Recent progress in edge reconstruction

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Abstract

This paper is a survey of some recent results on the edge reconstruction problem for graphs. Section 1 provides introductory definitions. Section 2 examines results proved by inclusion-exclusion techniques, including a theorem of Nash-Williams which extends the well-known results of Lovász and Müller, and some new results by Avellis and Borzacchini. Section 3 considers applications of these techniques to hamiltonian graphs, claw-free graphs, and degree conditions for edge reconstructibility. Section 4 describes some algebraic techniques, focusing on the recent result of Godsil, Krasikov and Roditty on k-edge reconstruction, using methods originally developed by Stanley. Section 5 contains some final remarks.

1. Introduction and definitions

This paper is a brief survey of some recent progress on the well-known edge reconstruction problem. We do not intend to cover all areas of research. Rather, our emphasis is on a particular class of results: those inspired by the 1972 paper of Lovász [**22**], who proved that a graph is edge reconstructible if it has more than half as many edges as a complete graph on the same vertex set.

It will be assumed that the reader is familiar with the basic concepts of graph theory; any terms not defined here can be found in [**5**]. All graphs are simple and finite unless specified otherwise. The *order* of a graph G is $\nu(G) = |V(G)|$, and the *size* of G is $\epsilon(G) = |E(G)|$. The *isomorphism class* $[G]$ of G is the set of all graphs isomorphic to G, and may be thought of as an "unlabelled version" of G.

The edge reconstruction problem is one of several reconstruction problems for graphs, and belongs to a family of reconstruction problems for various types of object. In each such problem we take an object, perform certain operations on it to form a collection of new

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objects, known as a *deck*, and then ask whether the original object is uniquely determined (perhaps up to some sort of isomorphism) by its deck. Two well-known problems for graphs which follow this pattern are the vertex and edge reconstruction problems. For general surveys of these problems see the papers by Bondy and Hemminger [**4**] and Nash-Williams [**26**].

In the vertex reconstruction problem we form the *vertex deck* of a graph G by taking the multiset

$$D_{\mathrm{V}}(G) = \{[G - v] \mid v \in V(G)\}.$$

We say that G is *vertex reconstructible* (or often just *reconstructible*) if $D_{\mathrm{V}}(G) = D_{\mathrm{V}}(H) \Rightarrow G \cong H$, and we write $G \in \mathcal{VR}$. The only known graphs which are not vertex reconstructible are $2K_1$ and K_2, which have the same vertex deck. This suggests the following conjecture, due to Kelly and Ulam (see [**4**]).

Vertex Reconstruction Conjecture: *If* $\nu(G) \geq 3$ *then* $G \in \mathcal{VR}$.

Classes of graphs known to satisfy this conjecture include regular graphs, trees, disconnected graphs (all due to Kelly - see [**4**]), graphs of order at most nine (McKay [**23**]), and maximal planar graphs (Fiorini and Lauri [**9, 20**]).

For the edge reconstruction problem, the *edge deck* of a graph G is the multiset

$$D_{\mathrm{E}}(G) = \{[G - e] \mid e \in E(G)\}.$$

We say that G is *edge reconstructible* if $D_{\mathrm{E}}(G) = D_{\mathrm{E}}(H) \Rightarrow G \cong H$, and write $G \in \mathcal{ER}$. The only known graphs which are not edge reconstructible occur in two pairs, $P_3 \cup K_1$ and $2K_2$, $K_{1,3}$ and $K_3 \cup K_1$. This suggests the conjecture with which we shall mostly be concerned, and which is due to Harary [**14**].

Edge Reconstruction Conjecture: *If* $\epsilon(G) \geq 4$ *then* $G \in \mathcal{ER}$.

A result of Greenwell [**13**] says that if I is the set of isolated vertices in G and $\epsilon(G) \geq 4$, then $G - I \in \mathcal{VR} \Rightarrow G \in \mathcal{ER}$. Therefore, regular graphs, trees, disoconnected graphs with at least two nontrivial components, graphs of order at most nine, and maximal planar graphs are all edge reconstructible as well as vertex reconstructible if their size is at least four. There are also classes of graphs which are known to satisfy the Edge Reconstrution Conjecture, but which are not known to satisfy the Vertex Reconstruction Conjecture. These include graphs with $\epsilon > \nu(\log_2 \nu - 1)$ (Müller [**24**]), hamiltonian graphs of sufficiently

large order (Pyber [**28**]), bidegreed graphs (Myrvold, Ellingham and Hoffman [**25**]) and claw-free graphs (Ellingham, Pyber and Yu [**8**]). We shall return to these classes in Sections 2 and 3.

A third interesting reconstruction problem for graphs has been posed by Stanley. Given a graph G and a vertex $v \in V(G)$, define $G * v$ to be the graph obtained from G by deleting all edges incident with v and adding edges from v to all vertices to which v was not adjacent in G. The operation $*$ is called *vertex switching.* We can formulate a vertex-switching reconstruction problem by defining the *vertex-switching deck* of G to be the multiset

$$D_{\mathrm{VS}}(G) = \{[G * v] \mid v \in V(G)\}.$$

We say that G is *vertex-switching reconstructible* if $D_{\mathrm{VS}}(G) = D_{\mathrm{VS}}(H) \Rightarrow G \cong H$, and we write $G \in \mathcal{VSR}$. The only known graphs which are not vertex-switching reconstructible have order four. For example, C_4 and $\overline{K_4}$ have the same vertex-switching deck, and there are three other such pairs of graphs with four vertices. This led Stanley [**30**] to make the following conjecture.

Vertex-Switching Reconstruction Conjecture: *If $\nu(G) \neq 4$ then $G \in \mathcal{VSR}$.*

Stanley proved this conjecture for graphs with order not divisible by 4, using methods discussed in Section 4.

The three problems above are probably the most interesting reconstruction problems for graphs. Variants of these problems have been posed for digraphs, hypergraphs, and other combinatorial structures - see [**4**] or [**26**] for some examples. We shall concentrate on the edge reconstruction problem and some of its extensions, with a brief mention of the vertex-switching problem in Section 4.

2. Inclusion-exclusion techniques

In this section we investigate some general results which can be proved using inclusion-exclusion techniques, including the result of Lovász's original paper [**22**]. In this paper Lovász showed that a graph is edge reconstructible if $\epsilon > \frac{1}{2}\binom{\nu}{2}$, using a very simple and clever inclusion-exclusion argument. Müller [**24**], also using an inclusion-exclusion argument, improved this to show that a graph is edge reconstructible if $2^{\epsilon-1} > \nu!$, or in particular if $\epsilon > \nu(\log_2 \nu - 1)$ when $\nu \geq 10$. Müller's result is significant because it implies

that "almost all" graphs (a proportion approaching 1 as $\nu \to \infty$) are edge reconstructible. Nash-Williams then found a general theorem from which the results of Lovász and Müller follow as corollaries. The full theorem appears in [**4**] and a special case in [**26**]. The proof in [**26**] does not explicitly use inclusion-exclusion, but the proof given in [**4**] does, and inclusion-exclusion seems to be the natural method here.

It is interesting to note that Lovász, Müller and Nash-Williams do not use any specifically graph-theoretical properties in their arguments. As we shall see, their results actually apply to a much more general reconstruction problem. We shall state Nash-Williams' result in terms of a framework proposed by Alon, Caro, Krasikov and Roditty [**1**], and then indicate how the results of Lovász and Müller can be derived from it.

Suppose that E is a finite set, whose elements we shall call *edges*. Let Π be a finite group which acts on E, so that for every $e \in E$ and $\pi \in \Pi$ we can define $\pi(e) \in E$, in such a way that

(1) $\delta(e) = e$ for every $e \in E$, where δ is the identity of Π, and

(2) $(\pi\phi)(e) = \pi(\phi(e))$ for all $e \in E$ and $\pi, \phi \in \Pi$.

The objects of interest to us will be the subsets of E, which we shall call *edgesets*. For any edgeset A and $\pi \in \Pi$ define $\pi(A)$ to be $\{\pi(e) \mid e \in A\}$. If A and B are edgesets and $\pi(A) = B$ then we say that π is an isomorphism from A to B, and write $A \cong B$. The *isomorphism class* of A is $[A] = \{A' \subseteq E \mid A' \cong A\}$. An *automorphism* of A is an isomorphism from A to itself; π is an automorphism of A if and only if $\pi(A) = A$. The set of automorphisms of A is denoted $\mathrm{Aut}A$.

Graphs can be represented by edgesets in a very natural way. Any graph of order ν can be considered as a subgraph of a complete graph of order ν. Fix some complete graph K_ν, and let $E = E(K_\nu)$. We can identify each graph $G \subseteq K_\nu$ with the edgeset $E(G)$. To obtain our usual concept of isomorphism between graphs we let Π be the group of all permutations of $V(K_\nu)$, with Π acting on $E = E(K_\nu)$ by $\pi(uv) = \pi(u)\pi(v)$. Many other classes of combinatorial objects can be represented by edgesets, including digraphs, hypergraphs, and various geometric figures.

The edge reconstruction problem now generalises easily to edgesets. Given an edgeset A, define the *edge deck* of A to be the multiset

$$D_{\mathrm{E}}(A) = \{[A - e] \mid e \in A\},$$

and say that A is *edge reconstructible* if $D_{\mathrm{E}}(A) = D_{\mathrm{E}}(B) \Rightarrow A \cong B$, writing $A \in \mathcal{ER}$.

After defining some notation, we can give a version of Kelly's counting lemma for this problem. For edgesets A and B with $C \subseteq A$, define

$$(A \to B)_C = \{\pi \in \Pi \mid \pi(A) \cap B = \pi(C)\},$$

and let $|A \to B|_C$ denote the cardinality of this set. Abbreviate $(A \to B)_A$ to $(A \to B)$ and $|A \to B|_A$ to $|A \to B|$; then $(A \to B) = \{\pi \in \Pi \mid \pi(A) \subseteq B\}$, the set of *embeddings* of A into B. Define $s(A, B)$ to be the number of subsets of A that are isomorphic to B; then $|A \to B| = |\mathrm{Aut}A| s(A, B)$.

Kelly's Lemma for Edges: *If $D_{\mathrm{E}}(A) = D_{\mathrm{E}}(B)$ then for any C with $|C| < |A|$, $s(C, A) = s(C, B)$ and $|C \to A| = |C \to B|$.*

The proof of this is similar to the proof of the corresponding results for vertex and edge reconstruction of graphs (see [**4**]).

The basic combinatorial tool in our proof will be the Principle of Inclusion-Exclusion, which may be found in any basic text on combinatorics (e.g. [**21**]). We shall use it in the following form.

Principle of Inclusion-Exclusion: *Suppose $\mathcal{P}$ is a finite set of properties defined on a finite set U. For any $\mathcal{S} \subseteq \mathcal{P}$, define $N(\mathcal{S})$ to be the number of elements of U having all the properties in $\mathcal{S}$ (and possibly more). Then the number of elements of U having none of the properties in $\mathcal{P}$ is*

$$\sum_{\mathcal{S} \subseteq \mathcal{P}} (-1)^{|\mathcal{S}|} N(\mathcal{S}).$$

Now we state and prove Nash-Williams' theorem for edgesets. Our proof follows that of [**4**], but uses simple inclusion-exclusion rather than Möbius functions.

Theorem 2.1 (Nash-Williams [**4**]) : *Suppose that A and B are edgesets, and $C \subseteq A$. If $D_{\mathrm{E}}(A) = D_{\mathrm{E}}(B)$ then*

$$|A \to B| \;=\; |A \to A| + (-1)^{|A|-|C|}(|A \to B|_C - |A \to A|_C) \qquad (*).$$

Proof: For each $e \in A - C$ define a property P_e on the set $U = (C \to B)$ by saying that π has P_e if and only if $\pi(e) \in B$. Then for any $\mathcal{S} = \{P_{e_1}, P_{e_2}, ..., P_{e_k}\}$, $N(\mathcal{S}) = |D \to B|$

where $D = C \cup \{e_1, e_2, ..., e_k\}$, and $|\mathcal{S}| = |D| - |C|$. Also, the number of elements of U with none of these properties is $|A \to B|_C$. So by the Principle of Inclusion-Exclusion,

$$|A \to B|_C = \sum_{C \subseteq D \subseteq A} (-1)^{|D|-|C|} |D \to B|.$$

Similarly,

$$|A \to A|_C = \sum_{C \subseteq D \subseteq A} (-1)^{|D|-|C|} |D \to A|.$$

Therefore,

$$|A \to B|_C - |A \to A|_C = \sum_{C \subseteq D \subseteq A} (-1)^{|D|-|C|} (|D \to B| - |D \to A|).$$

But by Kelly's Lemma, all the terms in the sum on the right are zero except for the one with $D = A$. Hence,

$$|A \to B|_C - |A \to A|_C = (-1)^{|A|-|C|} (|A \to B| - |A \to A|),$$

from which the theorem follows. ∎

To use this theorem to prove that an edgeset A is edge reconstructible, we find conditions under which the right side of $(*)$ is positive. For then $|A \to B|$ is positive, and since $|A| = |B|$, this implies that $A \cong B$.

Corollary 2.2: *An edgeset A is edge reconstructible if there exists $C \subseteq A$ so that*

(a) $|A| - |C|$ *is even and* $|A \to A|_C = 0$, *or*

(b) $|A \to B|_C = |A \to A|_C$ *whenever* $D_{\mathrm{E}}(A) = D_{\mathrm{E}}(B)$.

Proof: In either case the right side of $(*)$ is positive. ∎

Hong [**15**] has shown that the existence of C satisfying condition (b) of this corollary is actually a necessary and sufficient condition for A to be edge reconstructible.

Now Corollary 2.2 can be used to prove general versions of the results of Lovász and Müller.

Corollary 2.3 (Lovász): *If* $|A| > \frac{1}{2}|E|$ *then* $A \in \mathcal{ER}$.

Proof: If $|A| > \frac{1}{2}|E|$ then $|A \to D|_\emptyset = 0$ for any D with $|D| > \frac{1}{2}|E|$, by the Pigeonhole Principle. Therefore, by Corollary 2.2 (b) with $C = \emptyset$, $A \in \mathcal{ER}$. ∎

Corollary 2.4 (Müller) : *If* $2^{|A|-1} > |\Pi|$ *then* $A \in \mathcal{ER}$.

Proof: Suppose $A \notin \mathcal{ER}$. Then by Corollary 2.2 (a), $|A \to A|_C \geq 1$ for each of the $2^{|A|-1}$ sets $C \subseteq A$ with $|A| - |C|$ even. But since the sets $(A \to A)_C$ are disjoint,

$$|\Pi| \geq \sum_{C \subseteq A} |A \to A|_C \geq 2^{|A|-1},$$

which is a contradiction. ∎

Applied to the special case of edge reconstruction for graphs, these two corollaries give us the original results of Lovász and Müller, for $|E|$ is just $\binom{\nu}{2}$ and $|\Pi|$ is just $\nu!$. But they can also be applied to many other problems. For example, we now know that a digraph is arc-reconstructible if $\epsilon > \binom{\nu}{2}$ or $2^{\epsilon-1} > \nu!$. Notice that in general if $|\Pi|$ is large Corollary 2.3 may in fact give a better result than Corollary 2.4 (for graphs Corollary 2.4 always gives a better result).

A concept more general than graphs but less general than edgesets has been investigated by Avellis and Borzacchini [**2**, **6**]. They define a *g-graph* to consist of a set of vertices and a set of edges, where possible edges are defined recursively, as follows: any vertex is an edge, and any set of edges is an edge. Isomorphisms of g-graphs are defined in terms of a group which acts on the set of vertices. The definition of g-graphs allows both vertex and edge reconstruction problems to be defined for them, and some of the connections between the vertex and edge reconstruction problems for graphs can be generalised to g-graphs. Borzacchini has proved some edge reconstruction results for g-graphs which are special cases of the edgeset results above; in particular, Theorem 15 of [**6**] is a special case of Corollary 2.3.

Avellis and Borzacchini have also proved some new results for g-graphs, which can be generalised to edgesets. These new results seem interesting, although no applications have yet been given. For any edgesets $A_1, A_2, ..., A_k$ and B define the *covering number* $K(A_1, A_2, ..., A_k|B)$ to be the number of k-tuples $(A'_1, A'_2, ..., A'_k)$ with $A'_i \cong A_i$ for $1 \leq i \leq k$ and $\bigcup_{i=1}^{k} A'_i = B$. The following result is proved by an inclusion-exclusion argument.

Theorem 2.5 (Avellis and Borzacchini [**2**, Theorem 5]) :

$$K(A_1, A_2, ..., A_k|B) = \sum_{C \subseteq B} (-1)^{|B|-|C|} \prod_{i=1}^{k} s(A_i, C).$$

By applying Kelly's Lemma to this (in two ways) we get the following corollary.

Corollary 2.6 ([**2**, Theorem 10]) : *If $D_{\mathrm{E}}(A) = D_{\mathrm{E}}(B)$ and $|C_1|, |C_2|, ..., |C_k|$ are all less than $|A|$ then $K(C_1, C_2, ..., C_k|A) = K(C_1, C_2, ..., C_k|B)$.*

A further corollary may be obtained if we define $[A] - [B]$ to be the multiset

$$\{[A - B'] \mid B' \subseteq A, B' \cong B\}.$$

Corollary 2.7 ([**2**, Theorem 12]) : *If $D_{\mathrm{E}}(A) = D_{\mathrm{E}}(B)$ and $0 < |C| < |A|$ then $[A] - [C] = [B] - [C]$.*

Avellis and Borzacchini are also investigating some other ideas for determining relationships between sub-g-graphs from the edge deck. See [**2**] for details.

3. Applications

While the results in Section 2 are very general in nature, they can often be combined with knowledge about the structure of a particular class of graphs to prove more specific results. In this section we examine applications of this kind. Some of these applications are treated in greater detail in a paper by Bondy [**3**].

In order to use the results of Section 2 here we must express them in graph-theoretical language. The reader should note the following correspondences.

edgeset A	$\longleftrightarrow$	graph G
$\lvert A\rvert$	$\longleftrightarrow$	$\epsilon(G)$
$C \subseteq A$	$\longleftrightarrow$	F is a spanning subgraph of G

For graphs G and H, with F a spanning subgraph of G, $(G \to H)_F$ means the set of bijections $\pi : V(G) \to V(H)$ with $\pi(G) \cap H = \pi(F)$, where π extends to the edges of G in the natural way. Then $(G \to H) = (G \to H)_G$ is the set of embeddings of G into H.

The following lemmas will be useful in several of the examples below. The first is the form in which we shall employ the results of Section 2 when proving results about degree conditions and hamiltonian graphs. We state it in terms of graphs, but it can be generalised to edgesets. The second is a corollary of a very elegant result of D. G. Hoffman, and we refer the reader to [**3**] for its proof.

Lemma 3.1 (Lovász - see [**28**]) : *Suppose that G is a graph with a spanning subgraph F such that $|F \to G| < 2^{\epsilon(G)-\epsilon(F)-1}$. Then $G \in \mathcal{ER}$.*

Proof: Suppose that $G \notin \mathcal{ER}$. Then by Corollary 2.2 (a) each of the $2^{\epsilon(G)-\epsilon(F)-1}$ subgraphs H with $F \subseteq H \subseteq G$ and $\epsilon(G) - \epsilon(H)$ even must have $|G \to G|_H \geq 1$. But since these sets are disjoint subsets of $(F \to G)$, we have that $|F \to G| \geq 2^{\epsilon(G)-\epsilon(F)-1}$, which is a contradiction. ∎

Lemma 3.2 (Hoffman - see [3]) : *Suppose that G is a graph of minimum degree δ and average degree d. If*

$$d < \delta + 1 - \frac{1}{\delta + 1}$$

then $G \in \mathcal{ER}$.

Degree conditions

Caunter and Nash-Williams have found several conditions for a graph to be edge reconstructible based on the degrees of its vertices. They are based on the following upper bound for the number of embeddings of a spanning tree, whose proof involves a simple counting argument (see [3]).

Lemma 3.3 (Caunter and Nash-Williams) : *Let T be a spanning tree of G. Then*

$$|T \to G| \leq \nu\Delta!(\Delta - 1)^{\nu-\Delta-1}.$$

Since $\epsilon(T) = \nu - 1$ for any spanning tree T, the following is an immediate corollary of Lemmas 3.1 and 3.3.

Corollary 3.4: *A graph G is edge reconstructible if*

$$\nu\Delta!(\Delta - 1)^{\nu-\Delta-1} < 2^{\epsilon-\nu}.$$

Now if a graph has $\Delta \leq 2$ it is disconnected, a path or a cycle. In any of these cases the graph is known to be edge reconstructible provided its size is at least four. We may therefore assume that $\Delta \geq 3$. With this assumption, a simple sufficient condition for the inequality of Corollary 3.4 to hold can be given in terms of the average and maximum degrees of a graph. By combining this result with Lemma 3.2 we can also obtain a condition involving the maximum and minimum degrees.

Corollary 3.5: *A graph G is edge reconstructible if*

(a) $2\log_2(2\Delta) \leq d$, *or*

(b) $2\log_2(2\Delta) \leq \delta + 1 - 1/(\delta + 1)$.

Corollary 3.5 (b) can be applied to *bidegreed graphs*, graphs in which $\delta \neq \Delta$ and every vertex has degree either δ or Δ. It can easily be shown that a bidegreed graph is edge reconstructible unless $\Delta = \delta + 1$. Using this, Corollary 3.5 (b) implies that any bidegreed graph with $\delta \geq 8$ is edge reconstructible. This can be improved by applying Corollary 3.4 in the case when $\delta = 7$ to give the following theorem.

Theorem 3.6 (Caunter and Nash-Williams - see [**3**]) : *Any bidegreed graph with $\delta \geq 7$ is edge reconstructible.*

Using structural methods unrelated to the techniques under discussion here, Myrvold, Ellingham and Hoffman [**25**] have shown that all bidegreed graphs of size at least four are edge reconstructible.

Other degree results on edge reconstructibility have been obtained by Caro, Krasikov and Roditty [**7**] for $K_{1,k}$-free graphs. Their methods involve the use of specially constructed spanning trees, to which Lemma 3.1 is applied.

Hamiltonian graphs

Lovász in 1981 found a sufficient condition for a hamiltonian graph, or more precisely a graph with a hamiltonian path, to be edge reconstructible. His result is based on the following upper bound for the number of hamiltonian paths in a graph.

Lemma 3.7 (Lovász - see [**3**]) : *The number of hamiltonian paths in a graph is at most*

$$\frac{1}{2}\nu\left(\frac{\epsilon}{\nu - 1}\right)^{\nu-1}.$$

Theorem 3.8 (Lovász) : *If G has a hamiltonian path, $\nu \geq 41$ and either $\delta \geq 4$ or $d \geq 24/5$, then G is edge reconstructible.*

Sketch of Proof: If $\delta \geq 4$ then we may assume that $d \geq 24/5$, otherwise G is edge reconstructible by Lemma 3.2. So in either case we have $d \geq 24/5$, which is equivalent to $\epsilon \geq 12/5\nu$, since $d = 2\epsilon/\nu$. Let P be a hamiltonian path in G. Then by Lemma 3.7,

$$|P \to G| = |\mathrm{Aut}P|s(P,G) \leq \nu\left(\frac{\epsilon}{\nu - 1}\right)^{\nu-1}.$$

Since $\nu \geq 41$, $\epsilon \geq 12\nu/5$ and $\epsilon(P) = \nu - 1$, it can be shown that $|P \to G| < 2^{\epsilon(G)-\epsilon(P)-1}$, and thus $G \in \mathcal{ER}$ by Lemma 3.1. ∎

The above proof can in fact be modified to show that if d_0 is any fixed number greater than 4, then all graphs which have a hamiltonian path, $d \geq d_0$, and sufficiently many vertices are edge reconstructible.

Recently Pyber [28] has shown that the degree condition can be dropped. He proves a result which applies to a much larger class of graphs than just those with hamiltonian paths. To state it, we need some definitions. A *covering path system* for a graph is a set of vertex-disjoint paths containing all vertices of the graph. The *path covering number* $p(G)$ is the minimum number of paths in any path covering system for G. Obviously G has a hamiltonian path if and only if $p(G) = 1$.

To prove his result, Pyber used the following estimate on the number of hamiltonian cycles in a graph.

Lemma 3.9: *The number of hamiltonian cycles in a graph is at most* $c^{\epsilon-\nu}$, *where* $c = 2^{7/8}.6^{1/24} \simeq 1.976$.

Pyber uses this to give two results, the first for graphs in general, and the second a stronger result for graphs with $\delta \geq 2$. As it is easier to describe the proof in the case $\delta \geq 2$, and as all hamiltonian graphs have $\delta \geq 2$ anyway, we give only the second result. The reader may consult [28] for the first.

Theorem 3.10 (Pyber [28]) : *There exists a constant* α *so that if* $\delta \geq 2$ *and* $p \leq \alpha\nu/\log_e \nu$ *then a graph is edge reconstructible.*

Sketch of Proof: The proof assumes that we have a graph G satisfying the inequalities but which is not edge reconstructible. Since $\delta \geq 2$ it is easy to see that no two degree 2 vertices of G can be adjacent. This implies that $\epsilon \geq (5\nu - p)/4$ by a simple counting argument. Suppose that P is a spanning subgraph of G corresponding to a minimum path covering system. By examining the number of ways to add edges to G so that P becomes part of a hamiltonian cycle, and by using Theorem 3.10, it can be shown that $|P \to G| \leq 2p\nu^{2p}c^{\epsilon-\nu}$. The inequalities $\epsilon \geq (5\nu - p)/4$ and $p \leq \alpha\nu/\log_e \nu$ (where α is sufficiently large) can now be used to show that $|P \to G| < 2^{\epsilon(G)-\epsilon(P)-1}$, giving a contradiction to Lemma 3.1. Therefore no such graph G exists, proving the theorem. ∎

For graphs with hamiltonian paths, we have $p = 1$, and the theorem tells us that any sufficiently large graph with a hamiltonian path and $\delta \geq 2$ will be edge reconstructible. In

particular, any sufficiently large hamiltonian graph will be edge reconstructible. From the proof above, the number of vertices needed to guarantee that a hamiltonian graph is edge reconstructible is about 5800.

Both the degree condition and hamiltonian graph results depend on counting theorems (Lemmas 3.3, 3.7 and 3.9) to provide information which is used in collaboration with the results from Section 2. Our next example uses structural information about graphs in a more direct fashion.

Claw-free graphs

A graph is *claw-free* if it has no induced subgraph isomorphic to $K_{1,3}$. Ellingham, Pyber and Yu [8] have recently shown that all claw-free graphs satisfy the Edge Reconstruction Conjecture. The proof includes the application of Corollary 2.2 (a) to show that a claw-free graph which contains a subgraph isomorphic to K_4 is edge reconstructible. In fact, the following result can be proved for *$K_{1,k}$-free graphs*, graphs with no induced subgraph isomorphic to $K_{1,k}$.

Lemma 3.11: *If G is $K_{1,k}$-free and contains a subgraph isomorphic to K_{k+1} then $G \in \mathcal{ER}$.*

Proof: Suppose $u, v_1, v_2, ..., v_k$ are the vertices of a K_{k+1} in G. Let F be the spanning subgraph of G with $E(F) = \{uv_1, uv_2, ..., uv_k\}$ if $\epsilon(G) - k$ is even, or $E(F) = \{uv_1, uv_2, ..., uv_k, e\}$ if $\epsilon(G) - k$ is odd, where e is an edge not contained in the K_{k+1}. (If no such e exists then $G \cong K_{k+1}$ is edge reconstructible anyway.) Suppose that $|G \to G|_F \neq 0$. Then for some bijection $\pi : V(G) \to V(G)$, $\pi(G) \cap G = \pi(F)$. Thus $G \cap \pi^{-1}(G) = F$. But then the subgraph induced in $\pi^{-1}(G)$ by $\{u, v_1, v_2, ..., v_k\}$ must be isomorphic to $K_{1,k}$, which is impossible since $\pi^{-1}(G)$ is isomorphic to G. Therefore $|G \to G|_F = 0$ and $G \in \mathcal{ER}$ by Corollary 2.2 (a). ∎

As a corollary we obtain the following.

Corollary 3.12: *If G is $K_{1,k}$-free and Δ is greater than or equal to the Ramsey number $r(k,k)$, then $G \in \mathcal{ER}$.*

Proof: Suppose that v has degree Δ, and let N be the subgraph induced by the neighbours of v, $\nu(N) = \Delta$. Since G is $K_{1,k}$-free, N contains no induced $\overline{K_k}$. However, since $\nu(N) \geq$

$r(k,k)$, N must therefore contain a subgraph isomorphic to K_k, which together with v forms a subgraph isomorphic to K_{k+1} in G. Thus $G \in \mathcal{ER}$ by Lemma 3.11. ∎

In a very recent paper [**16**] Krasikov and Pyber claim to have improved Corollary 3.12 significantly, by showing that there is a constant C so that any $K_{1,k}$-free graph with $\Delta \geq Ck \log_e k$ is edge reconstructible.

Theorem 3.13 (Ellingham, Pyber and Yu [**8**]) : *Every claw-free graph of size at least four is edge reconstructible.*

Sketch of Proof: Since $r(3,3)$ is well-known to be 6, by Corollary 3.12 we may assume that $\Delta \leq 5$. It can be shown that any claw-free graph which is not edge reconstructible must have at most

$$1 + \Delta + \lfloor \frac{\Delta}{2} \rfloor + \lfloor \frac{\Delta}{4} \rfloor + \lfloor \frac{\Delta}{8} \rfloor + \ldots$$

vertices. Since $\Delta \leq 5$, this implies that such a graph has at most nine vertices. But by the result of McKay mentioned in Section 1, all such graphs are edge reconstructible. ∎

As mentioned earlier, Caro, Krasikov and Roditty [**7**] have some degree conditions for $K_{1,k}$-free graphs to be edge reconstructible.

4. Algebraic techniques

In this section we examine some algebraic techniques which have been used for the edge and vertex-switching reconstruction problems. The first use of an algebraic method to solve a reconstruction problem was by Stanley [**29**], who used an argument involving properties of Peck posets to prove the result of Lovász (Corollary 2.3). The essence of his argument was to define vector spaces composed of formal linear combinations of graphs, and then show that the linear transformation corresponding to the construction of the edge deck is injective. Peck posets merely provided a framework in which the injectivity of this linear transformation followed from known results. Stanley later used the same idea of representing graph operations as linear transformations to prove his vertex-switching result.

Stanley's edge reconstruction argument was generalised by Godsil, Krasikov and Roditty [**11**] to the k-edge reconstruction problem, where a graph is to be reconstructed from the subgraphs obtained by deleting k edges at a time. They obtained extensions not only of the result of Lovász, but also of the result of Müller. The extension of Lovász's

result to the k-edge reconstruction problem was also obtained independently by Ponzano [**27**]. Alon, Caro, Krasikov and Roditty [**1**] showed that the extension of Müller's result is in fact valid for edgesets, not just for graphs, and also obtained a generalisation of one of the corollaries to Nash-Williams' theorem, Corollary 2.2 (a).

We shall give the edgeset version of Godsil, Krasikov and Roditty's results. Let us first formally define the problem. Given a set E of edges and a group Π acting on E as in Section 2, define the *k-edge deck* of an edgeset A to be the multiset

$$D_{\mathrm{E}}^k(A) = \{[A - D] \mid D \subseteq A, |D| = k\}.$$

We say that A is k-edge-reconstructible if $D_{\mathrm{E}}^k(A) = D_{\mathrm{E}}^k(B) \Rightarrow A \cong B$, and write $A \in \mathcal{ER}^k$. Notice that $\mathcal{ER} = \mathcal{ER}^1 \supseteq \mathcal{ER}^2 \supseteq \mathcal{ER}^3 \supseteq \ldots$.

Suppose that $n \leq m \leq M = |E|$, and let H_{nm} be the $\binom{M}{n} \times \binom{M}{m}$ matrix $[h_{CA}]$, whose rows are indexed by the edgesets C of size n and whose columns are indexed by the edgesets A of size m, with

$$h_{CA} = \begin{cases} 1 & \text{if } C \subseteq A, \\ 0 & \text{otherwise.} \end{cases}$$

This matrix is known as the *incidence matrix* of m-subsets of E with n-subsets of E, and has been widely studied in design theory. It is known to have the following properties.

Lemma 4.1 (see [**12**] for (a); (b) is due to Frankl and Pach [**10**]) :

(a) *H_{nm} has full rank; in other words its rank is* $\min\{\binom{M}{n}, \binom{M}{m}\}$.

(b) *If $H_{nm}x = 0$ and $x \neq 0$ then x has at least 2^{n+1} nonzero entries.*

For any edgeset A of size m define an $\binom{M}{m}$-vector x^A indexed by the edgesets B of size m, where

$$x_B^A = |B \to A|.$$

Then x^A has an entry of 0 for each B not isomorphic to A, and an entry of $|\mathrm{Aut}A|$ for each B isomorphic to A. Hence $x^A = x^B$ if and only if $A \cong B$. For any edgeset C of size n, the C entry of $H_{nm}x^A$ is equal to

$$\begin{aligned} \sum_{B \subseteq E, |B| = m} h_{CB} x_B^A &= \sum_{C \subseteq B \subseteq E, |B| = m} |B \to A| \\ &= |\{\pi \in \Pi \mid \pi(B) = A \text{ for some } B \supseteq C\}| \\ &= |\{\pi \in \Pi \mid \pi(C) \subseteq A\}| \\ &= |C \to A|. \end{aligned}$$

Now we can state and prove the following result, which generalises Corollaries 2.3 and 2.4.

Theorem 4.2 (Godsil, Krasikov and Roditty [**11**]) : *An edgeset $|A|$ is k-edge reconstructible if*

(a) $|A| \geq (|E| + k)/2$, *or*

(b) $2^{|A|-k} > |\Pi|$.

Proof: Suppose that $D^k_{\mathrm{E}}(A) = D^k_{\mathrm{E}}(B)$, and let $m = |A| = |B|$. Then for every $C \subseteq E$ with $|C| = m-k$, $s(C, A) = s(C, B)$ and hence $|C \to A| = |C \to B|$. Consider the entry of $H_{m-k,m}(x^A - x^B)$ corresponding to C. From above, this is equal to $|C \to A| - |C \to B|$, which is zero. Therefore $H_{m-k,m}\,(x^A - x^B) = 0$. We show that if one of (a) or (b) holds then $x^A - x^B = 0$, and thus $A \cong B$.

(a) If $m = |A| \geq (|E| + k)/2 = (M + k)/2$ then $\binom{M}{m} \leq \binom{M}{m-k}$, and therefore the rank of $H_{m-k,m}$ is just its number of columns $\binom{M}{m}$, by Lemma 4.1 (a). Thus the linear transformation taking x to $H_{m-k,m}\,x$ is injective, and so $x^A - x^B = 0$.

(b) The vector x^A has exactly $|\Pi|/|\mathrm{Aut}A| \leq |\Pi|$ nonzero entries, and similarly x^B has at most $|\Pi|$ nonzero entries. Thus $x^A - x^B$ has at most $2|\Pi|$ nonzero entries. Therefore, if $2^{m-k} = 2^{|A|-k} > |\Pi|$, then $x^A - x^B$ has less than 2^{m-k+1} nonzero entries, and so must be zero by Lemma 4.1 (b). ∎

The graph version of this theorem states that a graph is k-edge reconstructible if $\epsilon \geq (\binom{\nu}{2} + k)/2$ or $2^{\epsilon-k} > \nu!$. As with the results in Section 2, this theorem can also be applied to a variety of other combinatorial reconstruction problems.

Our proof of Theorem 4.2 (a) involved the use of Lemma 4.1 (a) to prove that the linear transformation defined by $H_{m-k,m}$ was injective. Other results from algebra can be used instead of Lemma 4.1 (a), such as results from the theory of Peck posets (as in Stanley's proof of Lovász's result, mentioned earlier), or results from the theory of representations of abstract commutative algebras, as used by Ponzano [**27**].

Another method which has been applied by Krasikov and Roditty [**17**, **18**, **19**] to various reconstruction problems, including edge reconstruction, is the method of *balance equations*. This technique can be used to prove Theorem 4.1 (a), and also Stanley's result on vertex-switching reconstruction. It involves using counting methods to establish a system of linear equations (the balance equations) involving variables δ_i. Given objects S and T with the same deck, δ_i is defined to be $X_i(S \to S) - X_i(S \to T)$, where $X_i(S \to T)$ denotes the number of ways to obtain objects isomorphic to T from S by some change involving i subobjects. For example, for edge reconstruction $X_i(G \to H)$ is the number of

ways to obtain a graph isomorphic to H by deleting i edges of G and then replacing them with i *different* edges. If the set of balance equations can be shown to be inconsistent when S and T are nonisomorphic, then this proves that S is reconstructible.

5. Final comments

We hope we have shown that Lovász's result from 1972 has been the inspiration for a wide variety of results in the area of edge reconstruction. Much still remains to be done in this area, and there are surely applications of the methods discussed in this paper which are yet to be discovered. In our opinion, two promising avenues of research are finding applications of Nash-Williams' theorem to edge reconstruction of various classes of graphs, as illustrated in Section 3, and finding applications of the results of Avellis and Borzacchini mentioned at the end of Section 2. The generality of the results in Sections 2 and 4 serves as an indication that inclusion-exclusion and algebraic techniques will be either powerful enough to solve many reconstruction problems simultaneously, or too weak to solve any of them completely. If the latter is the case, then the best hope for solving the edge reconstruction problem for graphs lies with using a combination of techniques, as was done with the examples in Section 3.

Acknowledgements

The author wishes to thank R. G. Stanton and the organisers of the Seventeenth Manitoba Conference on Numerical Mathematics and Computing for inviting him to give the address which resulted in this paper, and J. A. Bondy for informing him of the results in [**27**].

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