Spanning trails with maximum degree at most 4 in $2K_2$-free graphs

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Abstract. A graph is called $2K_2$-free if it does not contain two independent edges as an induced subgraph. Mou and Pasechnik conjectured that every $\frac{3}{2}$-tough $2K_2$-free graph with at least three vertices has a spanning trail with maximum degree at most 4. In this paper, we confirm this conjecture. We also provide examples for all $t < \frac{5}{4}$ of t-tough graphs that do not have a spanning trail with maximum degree at most 4.

Keywords. Toughness; $2K_2$-free graph; 2-trail; dominating cycle

1 Introduction

Graphs considered in this paper are simple, undirected, and finite. Let G be a graph. Let $V(G)$ and $E(G)$ be the vertex set and edge set of G, respectively. For $v \in V(G)$, $N_G(v)$ denotes the set of neighbors of v in G, and $d_G(v) = |N_G(v)|$ the degree of v in G. If $S \subseteq V(G)$ then the subgraph induced by $V(G) - S$ is denoted by $G - S$. For notational simplicity we write $G - \{x\}$ for $G - x$. Let $u, v \in V(G)$ be two vertices. Then $\mathrm{dist}_G(u, v)$, the *distance between u and v in G*, is defined to be the length of a shortest path connecting u and v in G. If $uv \notin E(G)$, we write $G + uv$ for the new graph obtained from G by adding the edge uv. If $uv \in E(G)$, then $G - uv$ denotes the graph obtained from G by deleting the edge uv. Let $V_1, V_2 \subseteq V(G)$ be two disjoint sets. Then $E_G(V_1, V_2)$ is the set of edges of G with one end in V_1 and the other end in V_2. The graph G is called $2K_2$-free if it does not contain two independent edges as an induced subgraph.

The number of components of G is denoted by $c(G)$. Let $t \geq 0$ be a real number. The

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graph is said to be *t-tough* if $|S| \geq t \cdot c(G - S)$ for each $S \subseteq V(G)$ with $c(G - S) \geq 2$. The *toughness* $\tau(G)$ is the largest real number t for which G is t-tough, or is defined as ∞ if G is complete. This concept, a measure of graph connectivity and "resilience" under removal of vertices, was introduced by Chvátal [5] in 1973. It is easy to see that if G has a hamiltonian cycle then G is 1-tough. Conversely, Chvátal [5] conjectured that there exists a constant t_0 such that every t_0-tough graph is hamiltonian. Bauer, Broersma and Veldman [1] have constructed t-tough graphs that are not hamiltonian for all $t < \frac{9}{4}$, so t_0 must be at least $\frac{9}{4}$.

There are a number of papers on Chvátal's toughness conjecture, and it has been verified when restricted to a number of graph classes [2], including planar graphs, claw-free graphs, co-comparability graphs, and chordal graphs. Recently, Broersma, Patel and Pyatkin [3] proved that every 25-tough $2K_2$-free graph on at least three vertices is hamiltonian.

Another direction inspired by Chvátal's toughness conjecture is investigating the existence of spanning substructures weaker than hamiltonian cycles for a given toughness. For example, k-trees, k-walks, and k-trails are substructures of this kind. Let k be a positive integer. A *k-tree* is a tree with maximum degree at most k, and a *k-walk* is a closed walk with each vertex repeated at most k times. A k-walk can be obtained from a k-tree by visiting each edge of the tree twice. A *k-trail* is a k-walk with no repetition of edges. A graph has a spanning k-trail if and only if it has a spanning Eulerian subgraph with maximum degree at most $2k$. A spanning 2-tree is just a hamiltonian path and a spanning 1-walk/1-trail is a hamiltonian cycle.

In 1990, Jackson and Wormald [8] made the following conjecture.

Conjecture 1. *Let $k \geq 2$ be a positive integer. Then every $\frac{1}{k-1}$-tough graph has a spanning k-walk.*

Mou and Pasechnik [12, 11] confirmed Jackson and Wormald's conjecture for $2K_2$-free graphs. In [11], they proposed the following two conjectures.

Conjecture 2. *Every $\frac{3}{2}$-tough $2K_2$-free graph with at least three vertices has a spanning 2-trail.*

Conjecture 3. *Every 2-tough $2K_2$-free graph with at least three vertices is hamiltonian.*

The class of $2K_2$-free graphs is well studied, for instance, see [3, 4, 6, 10, 12, 11, 13]. It is a superclass of *split* graphs, where the vertices can be partitioned into a clique and an independent set. One can also easily check that every *cochordal* graph (i.e., a graph that is the complement of a chordal graph) is $2K_2$-free and so the class of $2K_2$-free graphs is at least as rich as the class of chordal graphs.

In this paper, we confirm Conjecture 2.

Theorem 1. *Let G be a $\frac{3}{2}$-tough $2K_2$-free graph with at least three vertices. Then G has a spanning 2-trail.*

There is a large literature proving the existence of a spanning closed trail under various conditions; a graph with a spanning closed trail is called *supereulerian*. A recent paper in this area, providing references to other papers, is [9]. However, apart from results on hamiltonicity there do not seem to be many results on spanning closed trails with bounded degree. Other than Theorem 1, the only one we are aware of is in [7], which proves that a 2-edge-connected n-vertex graph G with $n \geq 7$ and $\sigma_3(G) \geq n$ has a spanning 2-trail, where $\sigma_3(G)$ is the minimum degree sum over all triples of pairwise independent vertices.

We prove Theorem 1 in Section 2. In Section 3, we construct $2K_2$-free graphs with toughness close to $\frac{5}{4}$ but containing no spanning 2-trail.

2 Proof of Theorem 1

We need the following lemma in proving Theorem 1.

Lemma 2.1. *Let G be a bipartite graph with partite sets X and Y. If for every $S \subseteq X$, $|N_G(S)| \geq \frac{3}{2}|S|$, then G has a subgraph H covering X (meaning that $X \subseteq V(H)$) such that for every $x \in X$, $d_H(x) = 2$ and for every $y \in Y$, $d_H(y) \leq 2$.*

Proof. Form G' from G by replacing each $x \in X$ by x_1, x_2, x_3, each $y \in Y$ by y_1, y_2, and each $xy \in E(G)$ by six edges x_iy_j, $1 \leq i \leq 3$, $1 \leq j \leq 2$. Let π be the natural projection from G' to G with

$$\pi(x_i) = x, \quad \pi(y_j) = y, \quad \pi(x_iy_j) = xy.$$

Let $X' = \pi^{-1}(X)$ be the inverse image of X under π. For each $S' \subseteq X'$, let $S = \pi(S')$. Then $|S'| \leq 3|S| \leq 2|N_G(S)| = |N_{G'}(S')|$. Thus, by Hall's Theorem, G' has a matching M' covering X'. The projection $\pi(M')$ of M' is a graph containing all the vertices in X such that each vertex in X has degree 2 or 3, and each vertex in Y has degree at most 2. In $\pi(M')$, for each $x \in X$ with degree 3, delete one edge incident to x. Then the graph H induced by the remaining edges is the desired graph. □

We cannot reduce the number $\frac{3}{2}$ in Lemma 2.1. To see this, take $k \geq 1$, X with $|X| = 2k$, and $Y = Y_1 \cup Y_2$ with $|Y_1| = 2k$ and $|Y_2| = k$. To form G, join each vertex of X to a distinct vertex of Y_1 (giving a matching) and join every vertex of X to every vertex of Y_2. Then G has a subgraph H as described, but if we delete any $y \in Y$ then no such subgraph exists although $G - y$ satisfies the condition of Lemma 2.1 with $\frac{3k-1}{2k}$ instead of $\frac{3}{2}$.

A subgraph $G^* \subseteq G$ is called *dominating* if $G - V(G^*)$ is an edgeless graph. Mou and Pasechnik proved the existence of a dominating cycle in $2K_2$-free graphs. In fact, the proof of [12, Theorem 3] implies the following.

Lemma 2.2. *Let G be a $2K_2$-free graph containing a cycle. Then some longest cycle of G is dominating.*

Proof of Theorem 1. As G is $\frac{3}{2}$-tough, G is 3-connected. So G has a cycle. Let C be a dominating longest cycle of G, which exists by Lemma 2.2. Let $\overrightarrow{C}$ denote a forward orientation of C. For a vertex $x \in V(C)$, we let x^+ denote the successor of x on $\overrightarrow{C}$, and if $S \subseteq V(C)$ we define $S^+ = \{x^+ \mid x \in S\}$. We may assume $V(G) - V(C) \neq \emptyset$. Otherwise, C is a spanning 1-trail.

Claim A: Let $x \in V(G) - V(C)$.

(a) $N_G(x)$ does not contain two consecutive vertices on C.

(b) If $y, z \in N_G(x)$ with $y \neq z$ then there is no path from y^+ to z^+ that is that is internally disjoint from C; in particular, $y^+z^+ \notin E(G)$.

(c) C has at least 7 vertices.

Proof. Both (a) and (b) follow by standard arguments. We only prove (c) here. Since $G - V(C)$ is edgeless, $N_G(x) \subseteq V(C)$. By (a), $N_G(x)^+$ is disjoint from $N_G(x)$. As G is $\frac{3}{2}$-tough, $\delta(G) \geq 3$, so $|N_G(x)| = |N_G(x)^+| \geq 3$. Thus, $|V(C)| \geq 6$, and $|V(C)| = 6$ precisely when $|N_G(x)| = 3$ and $V(C) = N_G(x) \cup N_G(x)^+$. In that case, by (b) the vertices of $N_G(x)^+$ belong to separate components in $G - N_G(x)$. Thus, $c(G - N_G(x)) \geq 4$, and so $\frac{|N_G(x)|}{c(G-N_G(x))} \leq \frac{3}{4} < \frac{3}{2}$, contradicting the toughness of G. □

Let $G' = G - E(G[V(C)])$ with partite sets $X = V(G) - V(C)$ and $Y = V(C)$. Since G is $\frac{3}{2}$-tough and X is an independent set in G, we have that for any $S \subseteq X$, $|N_{G'}(S)| \geq \frac{3}{2}|S|$ (even when $|S| = 1$, because then $c(G - N_{G'}(S)) \geq 2$ by (a) of Claim A). Applying Lemma 2.1 to G', we see that G' (hence G) has a subgraph H such that for any $x \in X$, $d_H(x) = 2$ and for any $y \in Y \cap V(H)$, $d_H(y) = 1$ or $d_H(y) = 2$. Subject to this property, we choose a subgraph H of G such that the number of components in H is smallest. Let $H_1, \cdots, H_\ell$ be the components of H. Each H_i is either a path or a cycle. Assume, without loss of generality, that $H_1, \cdots, H_p$ are paths and $H_{p+1}, \cdots, H_\ell$ are cycles. For each path H_i $(1 \leq i \leq p)$, let u_i and v_i denote its endvertices (these two vertices are on C by the construction of H). Let s_i and t_i denote the neighbor of u_i and v_i in H, respectively. Note that s_i and t_i are vertices from $V(G) - V(C)$ and $s_i = t_i$ if H_i has length 2. Note also that $C \cup \left(\bigcup_{p+1 \leq i \leq \ell} H_i\right)$ is a spanning 2-trail if $p = 0$. Therefore, we assume $p \geq 1$.

Claim B: Each of the following holds.

(a) $s_iu_j, s_iv_j, t_iu_j, t_iv_j \notin E(G)$, for all i, j with $i \neq j$ and $i, j \in \{1, \cdots, p\}$.

(b) Let u be an endvertex of H_i and v be an endvertex of H_j, where $i \neq j$ and $i, j \in \{1, \cdots, p\}$. Then $uv \in E(G)$.

Proof. For (a), if say $s_iu_j \in E(G)$ then we could replace s_iu_i by s_iu_j in H to obtain fewer components. For (b), let s be the neighbor of u on H_i, and t be the neighbor of v on H_j. Note that $s, t \in V(G) - V(C)$. Since $i \neq j$, we have $s \neq t$. By (a), we have $sv, tu \notin E(G)$. Furthermore, $st \notin E(G)$ as $G - V(C)$ is edgeless. So $uv \in E(G)$ by the $2K_2$-freeness of G. $\square$

Claim C: Let q be an integer with $1 \leq q \leq p$, and let $V_q = \bigcup_{1\leq i\leq q} V(H_i)$. Then $G[V_q] - E(C)$ contains a path P_q with vertex set V_q such that for each i with $1 \leq i \leq q$, H_i is a subpath of P_q and both endvertices of P_q belong to $\{u_1, \cdots, u_q, v_1, \cdots, v_q\}$.

Proof. We show this claim by induction on q. For $q = 1$, H_1 itself is a desired path. So we assume that $q \geq 2$. By the induction hypothesis, $G[V_{q-1}] - E(C)$ contains a path P_{q-1} with the desired property. Assume, without loss of generality, that the two endvertices of P_{q-1} are u_a and v_b with $a, b \in \{1, \cdots, q-1\}$. As $|V(C)| \geq 7$ by (b) of Claim A, we see that one of $\mathrm{dist}_C(u_a, u_q), \mathrm{dist}_C(u_a, v_q), \mathrm{dist}_C(v_b, u_q), \mathrm{dist}_C(v_b, v_q)$ must be at least 2. Assume, without loss of generality, that $\mathrm{dist}_C(u_a, v_q) \geq 2$. Then $u_av_q \in E(G)$ by (b) of Claim B and $u_av_q \in E(G) - E(C)$ since $\mathrm{dist}_C(u_a, v_q) \geq 2$. Thus, $P_{q-1} \cup H_q + u_av_q$ is a desired path. $\square$

Let $D = \bigcup_{p+1\leq i\leq \ell} H_i$ be the union of the cycle components of H. Consider two cases.

Case 1: $p \geq 2$.

Let P_p be a path with the property stated in Claim C. Assume, without loss of generality, that the endvertices of P_p are u_1 and v_p. By (b) of Claim B, we have $v_pu_1 \in E(G)$. Let

$$T = \begin{cases} C \cup D \cup P_p - v_pu_1, & \text{if } v_pu_1 \in E(C); \\ C \cup D \cup P_p + v_pu_1, & \text{if } v_pu_1 \in E(G) - E(C). \end{cases}$$

Then T is a spanning 2-trail of G.

Case 2: $p = 1$.

Assume first that $|V(H_1)| \geq 4$. Consider the two edges s_1u_1 and t_1v_1. Again, we have $\{s_1v_1, t_1u_1, u_1v_1\} \cap E(G) \neq \emptyset$ by the $2K_2$-freeness of G. Let

$$T = \begin{cases} C \cup H - s_1u_1 + s_1v_1, & \text{if } s_1v_1 \in E(G); \\ C \cup H - t_1v_1 + t_1u_1, & \text{if } t_1u_1 \in E(G); \\ C \cup H - u_1v_1, & \text{if } u_1v_1 \in E(C); \\ C \cup H + u_1v_1, & \text{if } u_1v_1 \in E(G) - E(C). \end{cases}$$

Then T is a spanning 2-trail of G.

Assume now that $|V(H_1)| = 3$. Suppose that $\mathrm{dist}_C(u_1, v_1) \geq 3$. As $u_1^+v_1^+ \notin E(G)$ by (b) of Claim A, we have $\{u_1v_1, u_1v_1^+, v_1u_1^+\} \cap E(G) \neq \emptyset$ by the $2K_2$-freeness of G. Note that $\{u_1v_1, u_1v_1^+, v_1u_1^+\} \cap E(C) = \emptyset$ as $\mathrm{dist}_C(u_1, v_1) \geq 3$. Let

$$T = \begin{cases} C \cup H + u_1v_1^+ - v_1v_1^+, & \text{if } u_1v_1^+ \in E(G); \\ C \cup H + v_1u_1^+ - u_1u_1^+, & \text{if } v_1u_1^+ \in E(G); \\ C \cup H + u_1v_1, & \text{if } u_1v_1 \in E(G). \end{cases}$$

In the first case the vertex v_1^+ may also be contained in D, but when we add the edge $u_1v_1^+$ and remove the edge $v_1v_1^+$, the degree of v_1^+ in T is the same as in $C \cup H$. The same applies to u_1^+ in the second case. Thus the degree of each vertex in T is at most 4, and T is a spanning 2-trail of G.

Suppose that $N_G(s_1) - V(H) \neq \emptyset$. Then $N_G(s_1) - V(D)$, which includes u_1 and v_1, contains at least three vertices. By Claim A, these vertices are pairwise nonadjacent and $|V(C)| \geq 7$, so there are $u', v' \in N_G(s_1) - V(D)$ with $\mathrm{dist}_C(u', v') \geq 3$. We replace H_1 by the path $u's_1v'$ and apply the argument above.

Therefore, we assume all neighbors of s_1 not in H_1 lie in D, which must be nonempty. Suppose that $N_G(x') \subseteq V(H)$ for all $x' \in X \cap V(D)$. Then deleting all the $|X|+1$ neighbors of vertices in X on C results in at least $|X|$ components. Since $|V(D) \cap X| \geq 2$ and $s_1 \in X$, $|X| \geq 3$, so $\frac{|X|+1}{|X|} \leq \frac{4}{3} < \frac{3}{2}$, contradicting the toughness of G. Therefore there exist x' and u' with $x' \in X \cap V(D)$ and $u' \in N_G(x') - V(H)$. Let $x'v' \in E(D)$ and $D' = D - x'v' + x'u'$. Replacing D by D' in H, we see that the new graph has the same property as H, but it has two components that are paths, so we may apply Case 1.

The proof of Theorem 1 is now complete. □

3 An Extremal Example

In this section, we construct a family of $2K_2$-free graphs with toughness approaching $\frac{5}{4}$ that do not contain any spanning 2-trail.

Let $n \geq 2$ be an integer, $Q_1 = K_{4n}$, the complete graph on $4n$ vertices, $Q_2 = \overline{K_{4n}}$, the empty graph on $4n$ vertices, and $Q_3 = K_{n-1}$. Let G_n be a graph with $V(G_n) = V(Q_1) \cup V(Q_2) \cup V(Q_3)$ and $E(G_n)$ consisting of all edges in Q_1 and Q_3, all edges between $V(Q_3)$ and $V(Q_1) \cup V(Q_2)$, and a perfect matching between Q_1 and Q_2. It is easy to check that G is $2K_2$-free.

We claim that $\lim_{n\to\infty} \tau(G_n) = \frac{5}{4}$. Let $S \subseteq V(G_n)$ be a cutset such that $\tau(G_n) = \frac{|S|}{c(G_n - S)}$. Then $Q_3 \subseteq S$ as each vertex in Q_3 is adjacent to every other vertex of G_n. Also, $S \cap V(Q_2) = \emptyset$. Otherwise, as $c(G-(S-V(Q_2))) \geq c(G-S)$, we get $\frac{|S-V(Q_2)|}{c(G_n-(S-V(Q_2)))} < \frac{|S|}{c(G_n-S)} = \tau(G_n)$, contradicting the toughness of G. Thus, $c(G - S) = |S \cap V(Q_1)| + 1$ if $V(Q_1) \not\subseteq S$ and $c(G - S) = 4n$ otherwise. In the latter case, $\frac{|S|}{c(G_n-S)} = \frac{5n-1}{4n}$. So assume $V(Q_1) \not\subseteq S$ and $|V(Q_1) \cap S| = r$, where $1 \leq r \leq 4n - 1$. Then $\frac{n-1+r}{r+1}$ is a decreasing function of r which achieves its minimum when $r = 4n-1$. Hence, $\tau(G_n) = \frac{|S|}{c(G_n - S)} = \frac{5n-2}{4n}$, which approaches $\frac{5}{4}$ as $n \to \infty$.

We show now that G_n has no spanning 2-trail. Suppose on the contrary that T is a spanning 2-trail of G_n. Let $v \in V(Q_2)$ be a vertex. Then $d_T(v) \geq 2$. As $|N_G(v) \cap V(Q_1)| = 1$, $|N_T(v) \cap V(Q_3)| \geq 1$. Thus, $|E_T(V(Q_3), V(Q_2))| \geq 4n$. Since $|V(Q_3)| = n-1$, by the Pigeonhole Principle there is a vertex from Q_3 that has degree at least 5 in T. This contradicts the assumption that T is a 2-trail.

From the example above, we suspect the following might be true.

Conjecture 4. *Any $\frac{5}{4}$-tough $2K_2$-free graph with at least three vertices has a spanning 2-trail.*

Our proof of Theorem 1 relies on Lemma 2.1, which cannot be improved, so a new strategy will be needed to obtain a positive answer to this conjecture.

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