Quadrangular embeddings of complete graphs*

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Abstract

Hartsfield and Ringel proved that a complete graph K_n has an orientable quadrangular embedding if $n \equiv 5 \pmod 8$, and has a nonorientable quadrangular embedding if $n \geq 9$ and $n \equiv 1 \pmod 4$. We complete the characterization of complete graphs admitting quadrangular embeddings by showing that K_n has an orientable quadrilateral embedding if $n \equiv 0 \pmod 8$, and has a nonorientable quadrilateral embedding if $n \equiv 0 \pmod 4$. We also determine the order of minimal quadrangulations for some surfaces where the corresponding graphs are not complete.

Keywords: quadrangular embedding, complete graph, minimal quadrangulation.

1 Introduction

In this paper surfaces are compact 2-manifolds without boundary. The connected orientable surface of genus h is denoted S_h, and the connected nonorientable surface of genus k is denoted N_k. A disconnected surface is considered orientable if all of its components are orientable. The Euler characteristic of a surface Σ is denoted $\chi(\Sigma)$, which is $2-2h$ for S_h, and $2-k$ for N_k. All embeddings of graphs in surfaces are cellular (every face is homeomorphic to an open disk). For convenience we often identify a face by referring to its bounding cycle or bounding closed walk. Graphs may have loops or multiple edges; *simple* graphs have neither.

An embedding of a graph G in a surface Σ is *quadrangular*, or a *quadrangulation of* Σ, if every face is bounded by a 4-cycle. A *quadrilateral* is a face bounded by a 4-cycle. A quadrangulation of Σ

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is *minimal* if its underlying graph is simple and connected, and there is no quadrangular embedding of a simple graph of smaller order on Σ.

Similarly, a *triangulation* of a surface Σ is an embedding of a graph in Σ such that every face is bounded by a 3-cycle. Triangular embeddings of simple graphs are always minimum genus embeddings. Conditions for the existence of triangular embeddings of complete graphs were determined as part of the well-known Map Color Theorem (see [20]), which extended the Four Color Theorem to other surfaces. Subsequently there were a number of results providing lower bounds on the number of nonisomorphic triangular embeddings of K_n for certain families of n; see for example [1, 5, 6, 14].

Much less work has been done on quadrangular embeddings of complete graphs, which is our subject here. We begin with some simple observations. The first follows from Euler's formula and the fact that a quadrangulation with m edges has $m/2$ faces. The second follows from the first, the fact that $m = \binom{n}{2}$ for K_n, and because $\chi(\Sigma)$ is an integer, which is even when Σ is orientable.

Observation 1.1. (i) *If a connected graph G with n vertices and m edges has a quadrangular embedding on a surface Σ, then $m = 2n - 2\chi(\Sigma)$.*

(ii) *If K_n has a quadrangular embedding on Σ then $\chi(\Sigma) = n(5-n)/4$. Hence, if Σ is orientable then $n \equiv 0$ or $5 \pmod 8$ and if Σ is nonorientable then $n \equiv 0$ or $1 \pmod 4$.*

Hartsfield and Ringel [10, 11] obtained the following results, each of which covers half of the possible values of n for which quadrangular embeddings of K_n may exist, from Observation 1.1(ii).

Theorem 1.2 (Hartsfield and Ringel [10])**.** *A complete graph K_n with $n \equiv 5 \pmod 8$ admits a quadrangular embedding in an orientable surface.*

Theorem 1.3 (Hartsfield and Ringel [11])**.** *A complete graph K_n with $n \geq 9$ and $n \equiv 1 \pmod 4$ admits a quadrangular embedding in a nonorientable surface. However, there is no such embedding of K_5 (or of K_1).*

Hartsfield and Ringel also showed that the *generalized octahedron O_{2k}* (K_{2k} with a perfect matching removed) has orientable and nonorientable quadrangular embeddings for $k \geq 3$. In the orientable case this is a special case of an older result of White [21] discussed in Subsection 2.2 below.

Using current graphs, Korzhik and Voss [14] constructed exponentially many nonisomorphic orientable quadrangular embeddings of K_{8s+5} for $s \geq 1$, and Korzhik [13] constructed superexponentially many nonisomorphic orientable and nonorientable quadrangular embeddings of K_{8s+5} for $s \geq 2$. Grannell and McCourt [7] constructed many nonisomorphic orientable embeddings of complete graphs K_n with faces bounded by $4k$-cycles for $k \geq 2$, when $n = 8ks + 4k + 1$ for $s \geq 1$.

None of the above results, however, address the cases left open by Hartsfield and Ringel. We settle those cases, as follows.

Theorem 1.4. *A complete graph K_n with $n \equiv 0 \pmod 8$ admits a quadrangular embedding in an orientable surface.*

Theorem 1.5. *A complete graph K_n with $n \equiv 0 \pmod 4$ admits a quadrangular embedding in a nonorientable surface.*

Hartsfield [9, Lemmas 1 and 3] also claims to prove Theorem 1.5, by modifying a nonorientable quadrangular embedding of K_n to obtain an embedding of K_{n+8}. She provides the base cases, for K_8 and K_{12}, and gives a specific construction of an embedding of K_{16} from an embedding of K_8. However, the details of her construction for general $n \equiv 0 \pmod 4$ are not given, and do not appear to be straightforward. While her approach seems valid, her work cannot be regarded as a complete proof.

Before proving Theorems 1.4 and 1.5, we discuss some graph operations and constructions for embeddings in Section 2. Theorem 1.4 is proved in Section 3, using graphical surfaces and voltage graphs. We actually prove more general results on quadrangular embeddings of composition graphs $G[K_4]$. Theorem 1.5 is proved in Section 4, using the diamond sum operation. Our proof also gives a new proof for the existence result in Theorem 1.3. In Section 5 we use earlier sections to give some results on minimal quadrangulations. Section 6 gives some final remarks.

Combining our results with Theorems 1.2 and 1.3 gives a complete characterization.

Theorem 1.6. *The complete graph K_n has a quadrangular embedding in an orientable surface if and only if $n \equiv 0$ or $5 \pmod 8$, and in a nonorientable surface if and only if $n \equiv 0$ or $1 \pmod 4$ and $n \neq 1, 5$.*

2 Preliminaries

2.1 Graph operations

Let G and H be simple graphs. The *composition* (or *lexicographic product*) of G and H, denoted $G[H]$, has vertex set $V(G) \times V(H)$, with two vertices (v_1, u_1) and (v_2, u_2) adjacent if and only if either (i) $v_1v_2 \in E(G)$ or (ii) $v_1 = v_2$ and $u_1u_2 \in E(H)$. For example, $K_n[K_2]$ is the complete graph K_{2n} and $K_2[\overline{K_n}]$ is the complete bipartite graph $K_{n,n}$, where $\overline{K_n}$ represents a graph with n vertices and no edges. The *join* of G and H, denoted $G + H$, is the union of G and H together with one edge uv for each $u \in V(G)$ and $v \in V(H)$. For example, $K_4 + K_n$ is the complete graph K_{n+4}.

2.2 Graphical surfaces

White [21] showed that any composition $G[\overline{K_2}]$, where G is a simple graph without isolated vertices, has an orientable quadrangular embedding. Craft [3, 4] developed graphical surfaces, which yield a simple proof of this result. We outline his proof, since we need his construction in Section 3.

For a graph G, the *graphical surface* $S(G)$ derived from G is a surface obtained from an embedding of G in $\mathbb{R}^3$ by blowing up every vertex u into a sphere Σ_u and replacing every edge uv by a tube T_{uv} joining the spheres Σ_u and Σ_v. Since we work in $\mathbb{R}^3$, the resulting surface $S(G)$ is orientable.

Lemma 2.1 (Craft [3, 4])**.** *Let G be a simple graph without isolated vertices. Then $G[\overline{K_2}]$ has a quadrangular embedding on the graphical surface $S(G)$.*

Outline of proof. We embed $G[\overline{K_2}]$ in $S(G)$ as follows. For any vertex $u \in V(G)$, let u_{N} (north pole) and u_{S} (south pole) be two points in the sphere Σ_u; they represent the two vertices of $G[\overline{K_2}]$

corresponding to u. We may assume that all tubes joined to Σ_u are joined in some cyclic order around the equator of Σ_u. There are four edges of $G[\,\overline{K_2}\,]$ corresponding to each $uv \in E(G)$, which are $u_\mathrm{N}v_\mathrm{N}, u_\mathrm{N}v_\mathrm{S}, u_\mathrm{S}v_\mathrm{S}$ and $u_\mathrm{S}v_\mathrm{N}$. These can all be embedded along T_{uv} (there are two different ways to do this, but for our purposes it will not matter which is used). In the resulting embedding, every edge is contained in two quadrilaterals. For example, $u_\mathrm{S}v_\mathrm{N}$ is contained in quadrilaterals $Q_u = (t_X u_\mathrm{N} v_\mathrm{N} u_\mathrm{S})$ and $Q_v = (u_\mathrm{S} v_\mathrm{N} w_Y v_\mathrm{S})$ where $tu, vw \in E(G)$ and $X, Y \in \{\mathrm{N}, \mathrm{S}\}$. Note that if u has degree 1, then $t_X = v_\mathrm{S}$. □

White and Craft dealt only with orientable embeddings. However, we can also produce nonorientable embeddings. Given a graphical surface, we can replace a tube T_{uv} by a *twisted tube* $\widetilde{T}_{uv}$ by taking a simple closed curve γ around T_{uv} with a specified positive direction, cutting along it to produce two boundary curves γ_1 and γ_2, then re-identifying γ_1 with γ_2 so that the positive direction along γ_1 corresponds to the negative direction along γ_2 (this cannot be done in $\mathbb{R}^3$). We can still embed the four edges between $\{u_\mathrm{N}, u_\mathrm{S}\}$ and $\{v_\mathrm{N}, v_\mathrm{S}\}$ along $\widetilde{T}_{uv}$; they become orientation-reversing edges relative to the original orientation at each vertex. Depending on which tubes we replace, the resulting embedding may be nonorientable.

Lemma 2.2. *Let G be a simple graph with no isolated vertices and at least one cycle. Then $G[\,\overline{K_2}\,]$ has a quadrangular embedding on a nonorientable modified graphical surface $\widetilde{S}(G)$.*

Proof. Choose one edge uv belonging to a cycle and replace the tube T_{uv} by a twisted tube $\widetilde{T}_{uv}$ in the construction of Lemma 2.1. The resulting embedding is nonorientable because the cycle $(uvw\ldots z)$ in G gives an orientation-reversing cycle $(u_\mathrm{N}v_\mathrm{N}w_\mathrm{N}\ldots z_\mathrm{N})$ in the embedding of $G[\,\overline{K_2}\,]$ on the new surface $\widetilde{S}(G)$. □

2.3 Voltage graphs

We assume the reader is familiar with voltage graph constructions for embeddings. We summarize the main features; see [8] for more details.

Given a graph G, assign an arbitrary *plus direction* to each edge. A function α from the plus-directed edges of G to a group Γ is an *ordinary voltage assignment* on G. The pair $\langle G, \alpha\rangle$ is called an *ordinary voltage graph*. The *derived graph* G^α has vertex set $V(G) \times \Gamma$ and an edge from $u_a = (u, a)$ to $v_b = (v, b)$ whenever uv is a plus-directed edge in G and $b = a \cdot \alpha(uv)$.

If G has an embedding Φ, represented by a rotation of edges at each vertex and edge signatures, then G^α has a *derived embedding* Φ^α: for each $u_a \in V(G^\alpha)$ use the natural bijection between edges incident with u in G and edges incident with u_a in G^α to define the rotation at u_a from the rotation at u, and give each edge $u_a v_b$ of G^α the signature of the corresponding edge uv in G.

For each walk $W = v_0 e_1 v_1 e_2 v_2 \ldots e_k v_k$ in G define its *total voltage* to be $\alpha(e_1)^{\epsilon_1}\alpha(e_2)^{\epsilon_2}\ldots\alpha(e_k)^{\epsilon_k}$ where ϵ_i is $+1$ if W uses e_i in the plus direction and -1 otherwise. The faces of Φ^α come from the faces of Φ: each face in Φ with degree k whose boundary walk has total voltage of order r in Γ yields $|\Gamma|/r$ faces of degree kr in Φ^α. Also, Φ^α is nonorientable if and only if Φ is nonorientable and has an orientation-reversing closed walk whose total voltage is the identity of Γ.

2.4 The diamond sum

Let G and G' be two simple graphs with embeddings Φ and Φ' on disjoint surfaces Σ and Σ', respectively. Suppose that $k \geq 1$ and both G and G' have a vertex of degree k, say v and v' respectively. Let v have neighbors $v_0, v_1, ..., v_{k-1}$ in cyclic order around v in Φ, and let v' have neighbours $v'_0, v'_1, ..., v'_{k-1}$ in cyclic order around v' in Φ'. There is a closed disk D that intersects G in v and the edges $vv_0, vv_1, ..., vv_{k-1}$, and so that the boundary of D intersects G at $v_0, v_1, ..., v_{k-1}$. Similarly, there is a closed disk D' that intersects G' in v' and the edges $v'v'_0, v'v'_1, ..., v'v'_{k-1}$ and so that the boundary of D' intersects G' at $v'_0, v'_1, ..., v'_{k-1}$. Remove the interiors of D and D', and identify their boundaries so that v_i is identified with v'_i for $0 \leq i \leq k-1$. The resulting embedding is called a *diamond sum of* Φ *and* Φ' *at* v *and* v', denoted $\Phi \diamond_{v,v'} \Phi'$ or just $\Phi \diamond \Phi'$. Its graph is denoted $G \diamond G'$ and the surface is the connected sum $\Sigma \# \Sigma'$. Note that $\Phi \diamond \Phi'$ is orientable if and only if both Φ and Φ' are orientable.

The diamond sum was first used by Bouchet [2] in dual form to derive a new proof of the minimum genus of $K_{m,n}$. Bouchet's construction was later reinterpreted in more general situations in [12, 16, 17]. Our construction in Section 4 relies on the following observation.

Observation 2.3. *The diamond sum of two quadrangular embeddings is quadrangular.*

3 Embeddings from graphical surfaces and voltage graphs

In this section, we use graphical surfaces and voltage graphs to construct both orientable and nonorientable quadrangular embeddings of certain graphs of the form $G[K_4]$. Theorem 1.4 is a special case.

Theorem 3.1. *Let G be a connected simple graph with a perfect matching. Then $G[K_4]$ has an orientable quadrilateral embedding.*

Proof. Let G have perfect matching M, and let $S(G)$ be the graphical surface derived from G.

First, construct a quadrangular embedding Θ of $H = G[\,\overline{K_2}\,]$ on $S(G)$ as in Lemma 2.1. For each vertex $v \in V(G)$ there are two vertices $v_{\mathrm{N}}, v_{\mathrm{S}} \in V(H)$. For each $uv \in E(G)$, there is a tube T_{uv} in $S(G)$, along which run the edges $u_{\mathrm{N}}v_{\mathrm{N}}, u_{\mathrm{N}}v_{\mathrm{S}}, u_{\mathrm{S}}v_{\mathrm{S}}$ and $u_{\mathrm{S}}v_{\mathrm{N}}$ of H. Each edge $u_P v_Q$ of H belongs to two quadrilaterals of the form $(u_{\mathrm{N}} v_Q u_{\mathrm{S}} t_X)$ and $(u_P v_{\mathrm{N}} w_Y v_{\mathrm{S}})$ where $tu, vw \in E(G)$ and $P, Q, X, Y \in \{\mathrm{N}, \mathrm{S}\}$.

Modify the embedding Θ by splitting each edge into a digon (2-cycle) bounding a face. Let Ψ be the new embedding, with underlying graph J. The other faces of Ψ are quadrilaterals, in one-to-one correspondence with the quadrilaterals of Θ. We now assign voltages from the group $\mathbb{Z}_2$; since all elements of $\mathbb{Z}_2$ are self-inverse, the designation of plus directions for edges does not matter. Choose a voltage assignment $\alpha : E(J) \to \mathbb{Z}_2$ so that the voltages of the edges of J around each tube T_{uv} alternate between 0 and 1. Then each digon of Ψ has one edge of voltage 0 and one edge of voltage 1. Each quadrilateral of Ψ, which uses edges from two tubes T_{uv} and T_{vw}, has an edge of voltage 0 and an edge of voltage 1 on T_{uv}, and similarly for T_{vw}. Therefore, every digon has total voltage 1 and every quadrilateral has total voltage 0 in $\langle J, \alpha \rangle$. Thus, the derived embedding Ψ^α, with underlying graph $J^\alpha = H[\,\overline{K_2}\,] = G[\,\overline{K_2}\,][\,\overline{K_2}\,] = G[\,\overline{K_4}\,]$, is an orientable quadrangulation.

We could also have obtained a quadrangular embedding of $G[\overline{K_4}] = G[\overline{K_2}][\overline{K_2}]$ directly from Lemma 2.1, but that would not have had the special structure which we now exploit to obtain an embedding of $G[K_4]$. We work with the vertices of G in pairs specified by the perfect matching M.

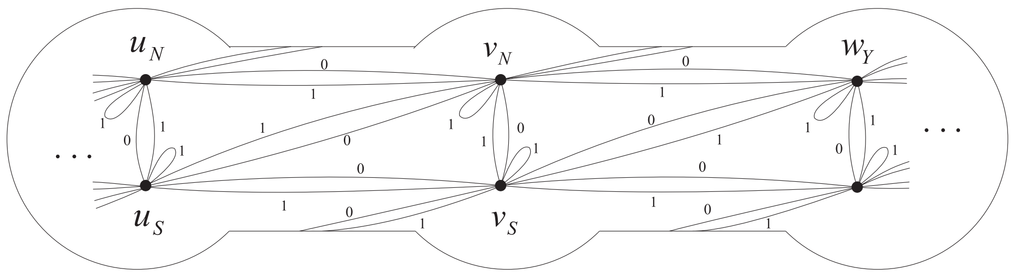

Figure 1: Votage graph $\langle J_1, \alpha_1 \rangle$ generated from graphical surface of $G[\overline{K_2}]$.

For each $uv \in M$, let e be one of the four edges of H on the tube T_{uv}, between $\{u_{\mathrm{N}}, u_{\mathrm{S}}\}$ and $\{v_{\mathrm{N}}, v_{\mathrm{S}}\}$. We choose $e = u_{\mathrm{S}}v_{\mathrm{N}}$, as this makes it easier to illustrate what is happening (see Figure 1). In Θ, $e = u_{\mathrm{S}}v_{\mathrm{N}}$ belongs to two quadrilaterals $Q_u = (t_X u_{\mathrm{N}} v_{\mathrm{N}} u_{\mathrm{S}})$ and $Q_v = (u_{\mathrm{S}} v_{\mathrm{N}} w_Y v_{\mathrm{S}})$ where $tu, vw \in E(G)$ and $X, Y \in \{\mathrm{N}, \mathrm{S}\}$. Let the two edges of the digon in J corresponding to e be e_1 and e_2, where e_1 belongs to the quadrilateral Q'_u of Ψ corresponding to Q_u and e_2 belongs to the quadrilateral Q'_v of Ψ corresponding to Q_v. Add a digon of two edges d_1 and d_2 in Q'_u between u_{N} and u_{S}, and a digon of two edges d_3 and d_4 in Q'_v between v_{N} and v_{S}, so we have four triangles $T_1(u) = (t_X u_{\mathrm{N}} u_{\mathrm{S}})$ using d_1, $T_2(u) = (u_{\mathrm{N}} v_{\mathrm{N}} u_{\mathrm{S}})$ using d_2, $T_1(v) = (u_{\mathrm{S}} v_{\mathrm{N}} v_{\mathrm{S}})$ using d_3 and $T_2(v) = (v_{\mathrm{S}} v_{\mathrm{N}} w_Y)$ using d_4. Assign voltage 1 to d_2 and d_3, and 0 to d_1 and d_4. Insert a loop with voltage 1 at each of $u_{\mathrm{N}}, u_{\mathrm{S}}, v_{\mathrm{N}}$ and v_{S} and put these loops in the four different triangles $T_1(u), T_2(u), T_1(v)$ and $T_2(v)$, respectively.

Everything up to this point could have been done using independently chosen quadrilaterals Q'_u containing u_{N} and u_{S} and Q'_v containing v_{N} and v_{S}. However, the total voltages for the degree 4 faces containing the loops at u_{S} and v_{N} are currently 1, so they will not generate quadrilaterals in the derived embedding. To fix this, swap the voltages on e_1 and e_2: this is where we use the pairing of vertices via M. Let Ψ_1, J_1 and α_1 be the final embedding, graph and voltage assignment, as shown in Figure 1.

In Ψ_1 there are four types of faces. Each face bounded by a digon has total voltage 1 in $\langle J_1, \alpha_1 \rangle$, and each quadrilateral and face of degree 4 containing a loop has total voltage 0. These three types of faces all lift to quadrilaterals in $\Psi_1^{\alpha_1}$. The final type of face is bounded by a loop of total voltage 1. This lifts to a face in $\Psi_1^{\alpha_1}$ bounded by a digon between $(u_X, 0)$ and $(u_X, 1)$, where $u \in V(G)$ and $X \in \{\mathrm{N}, \mathrm{S}\}$. Replacing each such digon in $\Psi_1^{\alpha_1}$ by a single edge generates the required orientable quadrangular embedding of $G[K_4]$. □

Proof of Theorem 1.4. Write $n = 8k$ and take $G = K_{2k}$, which has a perfect matching. Then $G[K_4] = K_n$ has an orientable quadrangular embedding by Theorem 3.1. □

We can also obtain a nonorientable version of Theorem 3.1. This provides an alternative proof of Theorem 1.5 in the case where $n \geq 16$ and $n \equiv 0 \pmod 8$.

Theorem 3.2. *Let G be a simple graph with a perfect matching and a cycle. Then $G[K_4]$ has a nonorientable quadrilateral embedding.*

Proof. Use Lemma 2.2 instead of Lemma 2.1 in the proof of Theorem 3.1. Replacing one or more tubes by twisted tubes does not affect the argument. Take the orientation-reversing cycle $C = (u_{\mathrm{N}}v_{\mathrm{N}}w_{\mathrm{N}}\ldots z_{\mathrm{N}})$ in $H = G[\overline{K_2}]$ from the proof of Lemma 2.2 and replace each edge of C by the edge of voltage 0 in the corresponding digon of J_1. This gives an orientation-reversing cycle of total voltage 0 in $\langle J_1, \alpha_1\rangle$, so the final embedding is nonorientable. □

4 Embeddings from diamond sums

In this section, we discuss quadrangular embeddings of complete graphs on nonorientable surfaces. Our construction is based on quadrangular embeddings of complete bipartite graphs and of K_7^+, the graph obtained from K_7 by subdividing an edge. A quadrangular embedding of K_7^+ in N_5 is shown in Figure 2.

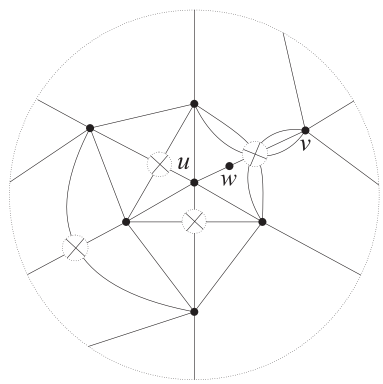

Figure 2: A quadrangular embedding of K_7^+ in N_5 (subdividing uv of K_7 by w).

As part of determining the orientable and nonorientable genera of $K_{m,n}$, Ringel showed that $K_{m,n}$ has quadrangular embeddings in certain cases. Bouchet [2] later provided a simpler proof.

Lemma 4.1 (Ringel [18, 19])**.** *The complete bipartite graph $K_{m,n}$ has an orientable quadrangular embedding whenever $(m-2)(n-2) \equiv 0 \pmod 4$ and $\min\{m,n\} \geq 2$, and a nonorientable quadrangular embedding whenever $mn \equiv 0 \pmod 2$ and $\min\{m,n\} \geq 3$.*

Lemma 4.2. *If a complete graph K_n admits an orientable or nonorientable quadrangular embedding, then K_{n+4} admits a nonorientable quadrangular embedding.*

Proof. Assume that K_7^+ is obtained from K_7 by subdividing an edge uv to create a degree 2 vertex w, as in Figure 2. We may interpret K_7^+ as a join $(K_1 \cup K_5) + \overline{K_2}$ where w is the vertex of the K_1, and u and v are the vertices of the $\overline{K_2}$. Let Φ_1 be the quadrangular embedding of K_7^+ on N_5 from Figure 2. We build the embedding of K_{n+4} in two steps from Φ_1, an orientable or nonorientable quadrangular embedding Φ_2 of $K_{6,n-1}$ from Lemma 4.1, and the assumed quadrangular embedding Φ_3 of K_n.

Let u' be a vertex of $K_{6,n-1}$ of degree 6. Apply the diamond sum to the embeddings Φ_1 and Φ_2 at vertices u and u'. By Observation 2.3 the resulting embedding Φ_4 is quadrangular; it is also nonorientable since Φ_1 is nonorientable. The underlying graph $K_7^+ \diamond K_{6,n-1}$ is $(K_1 \cup K_5) + \overline{K_{n-1}}$, where w is the vertex of the K_1, with degree $n-1$, and u is now a vertex of the $\overline{K_{n-1}}$.

Let w' be a vertex of K_n. Apply the diamond sum again to the embeddings Φ_3 and Φ_4 at w' and w. By Observation 2.3 the resulting embedding Φ_5 is quadrangular; it is also nonorientable since Φ_4 is nonorientable. The underlying graph $((K_1 \cup K_5) + \overline{K_{n-1}}) \diamond K_n$ is $K_5 + K_{n-1} = K_{n+4}$. □

Proof of Theorem 1.5. Apply Lemma 4.2 repeatedly starting from the quadrangular embedding of K_4 in the projective plane, which has the three hamilton cycles of K_4 as its face boundaries. □

We can also prove the existence part of Theorem 1.3 by applying Lemma 4.2 repeatedly starting from a quadrangular embedding of K_5 in the torus. For $n \geq 9$ the resulting embeddings of K_n are nonorientable.

5 Minimal quadrangulations

In this section we apply our results to determine the order of some minimal quadrangulations.

Hartsfield and Ringel [10, 11] gave lower bounds on the order of a minimal quadrangulation for a given surface, and showed that the quadrangular embeddings of complete graphs and generalized octahedra O_{2k}, $k \geq 4$, that they constructed were minimal. Lawrencenko [15] showed that certain orientable quadrangular embeddings of a graph $G[\overline{K_2}]$, which exist as described in Subsection 2.2, are minimal. The following lemma implies the minimality results both of Hartsfield and Ringel and of Lawrencenko.

Lemma 5.1. *Suppose L is obtained by deleting at most $n-4$ edges from the complete graph K_n, $n \geq 5$. Then any quadrangular embedding of L is minimal.*

Proof. Let $f(x) = x(x-5)/2$. Suppose that $x \geq 5$. If $2\frac{1}{2} \leq x' \leq x-1$, then because f is increasing on $[2\frac{1}{2}, \infty)$ we have $f(x) - f(x') \geq f(x) - f(x-1) = x-3$. If $1 \leq x' \leq 2\frac{1}{2}$, then because $2\frac{1}{2} \leq 5 - x' \leq x-1$ we have $f(x) - f(x') = f(x) - f(5-x') \geq x-3$. Thus, $f(x) - f(x') \geq x-3$ whenever $1 \leq x' \leq x-1$. If n is a nonnegative integer then $f(n) = \binom{n}{2} - 2n$.

Now suppose L has n vertices, m edges, and a quadrangular embedding in Σ. Since at most $n-4$ edges of K_n were deleted, L, and hence also Σ, is connected. If we have another quadrangulation of the same surface with $n' \leq n-1$ vertices and m' edges, then, using Observation 1.1(i),

$$\begin{aligned} m' - \binom{n'}{2} &= 2n' - 2\chi(\Sigma) - \binom{n'}{2} = -f(n') - 2\chi(\Sigma) = -f(n') + m - 2n \\ &\geq -f(n') + \binom{n}{2} - (n-4) - 2n = f(n) - f(n') - (n-4) \geq 1, \end{aligned}$$

proving that the other graph is not simple. □

Lemma 5.1 is sharp whenever K_{n-1} has a quadrangular embedding Φ of the appropriate orientability type (as in Theorem 1.6). Adding a new vertex of degree 2 adjacent to two opposite vertices of a face of Φ yields a quadrangular embedding of a graph obtained from K_n by deleting $n-3$ edges, but this is not minimal.

We can now apply Lemma 5.1 to Lemmas 2.1 and 2.2, and to Theorems 3.1 and 3.2. The orientable case of Corollary 5.2 is due to Lawrencenko [15, Theorem 2].

Corollary 5.2. *Let k and p be integers with $k \geq 4$ and $0 \leq p \leq k/4-1$. Suppose G is obtained from K_k by deleting at most p edges. Then $G[\,\overline{K_2}\,]$ has both orientable and nonorientable quadrangular embeddings that are minimal. Thus, minimal quadrangulations of the orientable surface of genus $k(k-3)/2-p+1$ and of the nonorientable surface of genus $k^2-3k-2p+2$ have order $2k$.*

Proof. Deleting p edges from K_k does not create isolated vertices or destroy all cycles. Thus, by Lemmas 2.1 and 2.2, $G[\,\overline{K_2}\,]$ has orientable and nonorientable quadrangular embeddings. These have order $2k$, and are minimal by Lemma 5.1 since we get $G[\,\overline{K_2}\,]$ by deleting $k+4p \leq 2k-4$ edges from K_{2k}. Observation 1.1(i) gives the genera of the surfaces. □

Corollary 5.3. *Let ℓ and q be integers with $\ell \geq 1$ and $0 \leq q \leq (\ell-1)/2$. Suppose G is obtained from $K_{2\ell}$ by deleting at most q edges. Then $G[K_4]$ has both orientable and nonorientable quadrangular embeddings that are minimal. Thus, minimal quadrangulations of the orientable surface of genus $8\ell^2-5\ell-4q+1$ and of the nonorientable surface of genus $16\ell^2-10\ell-8q+2$ have order 8ℓ.*

Proof. If $\ell = 1$ then $q = 0$ and $G[K_4] = K_{8k}$, so orientable and nonorientable quadrangular embeddings exist by Theorems 1.4 and 1.5. If $\ell \geq 2$ then the q edges deleted from $K_{2\ell}$ are incident with at most $\ell-1$ vertices, so G has $K_{2\ell} - E(K_{\ell-1})$ as a subgraph, and hence has a perfect matching and a cycle. Thus, by Theorems 3.1 and 3.2, $G[K_4]$ has the required embeddings.

For all ℓ these embeddings have order 8ℓ, and are minimal by Lemma 5.1 since we get $G[K_4]$ by deleting $16q < 8\ell-4$ edges from $K_{8\ell}$. Observation 1.1(i) gives the genera of the surfaces. □

There is some overlap here between the conclusions about the order of minimal quadrangulations. The case of Corollary 5.3 with $\ell/4 \leq q \leq (\ell-1)/2$ is also covered by Corollary 5.2 with $k = 4\ell$ and $p = 4q - \ell$.

6 Conclusion

We give some final remarks.

(1) In their work on quadrangular embeddings of complete graphs and minimal quadrangulations, Hartsfield and Ringel [10, 11] used a stricter definition of a quadrangular embedding than we do: they insisted that two distinct faces intersect in at most one edge and at most three vertices. The reason for this restriction is unclear. Perhaps they wished to make the embedding "polyhedral". However, an embedding is now usually considered polyhedral if it is a 3-*representative* (every noncontractible simple closed curve in the surface intersects the graph in at least three points) embedding of a 3-connected graph. A quadrangular embedding of K_n is never polyhedral in this sense: if we take a face bounded by $(uvwx)$ then the edge uw is part of the boundary of some other face, and using

these two faces we can find a simple closed curve intersecting the graph at just u and w, which must be noncontractible.

(2) We used diamond sums to prove the existence of nonorientable quadrangular embeddings of complete graphs. We could also use diamond sums to prove the existence of orientable quadrangular embeddings of complete graphs if we could find an orientable quadrangular embedding of K_{11}^{+}, the graph obtained by subdividing one edge of K_{11}, on S_9. We could then prove a lemma analogous to Lemma 4.2, saying that if K_n has an orientable quadrangular embedding then so does K_{n+8}, and apply this starting from quadrangular embeddings of K_5 on S_1 and K_8 on S_4.

(3) We can carry out the graphical surface/voltage graph construction from the proof of Theorem 3.1 with non-perfect matchings M of G as well as with perfect matchings, to give orientable and nonorientable quadrangular embeddings of some graphs L with $G[\,\overline{K_4}\,] \subseteq L \subseteq G[K_4]$. This provides some further results on minimal quadrangulations, but we omit the details.

(4) Our constructions have a lot of flexibility, particularly the constructions from Subsection 2.2 and Section 3. The graphical surface embeddings of $G[\,\overline{K_2}\,]$ in $S(G)$ (with twisted tubes allowed) require a cyclic order of tubes around the equator of each sphere, and a designation of which tubes are to be twisted. (This corresponds to choosing an arbitrary embedding of G, described by a rotation scheme with edge signatures.) There are two ways to run the edges along each tube. For Theorem 3.1 or 3.2 we may choose an arbitrary perfect matching M of G, and for each edge uv of M we may choose one of four possible edges along the corresponding tube to determine Q_u and Q_v. We also have two ways to assign the voltages for the digons of J running along each tube.

It therefore seems natural to ask whether our techniques can be used to provide useful lower bounds on the number of nonisomorphic quadrangular embeddings of K_n.

References

[1] C.P. Bonnington, M.J. Grannell, T.S. Griggs and J. Širáň, Exponential families of non-isomorphic triangulations of complete graphs, *J. Combin. Theory Ser. B* **78** (2000) 169–184.

[2] A. Bouchet, Orientable and nonorientable genus of the complete bipartite graph, *J. Combin. Theory Ser. B* **24** (1978) 24-33.

[3] D.L. Craft, *Surgical techniques for constructing minimal orientable imbeddings of joins and compositions of graphs*, Ph.D. Thesis, Western Michigan University, 1991.

[4] D.L. Craft, On the genus of joins and compositions of graphs, *Discrete Math.* **178** (1998) 25–50.

[5] M.J. Grannell, T.S. Griggs and J. Širáň, A lower bound for the number of triangular embeddings of some complete graphs and complete regular tripartite graphs, *J. Combin. Theory Ser. B* **98** (2008) 637–650.

[6] M.J. Grannell and M. Knor, A lower bound for the number of orientable triangular embeddings of some complete graphs, *J. Combin. Theory Ser. B* **100** (2010) 216–225.

[7] M.J. Grannell and T.A. McCourt, Doubly even orientable closed 2-cell embeddings of the complete graphs, *Electron. J. Combin.* **21** (2014) P1.22, 17pp.

[8] J. L. Gross and T. W. Tucker, *Topological Graph Theory*, Dover, Mineola, New York, 2001.

[9] Nora Hartsfield, Nonorientable quadrangular embeddings of complete multipartite graphs, Proc. 25th Southeastern Internat. Conf. on Combinatorics, Graph Theory and Computing (Boca Raton, FL, 1994), *Congr. Numer.* **103** (1994) 161–171.

[10] N. Hartsfield and G. Ringel, Minimal quadrangulations of orientable surfaces, *J. Combin. Theory Ser. B* **46** (1989) 84–95.

[11] N. Hartsfield and G. Ringel, Minimal quadrangulations of nonorientable surfaces, *J. Combin. Theory Ser. A* **50** (1989) 186–195.

[12] K. Kawarabayashi, C. Stephens and X. Zha, Orientable and nonorientable genera of some complete tripartite graphs, *SIAM J. Discrete Math.* **18** (2004) 479–487.

[13] V.P. Korzhik, Generating nonisomorphic quadrangular embeddings of a complete graph, *J. Graph Theory* **74** (2013) 133–142.

[14] V.P. Korzhik and H-J. Voss, On the number of nonisomorphic orientable regular embeddings of complete graphs, *J. Combin. Theory Ser. B* **81** (1) (2001) 58–76.

[15] Serge Lawrencenko, Realizing the chromatic numbers and orders of spinal quadrangulations of surfaces, *J. Combin. Math. Combin. Comput.* **87** (2013) 303–308.

[16] Z. Magajna, B. Mohar and T. Pisanski, Minimal ordered triangulations of surfaces, *J. Graph Theory* **10** (1986) 451–460.

[17] B. Mohar, T. D. Parsons, and T. Pisanski, The genus of nearly complete bipartite graphs, *Ars Combin.* **20** (1985) 173–183.

[18] G. Ringel, Das Geschlecht des vollständigen paaren Graphen, *Abh. Math. Sem. Univ. Hamburg.* **28** (1965) 139–150.

[19] G. Ringel, Der vollständige paare Graph auf nichtorientierbaren Flächen, *J. Reine Angew. Math.* **220** (1965) 88–93.

[20] G. Ringel, *Map Color Theorem*, Springer, Berlin, 1974.

[21] Arthur T. White, On the genus of the composition of two graphs, *Pacific J. Math.* **41** (1972) 275–279.