γ' with $\|\Gamma' \cap D_2\| = \|\Gamma \cap D_2\|$ and $\|\Gamma' \cap D_1\| = \|\Gamma \cap D_1\| - 2$, contradicting the minimality of Γ. ∎

Claim 2. At most one of 6 and 7 exists. By symmetry, at most one of 8 and 9 exists.

Proof. Suppose both 6 and 7 exist. By Lemma 2.3 (4), $2, 3, 6, 7$ are the only components of $\Gamma \cap D_2$ intersecting B_3, and Γ has (3627). To avoid an arc (not necessarily path) from 4 to 5 in B_5 violating $CS(6L_23a, 7L_22a)$ (i) in B_0, Γ must have (275) when it has (364), and must have (573) when it has (462). So Γ has either (364275) or (346257).

(2.1) Suppose Γ has (364275). If 8 does not exist, let $P = 3aL_24c$ and $P' = 3bL_14b$; if 9 does not exist, let $P = 4aL_23d$ and $P' = 4bL_13c$. In either case we have $OOP[2](P, P'; 3dL_27, 2cL_26)$.

Therefore, 8 and 9 exist. By Lemma 2.3, $3, 4, 8, 9$ are the only components of $\Gamma \cap D_2$ intersecting A_3, and Γ has (3849).

To avoid an arc from 1 to 2 in A_1 violating $CS(3aL_28, 4aL_29)$ (i) in A_0, Γ must have (429) when it has (318), and must have (923) when it has (814). So Γ has either (318429) or (381492). If Γ has (318429) we have $OOP[2](8L_24c, 9L_23d; 6L_23a, 7L_22a)$. If Γ has (381492) we have $OOP[5]$ $(8L_24c, 9L_23d; 6L_23a, 5L_24a)$.

(2.2) Suppose Γ has (346257). If $\Gamma \cap (3aL_24c)^\circ = \emptyset$ let $P = 3aL_24c$ and $P' = 3bL_14b$; if $\Gamma \cap (4aL_23d)^\circ = \emptyset$ let $P = 4aL_23d$ and $P' = 4bL_13c$. In either case we have $OOP[5](P, P'; 3dL_27, 5L_24a)$. ∎

Claim 3. If 7 exists then $7 = 7'$ and Γ has (275). By symmetry, if 8 exists then $8 = 8'$ and Γ has $(1'84)$.

Proof. We first show that Γ does not have $(57'3)$. Suppose Γ has $(57'3)$. Since at most one of 8 and 9 exists by Claim 2, we may take paths P, P' to be either $3aL_24c, 3bL_14b$ or $4bL_13c, 4aL_23d$. Then we have $OOP[5](P, P'; 3dL_27', 5L_24a)$.

If $7 \neq 7'$ then to avoid an arc from 4 to 5 violating $CS(3dL_27', 7L_22a)$(i) in B_0, Γ must have $(2757'3)$ and hence $(57'3)$. Thus, $7 = 7'$, and since Γ does not have $(57'3) = (573)$, it must have (275). ∎

Claim 4. If 6 exists then $6 = 6'$ and Γ has (462). By symmetry, if 9 exists then $9 = 9'$ and Γ has (492).

Proof. We first show that Γ does not have (364). Suppose Γ has (364). Since at most one of 8 and 9 exists by Claim 2, we may take paths P, P' to be either $3aL_24c, 3bL_14b$ or $4bL_13c, 4aL_23d$. Then we have $OOP[5](P, P'; 6L_23a, 5L_24a)$.

If $6 \neq 6'$ then to avoid an arc from 4 to 5 violating $CS(2cL_26', 6L_23a)$(i) in B_0, Γ must have $(3646'2)$ and hence (364). Thus, $6 = 6'$ and since Γ does not have (364) it must have (462). ∎

Now, from Claim 1 we may assume without loss of generality that 7 exists. By Claim 2, 6 does not exist, and at most one of 8 or 9 exists. If 8 exists, then Γ has $(31'84275)$ by Claim 3, and we get $OOP[1'](1'L_22c, 8L_24c; 5L_24a, 7L_22a)$. If 9 exists, then Γ has $(31'49275)$ by Claim 4, and we get $OOP[1'](1'L_22c, 4aL_29; 5L_24a, 7L_22a)$. Therefore, none of 6, 8 or 9 exists.

To summarise: Γ has (314275), 7 exists, $7 = 7'$, and none of 6, 8 or 9 exists. To find new NSCs in this situation, we use paths that may lie outside the disk D_2. By Lemma 5.1(ii), every edge of L_2 belongs to a face, contained in D_2, that includes a vertex of L_1. Applying this to an edge of L_2 with at least one end in 1, we obtain a face g with at least one vertex v_1 of 1 and at least one vertex v_2 of $2b$. The only components of $\Gamma \cap D_2$ that g may intersect are 1, 2 and (if $1 \neq 1'$) $1'$. By Lemma 5.1(ii), $g \cap 1$ and $g \cap 1'$ have at most one component each. Apply Lemma 5.1 to g, letting E_1 and E_2 be the disks, with boundaries M_1 and M_2.

Let O_{12} be an arc from v_1 to v_2 inside g, and let O_{34} be an arc from an interior point v_3 of xy to a vertex v_4 of $4b$ inside $f \cap A_0$. Cut A_0 along O_{12} and O_{34}; the result is a disk with clockwise boundary (in compressed notation)

$$v_1 O_{12} v_2 \Gamma^{-1} v_4 O_{34}^{-1} v_3 \Gamma^{-1} v_2 O_{12}^{-1} v_1 \Gamma^{-1} v_3 O_{34} v_4 \Gamma^{-1} v_1$$

(We do not distinguish between the two copies of $O_{12}, O_{34}, v_1, v_2, v_3, v_4$, since it will be clear which one we mean.) This disk contains a slightly smaller disk R, whose boundary we divide into left, right, top and bottom segments for convenience. Reading bottom to top, R has $Q_1 = v_1 \Gamma^{-1} 3d L_2^{-1} 4a \Gamma^{-1} v_1$ on the left, and $Q_2 = v_2 \Gamma 3a L_2 4c \Gamma v_2$ on the right. Reading left to right, R has O_{12} on the top and O_{12}^{-1} on the bottom. We use $>, <, \geq, \leq$ to denote order along Q_1 or Q_2, so that, for example, $u > v$ means u is above v. Along Q_1 we have $v_1 < 3d < 4a < v_1$, with $4a < 1' < v_1$ if $1' \neq 1$. Along Q_2 we have $v_2 < 2c < 7 < 5 < 3a < 4c < 2a < v_2$.

Now, examining $(M_1 \cup M_2) \cap R$, there must be vertices $x_1 > x_2 > x_3 > x_4$ on Q_1, $y_1 > y_2 > y_3 > y_4$ on Q_2, and paths P_1, P_2, P_3, P_4 in R, such that

- P_1, P_2, P_3, P_4 are vertex-disjoint, except that P_1 and P_4 may intersect at either or both of v_1, v_2;
- each P_i has ends x_i, y_i and is otherwise disjoint from ∂R;
- P_1 and P_4 are segments of M_1, while P_2 and P_3 are segments of M_2; and
- $x_1 \in 1$, $y_1 \in 2$, $x_4 \in 1$ or $1'$, $y_4 \in 2$.

The paths P_1 and P_4 are just the obvious segments of $\partial g = M_1$. If M_2 did not contain two disjoint paths P_2, P_3 as described, then we could find a circle in E_2 that was noncontractible in S_2, contradicting the fact that E_2 is a disk.

If an end of P_2 or P_3 belongs to $(4aL_2 3d)^\circ$ or $(3aL_2 4c)^\circ$, then the path is not essential because it does not have both ends on Γ. However, it can be extended to an essential path, in more than one way. Given $x_i > 3d$, define X_i^+ to be $4aL_2 x_i$ if $x_i < 4a$, or x_i if $x_i \geq 4a$. Given $x_i < 4a$, define X_i^- to be $x_i L_2 3d$ if $x_i > 3d$, or x_i otherwise. Define Y_i^+ and Y_i^- on the right similarly, based on the relationship of y_i to $4c$ and $3a$.

Let w_5 be the last vertex of 5 along Γ. Note that $5L_2 4a = w_5 L_2 4a$.

Suppose first that $x_3 < 4a$. Then necessarily $x_4 \in 1$. If $y_3 < w_5$ then $OP(P_1, X_3^- \cup P_3; 5L_2 4a, 7L_2 4a)$ produces a separating cycle which does not use $4b$; replace xy by $xL_1 y$ to obtain a separating cycle avoiding xy. If $w_5 \leq y_3 < 4c$, then since $x_4 \in 1$ we have $AOP(X_3^- \cup P_3 \cup Y_3^-, P_4; 2cL_2 3a, 3dL_2 7 \cup 7 \cup 7L_2 2a)$. If $y_3 \geq 4c$, then we have (the rather complicated) $AOP(P_1, X_3^- \cup P_3 \cup y_3\Gamma^{-1}4c \cup 4cL_2^{-1}3a; 2cL_2 3a, 3dL_2 7 \cup 7\Gamma 5 \cup 5L_2 4a)$.

Now suppose that $x_3 \geq 4a$, and that $y_2 \leq 3a$. If $\Gamma \cap (3cL_1 2b)^\circ = \emptyset$, then we have $AOP(P_2, P_3; 2cL_2 3a, 3cL_1 2b)$. Otherwise, 7 must intersect $3cL_1 2b$, so 7 has segments $7a, 7c$ on L_2 and $7b$ on L_1. Then we have $AOP(3aL_2 4c, 4bL_1 3c; 3cL_1 7b, 3dL_2 7a)$.

Finally, suppose that $x_3 \geq 4a$ and $y_2 > 3a$. If $x_4 \in 1$, then $OP(P_2 \cup Y_2^+, P_4; 5L_2 4a, 7L_2 2a)$ produces a separating cycle that does not use $4b$; replace xy by $xL_1 y$ to obtain a separating cycle avoiding xy. If $x_4 \notin 1$, then $1 \neq 1'$ and $x_4 \in 1'$. Let $P_2' = P_2 \cup y_2\Gamma^{-1}4c \cup 4cL_2^{-1}3a$ if $y_2 \geq 4c$, and $P_2' = P_2 \cup Y_2^-$ if $3a < y_2 < 4c$. Then we have $AOP(P_4, P_2'; 2cL_2 3a, 3dL_2 7 \cup 7\Gamma 5 \cup 5L_2 4a)$.

We have covered all cases, so this concludes the proof of the theorem. ∎

Since representativity for triangulations is the same as the *edgewidth*, or length of the shortest noncontractible cycle, we observe the following corollary of Theorem 5.2.

Corollary 5.3. *Any triangulation of the double torus whose shortest noncontractible cycle has length at least* 4 *has a noncontractible separating cycle.*

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