Quadrangular embeddings of complete graphs and the Even Map Color Theorem (with details)[1]

Wenzhong Liu[2], Serge Lawrencenko[3], Beifang Chen[4], M. N. Ellingham[5], Nora Hartsfield[6], Hui Yang[7], Dong Ye[8] and Xiaoya Zha[9]

2 June 2016; revised 19 October 2018

Abstract

Hartsfield and Ringel constructed orientable quadrangular embeddings of the complete graph K_n for $n \equiv 5 \pmod 8$, and nonorientable ones for $n \geq 9$ and $n \equiv 1 \pmod 4$. These provide minimal quadrangulations of their underlying surfaces. We extend these results to determine, for every complete graph K_n, $n \geq 4$, the minimum genus, both orientable and nonorientable, for the surface in which K_n has an embedding with all faces of degree at least 4, and also for the surface in which K_n has an embedding with all faces of even degree. These last embeddings provide sharpness examples for a result of Hutchinson bounding the chromatic number of graphs embedded with all faces of even degree, completing the proof of the Even Map Color Theorem. We also show that if a connected simple graph G has a perfect matching and a cycle then the lexicographic product $G[K_4]$ has orientable and nonorientable quadrangular embeddings; this provides new examples of minimal quadrangulations.

Keywords: quadrangular embedding, complete graph, minimal quadrangulation, 4-genus, even-faced embedding, map coloring, chromatic number.

---

[2]Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, China. Email: `wzhliu7502@nuaa.edu.cn`. Partially supported by Fundamental Research Funds for the Central Universities (NZ2015106) and NSFC grant (11471106).

[3]Russian State University of Tourism and Service, Institute for Tourism and Hospitality, Bldg. 32A, Kronstadt Boulevard, Moscow, 125438, Russia. Email: `lawrencenko@hotmail.com`.

[4]Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong, China. Email: `mabfchen@ust.hk`.

[5]Department of Mathematics, Vanderbilt University, Nashville, TN 37240, USA, Email: `mark.ellingham@vanderbilt.edu`. Partially supported by National Security Agency grant H98230–13–1–0233 and Simons Foundation award no. 429625.

[6]Deceased.

[7]Department of Mathematics, Guizhou University, Guiyang, 550025, China. Email: `hui-yang@163.com`.

[8]Department of Mathematical Sciences, Middle Tennessee State University, Murfreesboro, TN 37132, USA, Email: `dong.ye@mtsu.edu`. Partially supported by Simons Foundation award no. 359516.

[9]Department of Mathematical Sciences, Middle Tennessee State University, Murfreesboro, TN 37132, USA, Email: `xiaoya.zha@mtsu.edu`. Partially supported by National Security Agency grant H98230–13–1–0216.

1 Main results

In this paper surfaces are connected compact 2-manifolds without boundary. The orientable surface of genus h is denoted S_h, and the nonorientable surface of genus k is denoted N_k. The Euler characteristic of a surface Σ is denoted $\varepsilon(\Sigma)$, which is $2-2h$ for S_h, and $2-k$ for N_k.

Frequently we want embeddings of a given graph with minimum genus, which have faces that are small, often triangular faces. In particular, the determination of the minimum genus of complete graphs as part of the Map Color Theorem [30] was one of the driving forces behind the development of topological graph theory. However, we can also consider the minimum genus of embeddings with restrictions on face degrees. In this paper we consider embeddings where all faces have degree at least 4, or all faces have even degree. Euler's formula and face/edge counting imply the following.

Observation 1.1. *If Φ is an embedding of an n-vertex m-edge graph in a surface Σ with all faces of degree at least 4, then $m \le 2n - 2\varepsilon(\Sigma)$, with equality if and only if the embedding is cellular and every face degree is 4. For a complete graph K_n, $n \ge 4$, such an embedding has $n(n-5) \le -4\varepsilon(\Sigma)$, with equality if and only if the embedding is quadrangular.*

In this paper we completely resolve the question of the minimum genus of a surface in which K_n has an embedding with all faces of degree at least 4, or with all faces of even degree. These results also complete the proof of a coloring result, the Even Map Color Theorem. In 1975 Hutchinson [16] showed that the chromatic number bound of the Map Color Theorem can be significantly improved for even-faced embeddings; our results improve her bound in one case and provide sharpness examples. We also provide some constructions for minimal quadrangulations, simple quadrangulations with a minimum number of vertices in a given surface.

Our main results are as follows (see Section 3 for definitions not stated here). For a graph G and positive integer d, the *orientable d-genus* $g_d(G)$ and the *orientable even-faced genus* $g_{\text{even}}(G)$ are the smallest $h \ge 0$ for which G has a cellular embedding in S_h with all face degrees at least d, or with all face degrees even, respectively. We can similarly define the *nonorientable d-genus* $\widetilde{g}_d(G)$ and the *nonorientable even-faced genus* $\widetilde{g}_{\text{even}}(G)$ (for convenience we take $\widetilde{g}_d(G)$ or $\widetilde{g}_{\text{even}}(G)$ to be 0 if G has a suitable planar embedding).

Theorem 1.2. *Let $f(n) = 1 + \lceil n(n-5)/8 \rceil$. Then*

$$g_4(K_n) = f(n) \quad \text{if } n \ge 4, \qquad \text{and} \qquad g_{\text{even}}(K_n) = \begin{cases} f(n) & \text{if } n \ge 4 \text{ and } n \ne 6, \\ f(6)+1 = 3 & \text{if } n = 6. \end{cases}$$

For $n = 5$, for $n \ge 7$, and for $g_4(K_6)$ there is a face-simple closed-2-cell embedding of K_n realizing each equation. Such an embedding is quadrangular if and only if $n \equiv 0$ or $5 \pmod 8$.

Theorem 1.3. *Let $\widetilde{f}(n) = 2 + \lceil n(n-5)/4 \rceil$. Then*

$$\widetilde{g}_4(K_n) = \widetilde{g}_{\text{even}}(K_n) = \begin{cases} \widetilde{f}(n) & \text{if } n \ge 4 \text{ and } n \ne 5, \\ \widetilde{f}(5)+1 = 3 & \text{if } n = 5. \end{cases}$$

For $n = 4$ and for $n \ge 6$ there is a closed-2-cell embedding of K_n realizing this pair of equations, that is face-simple if $n \ge 6$. Such an embedding is quadrangular if and only if $n \equiv 0$ or $1 \pmod 4$.

Let $\chi(\Phi)$ and $\chi^*(\Phi)$ denote the number of colors needed to properly vertex-color or face-color, respectively, a graph embedding Φ. We will ignore loops when vertex-coloring and *monofacial* edges (with the same face on both sides) when face-coloring, so χ and χ^* are defined for all embeddings.

Theorem 1.4 (Even Map Color Theorem)**.** *Let Φ be a (not necessarily cellular) embedding of a (not necessarily connected) graph (loops and multiple edges allowed) in a surface Σ. Define*

$$H_{\text{even}}(\Sigma) = \left\lfloor \frac{5+\sqrt{25-16\varepsilon(\Sigma)}}{2} \right\rfloor \text{ if } \Sigma \neq S_0, \quad \text{and} \quad c(\Sigma) = \begin{cases} 2 & \text{if } \Sigma = S_0, \\ H_{\text{even}}(\Sigma) - 1 & \text{if } \Sigma = N_2 \text{ or } S_2, \\ H_{\text{even}}(\Sigma) & \text{otherwise.} \end{cases}$$

(a) If every face of Φ has even degree (individual face boundary components may have odd length), then (ignoring loops when coloring) $\chi(\Phi) \leq c(\Sigma)$.
(b) If every vertex of Φ has even degree, then (ignoring monofacial edges when coloring) $\chi^*(\Phi) \leq c(\Sigma)$.

Moreover, for every surface Σ there exist face-simple closed-2-cell embeddings of connected simple graphs, that are quadrangular for (a) and 4-regular for (b), which show that these bounds are sharp.

The following provides new constructions of minimal quadrangulations, as well as giving an alternative proof of some cases of Theorems 1.2 and 1.3.

Theorem 1.5. *Let G be a connected simple graph with a perfect matching. Then $G[K_4]$ has a face-simple orientable quadrangular embedding. Moreover, if G also has a cycle, then $G[K_4]$ also has a face-simple nonorientable quadrangular embedding.*

Section 2 provides some background to our results, and Section 3 provides precise definitions and preliminary results. Section 4 proves Theorems 1.2 and 1.3, and Section 5 proves the Even Map Color Theorem. Section 6 proves Theorem 1.5, and Section 7 shows that results from Sections 4 and 6 yield minimal quadrangulations. Section 8 contains some final remarks.

This version of this paper contains some details not included in the version submitted for publication.

2 Background

The minimum genus of the complete graph K_n and conditions for the existence of triangular embeddings of K_n were determined as part of the well-known Map Color Theorem [30], which extended the Four Color Theorem to other surfaces. Subsequently there were a number of results showing existence of multiple triangular embeddings of certain complete graphs, such as [1, 24, 33], and then providing lower bounds on the number of nonisomorphic triangular embeddings of K_n for certain families of n, such as [2, 8, 9, 20].

Less work has been done on quadrangular embeddings of complete graphs, or embeddings of complete graphs with all faces of degree at least 4, or all faces of even degree. By Observation 1.1, $n(n-5) = -4\varepsilon(\Sigma)$ when there is a quadrangulation of K_n in Σ, which means that $n \equiv 0$ or

5 (mod 8) in the orientable case, and $n \equiv 0$ or 1 (mod 4) in the nonorientable case. Hartsfield and Ringel [14, 15] obtained the following results, mostly using current graphs, covering half of the possible values of n for which quadrangular embeddings of K_n may exist.

Theorem 2.1 (Hartsfield and Ringel [14, 15])**.** *A complete graph K_n with $n = 8$ or $n \equiv 5 \pmod 8$ has a face-simple orientable quadrangular embedding. A complete graph K_n with $n \geq 9$ and $n \equiv 1 \pmod 4$ has a face-simple nonorientable quadrangular embedding. However, K_5 has no nonorientable quadrangular embedding.*

The embeddings in Theorem 2.1 are minimal quadrangulations. Hartsfield and Ringel also constructed quadrangular embeddings of the generalized octahedron $O_{2k} = K_k[\overline{K_2}]$ that are minimal. We discuss minimal quadrangulations in more detail in Section 7. The fact that K_5 has no nonorientable quadrangular embedding was also proved earlier (in dual form) by Hutchinson [16].

Using current graphs, Korzhik and Voss [20] constructed exponentially many nonisomorphic orientable quadrangular embeddings of K_{8s+5} for $s \geq 1$, and Korzhik [19] constructed superexponentially many nonisomorphic orientable and nonorientable quadrangular embeddings of K_{8s+5} for $s \geq 2$. Grannell and McCourt [10] constructed many nonisomorphic orientable embeddings of complete graphs K_n with faces bounded by $4k$-cycles for $k \geq 2$, when $n = 8ks + 4k + 1$ for $s \geq 1$.

It is natural to ask whether the results in Theorem 2.1 can be extended to the other cases where quadrangular embeddings of K_n might exist, namely $n \equiv 0 \pmod 8$ for orientable embeddings, and $n \equiv 0 \pmod 4$ for nonorientable embeddings. When K_n has a quadrangular embedding it is a minimal quadrangulation, and realizes $g_4(K_n)$ and $g_{\text{even}}(K_n)$, or $\widetilde{g}_4(K_n)$ and $\widetilde{g}_{\text{even}}(K_n)$. But we can also try to determine these parameters even if K_n does not have a quadrangular embedding. Our Theorems 1.2 and 1.3 resolve all of these questions, and we provide new proofs for the existence results in Theorem 2.1.

Some explanation of the origins of this paper is appropriate. In the early 1990s one of us, Hartsfield, developed a technique for constructing quadrangulations by "adding handles using diagonals". She used this technique to derive a number of results on quadrangular embeddings, including that K_n has a nonorientable quadrangular embedding when $n \equiv 0 \pmod 4$ [12]. She also applied this to derive results on $\widetilde{g}_4(K_n)$ and $\widetilde{g}_{\text{even}}(K_n)$ in a paper that was submitted for publication in 1994 [13]. As indicated in [17], Hartsfield was aware that her results would give sharpness examples for Hutchinson's coloring results [16]. Hartsfield's papers [12, 13] outlined proofs (providing basis cases and examples of inductive steps, such as from K_8 to K_{16}) but did not give complete general arguments.

In the late 1990s three of us, Chen, Lawrencenko and Yang (CLY), derived results on $g_4(K_n)$ using current graphs [4, 22]. These were submitted for publication in 1998. When Hartsfield and CLY discovered they had been working on similar results, they decided to combine their results into a single paper. Unfortunately, this single paper was never finished. Some researchers were aware of the results of Hartsfield (cited in [17]) and of CLY (cited in [31]) but they were not publicly available.

Around 2015 the remaining four authors, Ellingham, Liu, Ye and Zha (ELYZ), worked on some problems of Craft [6] on quadrangular embeddings of composition graphs. ELYZ realized that their constructions (see Section 6) provided orientable quadrangular embeddings for K_n with $n \equiv 0$

(mod 8), which did not seem to be in the literature. ELYZ also came up with a diamond sum construction (see Section 4) for nonorientable quadrangular embeddings of K_n for $n \equiv 0 \pmod 4$. ELYZ's results were written up [25] and submitted in 2016. After submission of their paper ELYZ were informed of the earlier unpublished results of Hartsfield and CLY. It was decided to combine all of the results into the present joint paper. Although Nora Hartsfield died in 2011 we think it is appropriate to include her as an author.

We hope that eventually the other proofs of Theorems 1.2 and 1.3 using Hartsfield's diagonal technique and current graphs will also appear. For the current graph results, some modification of the index 2 current graphs in [4, 22] is required, and we hope to provide nonorientable constructions as well as orientable ones. A paper using a combination of current graphs and Hartsfield's diagonal technique is in preparation [23] and additional papers may follow.

3 Preliminaries

3.1 Graph embeddings

Our graphs may have loops or multiple edges; *simple* graphs have neither. We say a graph embedding has some graph property (such as bipartiteness) if the underlying graph has this property. A face of a graph embedding is *cellular* if it is homeomorphic to an open disk. We often identify a cellular face by referring to its bounding cycle or bounding closed walk. A graph embedding is *cellular* if every face is cellular, *closed-2-cell* if it is cellular and every face is bounded by a cycle (with no repeated vertices), and *face-simple* if every two distinct faces share at most one boundary edge. In this paper all embeddings are cellular unless we specifically refer to a *general* embedding, which means that faces may have multiple boundary components and internal handles or crosscaps.

Suppose Φ is a graph embedding in surface Σ. The *degree* of a face is the number of sides of edges with which it is incident. A k-face is a face of degree k, and a C_k*-face* is a cellular face bounded by a k-cycle. A cellular k-face is bounded by a single closed walk of length k, which may or may not be a k-cycle. The minimum vertex degree and minimum face degree of Φ are denoted $\delta(\Phi)$ and $\delta^*(\Phi)$, respectively. The embedding Φ is *even-vertexed* or *even-faced* if every vertex or every face, respectively, has even degree. An even-faced noncellular embedding may have individual face boundary component walks of odd length, as long as the overall degree of each face is even. We say Φ is *quadrangular*, or a *quadrangulation of* Σ, if every face is a C_4-face. We also refer to a C_4-face as a *quadrilateral.* A quadrangulation of Σ is *minimal* if its underlying graph is simple and connected, and there is no quadrangular embedding of a simple graph of smaller order in Σ. Similarly, a *triangulation* of a surface Σ is an embedding of a graph in Σ such that every face is a C_3-face.

Lemma 3.1 (Euler's inequality)**.** *Suppose we have a general embedding of a graph G in a surface of Euler characteristic ε, with n vertices, m edges and r faces. Then $n - m + r \geq \varepsilon$, with equality if and only if the embedding is cellular,*

Observation 3.2. *Suppose Φ is a general embedding of a simple graph G and $\delta(\Phi) \geq 2$. Then every face of G of degree at most 5 is bounded by a single cycle.*

Observation 3.3. *Suppose Φ is a general even-faced embedding of a simple connected graph on at least three vertices. Then every face boundary walk has length at least 3, and hence $\delta^*(\Phi) \geq 4$.*

Observation 3.4. *Suppose Φ is a quadrangular embedding of a simple connected graph and $\delta(\Phi) \geq 3$. If Φ is not face-simple then it contains two faces of the form $(uvwx)$ and $(uvxw)$. Thus, if Φ is orientable or bipartite then it is face-simple.*

3.2 Graph operations

Let G and H be simple graphs. The complement of G is denoted $\overline{G}$. The *composition* (or *lexicographic product*) of G and H, denoted $G[H]$, has vertex set $V(G) \times V(H)$, with two vertices (v_1, w_1) and (v_2, w_2) adjacent if and only if either (i) $v_1v_2 \in E(G)$ or (ii) $v_1 = v_2$ and $w_1w_2 \in E(H)$. For example, $K_n[K_2]$ is the complete graph K_{2n}, and $K_n[\,\overline{K_2}\,]$ is the generalized octahedron $O_{2n} = K_{2n} - nK_2$. The *join* of G and H, denoted $G + H$, is the union of G and H together with one edge uv for each $u \in V(G)$ and $v \in V(H)$. For example, $K_4 + K_n$ is the complete graph K_{n+4}.

3.3 The diamond sum

Let G and G' be two simple graphs with embeddings Φ and Φ' in disjoint surfaces Σ and Σ', respectively. Suppose that $k \geq 1$ and both G and G' have a vertex of degree k, say v and v' respectively. Let v have neighbors $v_0, v_1, ..., v_{k-1}$ in cyclic order around v in Φ, and let v' have neighbours $v'_0, v'_1, ..., v'_{k-1}$ in cyclic order around v' in Φ'. There is a closed disk D that intersects G in v and the edges $vv_0, vv_1, ..., vv_{k-1}$, and so that the boundary of D intersects G at $v_0, v_1, ..., v_{k-1}$. Similarly, there is a closed disk D' that intersects G' in v' and the edges $v'v'_0, v'v'_1, ..., v'v'_{k-1}$ and so that the boundary of D' intersects G' at $v'_0, v'_1, ..., v'_{k-1}$. Remove the interiors of D and D', and identify their boundaries so that v_i is identified with v'_i for $0 \leq i \leq k-1$. The resulting embedding is called a *diamond sum of Φ and Φ' at v and v'*, denoted $\Phi\Diamond_{v,v'}\Phi'$ or just $\Phi\Diamond\Phi'$. Its graph is denoted $G\Diamond G'$ and the surface is the connected sum $\Sigma\#\Sigma'$. Note that $\Phi\Diamond\Phi'$ is orientable if and only if both Φ and Φ' are orientable.

The diamond sum was first used by Bouchet [3] in dual form to derive a new proof of the minimum genus of $K_{m,n}$. Bouchet's construction was later reinterpreted in more general situations in [18, 26, 27].

The diamond sum of two cellular embeddings is cellular. It is also not difficult to see that if Φ and Φ' are quadrangular and the diamond sum $\Phi\Diamond\Phi'$ is simple, then $\Phi\Diamond\Phi'$ is also quadrangular. To build embeddings in Section 4 that are face-simple and closed-2-cell, we rely on the following technical extension of this observation, which allows Φ' to contain non-C_4-faces.

Lemma 3.5. *Suppose Φ is a face-simple quadrangular embedding of a simple graph G, $\delta(\Phi) \geq 3$, $v \in V(G)$, and the neighbors of v in G are independent. Suppose Φ' is a closed-2-cell embedding of a simple graph G', $v' \in V(G')$, and every pair of distinct faces of Φ' shares at most one edge of $G' - v'$. Then $\Phi'' = \Phi\Diamond_{v,v'}\Phi'$ is a face-simple closed-2-cell embedding and there is a degree-preserving bijection between the non-C_4-faces in Φ' and the non-C_4-faces in Φ''.*

Note that the condition on pairs of distinct faces of Φ' holds if Φ' is face-simple.

Proof. There are three types of faces in Φ'': (1) those that use only edges of $G-v$; (2) those that use edges of both $G-v$ and $G'-v'$; and (3) those that use only edges of $G'-v'$. Represent the faces of Φ using v as $Z_i = (vv_iw_iv_{i+1})$ and the faces of Φ' using v' as $Z'_i = (v'v'_i \ldots v'_{i+1})$, for $0 \le i \le k-1$, taking subscripts modulo k. Since the neighbors of v are independent, w_i is not a neighbor of v.

All faces in Φ and Φ' are bounded by cycles since Φ and Φ' are closed-2-cell, which implies that faces of type (1) and (3) are bounded by cycles, and all Z_i and Z'_i are cycles. Thus, every face of type (2) is a face Z''_i obtained by combining paths $Z_i - v = v_iw_iv_{i+1}$ and $Z'_i - v'$ by identifying v_i with v'_i and v_{i+1} with v'_{i+1}. Since w_i is not a neighbor of v, it is not identified with any vertex of $Z'_i - v'$, so Z''_i is a cycle, and of the same length as Z'_i. Thus, Φ'' is closed-2-cell.

Since all faces of type (1) are C_4-faces, mapping each non-C_4-face Z'_i to Z''_i and each non-C_4-face of type (3) to itself gives the required degree-preserving bijection for non-C_4-faces.

Let m_{st} be the maximum number of edges shared by a face of type (s) and a distinct face of type (t). Clearly $m_{13} = 0$; since Φ is face-simple, $m_{11}, m_{12} \le 1$; and by the hypothesis on Φ', $m_{32}, m_{33} \le 1$. Consider two arbitrary distinct faces Z''_i, Z''_j of type (2). Suppose $Z_i - v$ and $Z_j - v$ share an edge ab. Since Φ is simple and $\delta(\Phi) \ge 3$, we cannot have $Z_i = (vabc)$ and $Z_j = (vabd)$, so we may assume that we have $Z_i = (vabc)$ and $Z_j = (vbad)$. But then a and b are adjacent neighbors of v, a contradiction. Therefore, Z''_i and Z''_j share no edges of $G-v$, and by the hypothesis on Φ' they share at most one edge of $G'-v'$, so $m_{22} \le 1$, and Φ'' is face-simple. □

3.4 Graphical surfaces

White [32] showed that any composition $G[\overline{K_2}\,]$, where G is a simple graph without isolated vertices, has an orientable quadrangular embedding. Craft [5, 7] developed graphical surfaces, which yield a simple proof of this result. We outline his proof, since we need his construction in Section 6.

For a graph G, the *graphical surface* $S(G)$ derived from G is a surface obtained from an embedding of G in $\mathbb{R}^3$ by blowing up every vertex u into a sphere Σ_u and replacing every edge uv by a tube T_{uv} joining the spheres Σ_u and Σ_v. Since we work in $\mathbb{R}^3$, the resulting surface $S(G)$ is orientable.

Lemma 3.6 (Craft [5, 7])**.** *Let G be a connected simple graph. Then $G[\overline{K_2}\,]$ has a quadrangular embedding in the graphical surface $S(G)$.*

Outline of proof. We embed $G[\overline{K_2}\,]$ in $S(G)$ as follows. For any vertex $u \in V(G)$, let u_{N} (north pole) and u_{S} (south pole) be two points in the sphere Σ_u; they represent the two vertices of $G[\overline{K_2}\,]$ corresponding to u. We may assume that all tubes joined to Σ_u are joined in some cyclic order around the equator of Σ_u. There are four edges of $G[\overline{K_2}\,]$ corresponding to each $uv \in E(G)$, which are $u_{\mathrm{N}}v_{\mathrm{N}}, u_{\mathrm{N}}v_{\mathrm{S}}, u_{\mathrm{S}}v_{\mathrm{S}}$ and $u_{\mathrm{S}}v_{\mathrm{N}}$. These can all be embedded along T_{uv} (there are two different ways to do this, but for our purposes it will not matter which is used). In the resulting embedding, every edge is contained in two quadrilaterals. For example, $u_{\mathrm{S}}v_{\mathrm{N}}$ is contained in quadrilaterals $Q_u = (t_X u_{\mathrm{N}} v_{\mathrm{N}} u_{\mathrm{S}})$ and $Q_v = (u_{\mathrm{S}} v_{\mathrm{N}} w_Y v_{\mathrm{S}})$ where $tu, vw \in E(G)$ and $X, Y \in \{\mathrm{N}, \mathrm{S}\}$. Note that if u has degree 1, then $t_X = v_{\mathrm{S}}$. □

White and Craft dealt only with orientable embeddings. However, we can also produce nonorientable embeddings. Given a graphical surface, we can replace a tube T_{uv} by a *twisted tube* $\widetilde{T}_{uv}$ by taking a simple closed curve γ around T_{uv} with a specified positive direction, cutting along it to produce two boundary curves γ_1 and γ_2, then re-identifying γ_1 with γ_2 so that the positive direction along γ_1 corresponds to the negative direction along γ_2 (this cannot be done in $\mathbb{R}^3$). We can still embed the four edges between $\{u_\mathrm{N}, u_\mathrm{S}\}$ and $\{v_\mathrm{N}, v_\mathrm{S}\}$ along $\widetilde{T}_{uv}$; they become orientation-reversing edges relative to the original orientation at each vertex. Depending on which tubes we replace, the resulting embedding may be nonorientable.

Lemma 3.7. *Let G be a connected simple graph with at least one cycle. Then $G[\,\overline{K_2}\,]$ has a quadrangular embedding in a nonorientable modified graphical surface $\widetilde{S}(G)$.*

Proof. Choose one edge uv belonging to a cycle and replace the tube T_{uv} by a twisted tube $\widetilde{T}_{uv}$ in the construction of Lemma 3.6. The resulting embedding is nonorientable because the cycle $(uvw\ldots z)$ in G gives an orientation-reversing cycle $(u_\mathrm{N} v_\mathrm{N} w_\mathrm{N} \ldots z_\mathrm{N})$ in the embedding of $G[\,\overline{K_2}\,]$ in the new surface $\widetilde{S}(G)$. □

3.5 Voltage graphs

We assume the reader is familiar with voltage graph constructions for embeddings. We summarize the main features; see [11] for more details.

Given a graph G, assign an arbitrary *plus direction* to each edge. A function α from the plus-directed edges of G to a group Γ is an *ordinary voltage assignment* on G. The pair $\langle G, \alpha\rangle$ is called an *ordinary voltage graph*. The *derived graph* G^α has vertex set $V(G) \times \Gamma$ and an edge from $u_a = (u, a)$ to $v_b = (v, b)$ whenever uv is a plus-directed edge in G and $b = a \cdot \alpha(uv)$.

If G has an embedding Φ, represented by a rotation of edges at each vertex and edge signatures, then G^α has a *derived embedding* Φ^α: for each $u_a \in V(G^\alpha)$ use the natural bijection between edges incident with u in G and edges incident with u_a in G^α to define the rotation at u_a from the rotation at u, and give each edge $u_a v_b$ of G^α the signature of the corresponding edge uv in G.

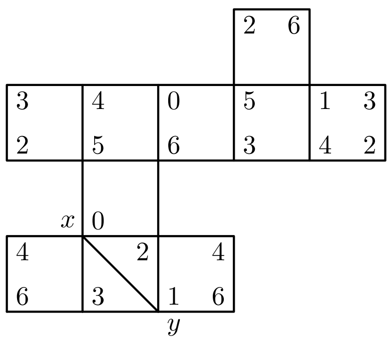
For each walk $W = v_0 e_1 v_1 e_2 v_2 \ldots e_k v_k$ in G define its *total voltage* to be $\alpha(e_1)^{\epsilon_1}\alpha(e_2)^{\epsilon_2}\ldots\alpha(e_k)^{\epsilon_k}$ where ϵ_i is $+1$ if W uses e_i in the plus direction and -1 otherwise. The faces of Φ^α come from the faces of Φ: each face in Φ with degree k whose boundary walk has total voltage of order r in Γ yields $|\Gamma|/r$ faces of degree kr in Φ^α. Also, Φ^α is nonorientable if and only if Φ is nonorientable and has an orientation-reversing closed walk whose total voltage is the identity of Γ.

4 Embeddings from diamond sums

In this section we prove Theorems 1.2 and 1.3 by constructing embeddings of minimum genus with all face degrees at least 4, and with all face degrees even, for each complete graph K_n, $n \geq 4$. Our constructions are inductive. The base cases are provided in Appendix A.

The induction steps use quadrangular embeddings of complete bipartite graphs and of K_7^+ and K_{11}^+, where K_n^+ denotes the graph obtained from K_n by subdividing an edge. In Figure 1 we provide embeddings $\widetilde{\Psi}_7$ of K_7 in N_5 (as a polygon with labeled vertices, indicating how edges are to be identified around the boundary) and Ψ_{11} of K_{11} in S_9 (as a rotation system; see [11, Section

3.2]). Each embedding is face-simple and all faces are C_4-faces apart from two C_3-faces that share an edge xy ($xy = 01$ for $\widetilde{\Psi}_7$ and $xy = 56$ for Ψ_{11}). Nonorientability of $\widetilde{\Psi}_7$ follows from the fact that there are edges, such as 05, used twice in the same direction around the outer boundary of the polygon. By subdividing xy with a vertex z in each case, we obtain embeddings $\widetilde{\Psi}_7^+$ and Ψ_{11}^+ of graphs K_7^+ and K_{11}^+. These embeddings are not face-simple, but with the choice $v' = x$ they satisfy the hypotheses for Φ' in Lemma 3.5.



0. 1 3 4 6 a 9 8 7 5 2
1. 0 8 7 a 9 3 2 5 6 4
2. 0 1 5 6 4 3 9 a 7 8
3. 0 4 6 5 1 2 a 9 7 8
4. 0 8 6 5 7 9 a 2 3 1
$x =$ 5. 0 9 a 4 6 2 3 7 8 1
$y =$ 6. 0 1 8 7 3 2 5 4 a 9
7. 0 2 1 5 6 3 4 a 8 9
8. 0 7 9 a 4 3 6 5 1 2
9. 0 4 3 8 a 6 2 1 5 7
a. 0 9 7 5 1 2 6 8 3 4

Figure 1: Embeddings $\widetilde{\Psi}_7$ of K_7 in N_5 (left) and Ψ_{11} of K_{11} in S_9 (right).

As part of determining the orientable and nonorientable genera of $K_{m,n}$, Ringel showed that $K_{m,n}$ has quadrangular embeddings in certain cases. Bouchet [3] later provided a simpler proof.

Lemma 4.1 (Ringel [28, 29])**.** *The complete bipartite graph $K_{m,n}$ has an orientable quadrangular embedding whenever $(m-2)(n-2) \equiv 0 \pmod 4$ and $\min\{m,n\} \geq 2$, and a nonorientable quadrangular embedding whenever $mn \equiv 0 \pmod 2$ and $\min\{m,n\} \geq 3$.*

First we consider nonorientable embeddings. The following two lemmas provide the induction steps in our proof.

Lemma 4.2. *If a complete graph K_n, $n \geq 4$, admits a face-simple orientable or nonorientable quadrangular embedding, then K_{n+4} admits a face-simple nonorientable quadrangular embedding.*

Proof. Take K_7^+ as described above, with special vertices x, y, z. We may interpret K_7^+ as a join $(K_1 \cup K_5) + \overline{K_2}$ where z is the vertex of the K_1, and x and y are the vertices of the $\overline{K_2}$. We build the embedding of K_{n+4} in two steps from the embedding $\Phi_1 = \widetilde{\Psi}_7^+$ of K_7^+ as described above, an orientable or nonorientable quadrangular embedding Φ_2 of $K_{6,n-1}$ from Lemma 4.1, and the assumed quadrangular embedding Φ_3 of K_n.

Let x' be a vertex of $K_{6,n-1}$ of degree 6. Since $\delta(\Phi_2) = \min(6, n-1) \geq 3$, $K_{6,n-1}$ is bipartite, and using Observation 3.4, we satisfy the hypotheses for Φ in Lemma 3.5 by taking $\Phi = \Phi_2$ and $v = x'$. From above we also satisfy the hypotheses for Φ' in Lemma 3.5 by taking $\Phi' = \Phi_1 = \widetilde{\Psi}_7^+$ and $v' = x$. Therefore, by Lemma 3.5, applying the diamond sum to Φ_1 and Φ_2 at x and x' yields a face-simple quadrangulation Φ_{12}, which is nonorientable since Φ_2 is nonorientable. The underlying graph $G_{12} = K_7^+ \Diamond K_{6,n-1}$ is $(K_1 \cup K_5) + \overline{K_{n-1}}$, where z is the vertex of the K_1, with degree $n-1$, and y is now a vertex of the $\overline{K_{n-1}}$.

Since $\delta(\Phi_{12}) = \min(6, n-1) \geq 3$ and the neighbors of z in G_{12} are independent, we satisfy the hypotheses for Φ in Lemma 3.5 by taking $\Phi = \Phi_{12}$ and $v = z$. Let z' be a vertex of K_n. We also satisfy the hypotheses for Φ' in Lemma 3.5 by taking $\Phi' = \Phi_3$ and $v' = z'$. Therefore, by Lemma 3.5, applying the diamond sum to Φ_{12} and Φ_3 at z and z' yields a face-simple quadrangulation Φ_{123}, which is nonorientable since Φ_{12} is nonorientable. The underlying graph $((K_1 \cup K_5) + \overline{K_{n-1}}) \Diamond K_n$ is $K_5 + K_{n-1} = K_{n+4}$. □

The same proof also shows the following. The C_p-face in the embedding Φ_3 of K_n corresponds to a C_p-face in the embedding $\Phi_{123} = \Phi_{12} \Diamond \Phi_3$ of K_{n+4} by the degree-preserving bijection of Lemma 3.5.

Lemma 4.3. *Suppose that $n \geq p \geq 5$. If a complete graph K_n admits a face-simple orientable or nonorientable embedding in which all faces are C_4-faces except for one C_p-face, then K_{n+4} has a face-simple nonorientable embedding in which all faces are C_4-faces except for one C_p-face.*

Theorem 4.4. *Given an integer n, let $k = 2 + \lceil n(n-5)/4 \rceil$.*

Suppose that $n \geq 6$. If $n \equiv 0$ or $1 \pmod 4$ then K_n has a face-simple quadrangular embedding in N_k. If $n \equiv 2$ or $3 \pmod 4$ then K_n has a face-simple embedding in N_k in which every face is a C_4-face except for one C_6-face.

For $n = 4$, K_4 has a quadrangular embedding in $N_k = N_1$ that is closed-2-cell but not face-simple. For $n = 5$, K_5 has no quadrangular embedding in $N_k = N_2$, but has an embedding in N_3 with three C_4-faces and one 8-face.

Proof. In each case the genus will follow by simple face/edge counting and Euler's formula, so we focus on the other properties. For $n \not\equiv 1 \pmod 4$, Appendix A gives the required embeddings $\widetilde{\Theta}_n$ for $n \in \{4, 6, 7, 8\}$, and we then repeatedly apply Lemma 4.2 or 4.3. (We need $n = 8$ because the embedding for $n = 4$ is not face-simple.)

Suppose that $n \equiv 1 \pmod 4$. For $n = 5$, Hutchinson [16, proof of Theorem 2] and Hartsfield and Ringel [15, Theorem 2] showed that there is no quadrangular embedding of K_5 in the Klein bottle N_2. However, there is a face-simple quadrangular embedding $\widetilde{\Psi}_6^-$ of $K_6 - e$ in N_3 given in Appendix A, and deleting vertex 0 gives the required embedding of K_5 in N_3. For $n \geq 9$, applying Lemma 4.2 to the orientable quadrangular embedding Θ_5 of K_5 from Appendix A gives a nonorientable quadrangular embedding of K_9, and we then repeatedly apply Lemma 4.2. □

Theorem 1.3 follows because every embedding given in Theorem 4.4 is even-faced.

By adding chords (carefully, for K_5) or a single vertex inside the face of degree greater than 4 we also obtain the following.

Corollary 4.5. *Suppose that $n \geq 4$ and $k = \lceil n(n-5)/4 \rceil + 2$. If $n \equiv 0$ or $1 \pmod 4$ and $n \neq 5$, then K_n has a quadrangular embedding in N_k, which is face-simple if $n \geq 8$. For $n = 5$, K_5 is a subgraph of a quadrangular 5-vertex embedding with multiple edges in N_3, and of a face-simple simple 6-vertex quadrangular embedding in N_3. If $n \equiv 2$ or $3 \pmod 4$, K_n is a subgraph of a quadrangular n-vertex embedding with multiple edges in N_k, and of a face-simple simple $(n+1)$-vertex quadrangular embedding in N_k.*

Proof. For $n = 5$ we take $\widetilde{\Psi}_6^-$ from Appendix A as the simple 6-vertex embedding. Deleting vertex 0 from $\widetilde{\Psi}_6^-$ leaves an 8-face $(1_1 4_1 5_1 2 1_2 3 4_2 5_2)$ (subscripting occurrences of the same vertex to distinguish them). We can add multiple edges $1_1 3$ and $5_1 3$. □

Now we turn to orientable embeddings. The following two lemmas provide the induction steps in our proof. They are proved in exactly the same way as Lemmas 4.2 and 4.3, except that we take Φ_1 to be the orientable quadrangular embedding Ψ_{11}^+ of K_{11}^+ instead of the nonorientable embedding $\widetilde{\Psi}_7^+$ of K_7^+, and Φ_2 to be an orientable quadrangular embedding of $K_{10,n-1}$, instead of a nonorientable embedding of $K_{6,n-1}$.

Lemma 4.6. *If a complete graph K_n, $n \geq 4$, admits a face-simple orientable quadrangular embedding, then K_{n+8} admits a face-simple orientable quadrangular embedding.*

Lemma 4.7. *Suppose that $5 \leq p \leq n$. If a complete graph K_n admits a face-simple orientable embedding in which all faces are C_4-faces except for one C_p-face, then K_{n+8} has a face-simple orientable embedding in which all faces are C_4-faces except for one C_p-face.*

There is one case where we cannot find an embedding with all of the properties we would like.

Proposition 4.8. *Every general embedding of K_6 in S_2 is cellular with five C_4-faces and two C_5-faces, and such an embedding exists. Thus, K_6 has no general even-faced embedding in S_2.*

Outline of proof. Let Φ be a general embedding of K_6 in S_2, with $n = 6$ vertices, $m = 15$ edges, r faces, and r_i faces of degree i. By Euler's inequality, $r \geq \varepsilon - n + m = -2 - 6 + 15 = 7$. By Observation 3.3, $\delta^*(\Phi) \geq 4$, and so $30 = 2m = 4r_4 + 5r_5 + 6r_6 + \ldots \geq 4r$. Hence $r = 7$, with either $r_4 = 6$ and $r_6 = 1$, or $r_4 = 5$ and $r_5 = 2$. Since $r = 7$, Φ is cellular. By Observation 3.2 the 4-faces and any 5-faces are bounded by cycles; any 6-face is bounded by a cycle or a 'bowtie' walk (*abcade*).

Now that we have restricted the structure of Φ, we can perform a case analysis to show that the embedding is as described. Details may be found in Appendix B. It is also easy to generate and check all rotation systems for K_6 (up to isomorphism) by computer. We did this; it ran in less than a minute. The embedding Θ_6 from Appendix A demonstrates existence. □

Theorem 4.9. *Given an integer n, let $h = 1 + \lceil n(n-5)/8 \rceil$.*

Suppose that $n = 5$ or $n \geq 7$. If $n \equiv 0$ or $5 \pmod 8$, then K_n has a face-simple quadrangular embedding in S_h. If $n \not\equiv 0$ and $5 \pmod 8$ then K_n has a face-simple embedding in which every face is a C_4-face except for one C_p-face, where $p \in \{6, 8, 10\}$ (specifically, $p = 12 - (n(n-5) \bmod 8)$).

For $n = 4$, K_4 has an embedding in $S_h = S_1$ with one C_4-face and one 8-face. For $n = 6$, K_6 has no even-faced embedding in $S_h = S_2$, but has an embedding in S_2 with five C_4-faces and two C_5-faces, and an embedding in S_3 with four C_4-faces and one 14-face.

Proof. In each case the genus will follow by simple face/edge counting and Euler's formula, so we focus on the other properties. For $n \neq 4$ and 6, Appendix A gives the required embeddings Θ_n of K_n for $n \in \{5, 7, 8, 9, 10, 11, 12, 14\}$, covering all classes modulo 8, and we then repeatedly apply Lemma 4.6 or 4.7. For $n = 4$, there cannot be an embedding with a C_8-face, but an embedding Θ_4 with an 8-face is given in Appendix A.

For $n = 6$, see Proposition 4.8. To obtain the embedding of K_6 in S_3 with a 14-face, take Θ_6 from Appendix A and swap the positions of 1 and 2 in the rotation of vertex 0. This replaces face boundaries (01234), (03142) and (0251) by a single closed walk (01234025103142) of length 14. $\square$

Theorem 1.2 follows because every embedding given in Theorem 4.9, except for the embedding of K_6 in S_2, is even-faced.

By adding chords (carefully, for K_4 and K_6) or a single vertex (or two vertices, for K_6) inside the face of degree greater than 4 we also obtain the following.

Corollary 4.10. *Suppose that $n \geq 4$ and $h = \lceil n(n-5)/8 \rceil + 1$. If $n \equiv 0$ or $5 \pmod 8$, then K_n has a quadrangular embedding in S_h. If $n \not\equiv 0$ and $5 \pmod 8$ and $n \neq 6$, K_n is a subgraph of a quadrangular n-vertex embedding with multiple edges in S_h, and of a simple $(n+1)$-vertex quadrangular embedding in S_h. For $n = 6$, K_6 is a subgraph of a quadrangular 6-vertex embedding with multiple edges in S_3 and a simple 8-vertex quadrangular embedding in S_3.*

Proof. For K_4, the 8-face in Θ_4 is $(0_1 1_1 2_1 3_1 1_2 0_2 3_2 2_2)$ (subscripting occurrences of the same vertex to distinguish them) and we can add multiple edges $0_1 3_1, 1_2 2_2$. Adding a new vertex 4 adjacent to $0_1, 2_1, 1_2, 3_2$ gives an embedding isomorphic to the the quadrangular embedding Θ_5 of K_5 in S_1.

For K_6, the 14-face from the proof of Theorem 4.9 is $(0_1 1_1 2_1 3_1 4_1 0_2 2_2 5 1_2 0_3 3_2 1_3 4_2 2_3)$ and we can add multiple edges $0_1 3_1, 3_1 2_2, 3_1 4_2, 2_2 0_3, 0_3 4_2$ or new vertices 6 adjacent to $2_3, 1_1, 3_1, 0_2, 5$ and then 7 adjacent to $1_2, 3_2, 4_2, 6$. (Or the 8-vertex simple quadrangulation of S_3 containing K_7 also contains K_6.) $\square$

5 Proof of the Even Map Color Theorem

The Map Color Theorem says that for a graph embedding Φ in a surface $\Sigma \neq S_0$, $\chi(\Phi) \leq H(\Sigma) = \left\lfloor \left(7 + \sqrt{49 - 24\varepsilon(\Sigma)}\right)/2 \right\rfloor$ (the *Heawood number* of Σ), which can be improved to $\chi(\Phi) \leq H(\Sigma) - 1$ if $\Sigma = N_2$, and these bounds are sharp. Hutchinson showed that this can be significantly improved if the embedding is even-faced. In this section we strengthen her bound in one case, and use the embeddings constructed in Section 4 to show that the bounds are sharp.

Theorem 5.1 (Hutchinson [16, Theorems 1 and 2 and Corollary 2])**.** *For a surface Σ define $H_{\text{even}}(\Sigma) = \left\lfloor \left(5 + \sqrt{25 - 16\varepsilon(\Sigma)}\right)/2 \right\rfloor$. If Φ is an even-faced graph embedding in a surface $\Sigma \neq S_0$ then $\chi(\Phi) \leq H_{\text{even}}(\Sigma)$. If $\Sigma = N_2$ this can be improved to $\chi(\Phi) \leq H_{\text{even}}(N_2) - 1 = 4$. These results are sharp when $\Sigma = N_1$, N_2 or S_1.*

Hutchinson's proof requires Φ to be cellular, because she first proves a face-coloring result and applies that to the dual Φ^*; each vertex of Φ needs to correspond to a distinct face of Φ^*. One way to extend the result to general embeddings is by first applying the following lemma.

Lemma 5.2. *Suppose Φ is a general embedding. Then we can construct a new embedding Φ' in the same surface by adding edges, such that (a) Φ' is cellular, (b) each face of Φ corresponds to a distinct face of Φ', (c) if two faces are adjacent in Φ then the corresponding faces are adjacent in Φ', (d) if Φ is even-faced then so is Φ', and (e) if Φ is even-vertexed then so is Φ'.*

Proof. We can add edges inside faces to destroy each internal handle or crosscap (run an edge along the handle or across the crosscap) and connect different boundary components without creating any new faces, satisfying (a)–(d). If we replace each new edge by two parallel edges bounding a 2-face we still satisfy (a)–(d) and also satisfy (e). In particular, since new edges have the same face on both sides, this does not violate (c). □

We can also prove Theorem 5.1 directly for general embeddings. We outline a proof of the general inequality in Theorem 5.1, based on translating and simplifying the proof of [16, Theorem 1], as we need some details later. We use the following preliminary results, which are implicit in the arguments of [16].

Observation 5.3. *If we remove edges or vertices from an even-faced embedding it remains even-faced.*

Lemma 5.4. *If Ψ is a general n-vertex embedding in a surface Σ with $\delta(\Psi) \geq d \geq 4$ and $\delta^*(\Psi) \geq 4$, then $n(d-4) \leq -4\varepsilon(\Sigma)$.*

Proof. Suppose Ψ has m edges and r faces, and let $\varepsilon = \varepsilon(\Sigma)$. By Euler's inequality, $dn - dm + dr \geq d\varepsilon$ so $dm \leq d(r-\varepsilon) + dn$ (1). Since $\delta(\Psi) \geq d$, $dn \leq 2m$ (2). From (1) and (2), $dm \leq d(r-\varepsilon) + 2m$ so that $(d-2)m \leq d(r-\varepsilon)$ (3). Since $\delta^*(\Psi) \geq 4$, $4r \leq 2m$ so that $2r \leq m$ (4). From (3) and (4), $2(d-2)r \leq (d-2)m \leq d(r-\varepsilon)$, hence $(d-4)r \leq -d\varepsilon$, and thus $(d-4)(r-\varepsilon) \leq -d\varepsilon - (d-4)\varepsilon$, i.e., $(d-4)(r-\varepsilon) \leq -2(d-2)\varepsilon$ (5). By (3) and (5) and since $d-4 \geq 0$, $2(d-4)(d-2)m \leq 2(d-4)d(r-\varepsilon) \leq -4d(d-2)\varepsilon$, and dividing by $d-2 > 0$ gives $2(d-4)m \leq -4d\varepsilon$ (6). By (2) and (6) and since $d-4 \geq 0$, $dn(d-4) \leq 2m(d-4) \leq -4d\varepsilon$ and dividing by $d > 0$ gives $n(d-4) \leq -4\varepsilon$, as required. □

Lemma 5.5. *Let Σ be a surface with $\Sigma \neq S_0$, and let $d = H_{\mathrm{even}}(\Sigma)$. Then $d \geq 4$ and d is the smallest positive integer such that $(d+1)(d-4) > -4\varepsilon(\Sigma)$.*

Proof. Consider $p(x) = (x+1)(x-4) + 4\varepsilon(\Sigma) = x^2 - 3x - 4 + 4\varepsilon(\Sigma)$. Since $\varepsilon(\Sigma) \leq 1$, $p(1) = p(2) < 0$, and so $p(x)$ has two real roots $\alpha_1 < 1$ and $\alpha_2 > 2$. By the quadratic formula, $\alpha_2 = (3 + \sqrt{25 - 16\varepsilon(\Sigma)})/2$. A positive integer d has $p(d) > 0$ if and only if $d > \alpha_2$. But since d is an integer, $d > \alpha_2$ is equivalent to $d \geq \lfloor \alpha_2 + 1 \rfloor = H_{\mathrm{even}}(\Sigma)$. So the smallest positive integer d with $(d+1)(d-4) > -4\varepsilon(\Sigma)$, or $p(d) > 0$, or $d > \alpha_2$, is exactly $H_{\mathrm{even}}(\Sigma)$.

Since $\varepsilon(\Sigma) \leq 1$, if $d = H_{\mathrm{even}}(\Sigma)$ then the formula for $H_{\mathrm{even}}(\Sigma)$ yields $d \geq 4$. □

Outline of proof that $\chi(\Phi) \leq H_{\mathrm{even}}(\Sigma)$ for general embeddings. We label the steps here for reference. Let $\varepsilon = \varepsilon(\Sigma) \leq 1$. (A) Let $d = H_{\mathrm{even}}(\Sigma) \geq 4$. (B) Let Φ be an even-faced embedding of minimum order n in Σ whose graph G is not d-colorable. (C) If $n \leq d$, then G is d-colorable, so $n \geq d+1$. (D) Remove all loops from Φ to obtain an even-faced embedding Φ_1 of G_1 with $\chi(G_1) = \chi(G)$. (E) Starting with Φ_1 repeatedly remove one edge from each 2-face until no 2-faces remain, giving an even-faced embedding Φ_2 of a graph G_2 with $\chi(G_2) = \chi(G)$ and $\delta^*(\Phi_2) \geq 4$. (F) If $\delta(\Phi_2) \leq d-1$, then we can remove a vertex x of degree at most $d-1$, d-color $G_2 - x$ by minimality since $\Phi_2 - x$ remains even-faced, then color x, so G_2, and hence G, is d-colorable. Thus, $\delta(\Phi_2) \geq d$. (G) By (E), (F), Lemma 5.4 and (C), $(d+1)(d-4) \leq n(d-4) \leq -4\varepsilon$, contradicting Lemma 5.5. □

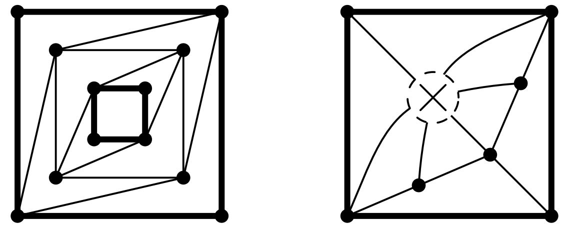


Figure 2: Adding a quadrangular handle (left) or crosscap (right).

Working with general embeddings, rather than cellular embeddings, simplifies the above proof in step (F): we do not have to worry about losing cellularity when we delete a vertex.

We now show that the bound of Theorem 5.1 can be improved by 1 when the surface is S_2, and then show that with this improvement the result is sharp.

Proposition 5.6. *Suppose G is a graph (multiple edges and loops allowed) with a general even-faced embedding Φ in S_2. Then (ignoring loops when coloring) $\chi(G) \le H_{\text{even}}(S_2) - 1 = 5$.*

Proof. We modify the above proof that $\chi(\Phi) \le H_{\text{even}}(\Sigma)$. In (A) take $d = 5$ instead of $d = H_{\text{even}}(S_2) = 6$. As in (B), let Φ be an even-faced embedding of minimum order n in $\Sigma = S_2$ ($\varepsilon = -2$) of a graph G with $\chi(G) > d = 5$. Apply Lemma 5.2 to convert Φ into an even-faced cellular embedding Φ_0 of a necessarily connected graph G_0. Apply (D) and (E) to Φ_0 to construct Φ_1 and Φ_2 as above. Then G_2 is loopless and connected, $\chi(G_2) = \chi(G_1) = \chi(G_0) \ge \chi(G)$, and $\delta^*(\Phi_2) \ge 4$. From (C) and (F), $n \ge d+1 = 6$ and $\delta(\Phi_2) \ge d = 5$.

Delete edges from Φ_2 to obtain an even-faced embedding Φ_3 in S_2 of an underlying connected simple graph G_3 of G_2. We have $\chi(G_3) = \chi(G_2) \ge \chi(G) > 5$. Let m and r be the number of edges and faces of Φ_3, respectively.

Suppose that $n = 6$. If $G_3 \ne K_6$, then G_3, and hence G, is 5-colorable, so $G_3 = K_6$. But then, by Proposition 4.8, Φ_3 cannot exist, a contradiction.

So $n \ge 7$. If $\Delta(G_3) = 5$, then since G_3 is simple, connected and not equal to K_6, by Brooks' Theorem, G_3, and hence G, is 5-colorable. Thus, G_3 has a vertex of degree 6 or more, from which $2m \ge 5(n-1)+6 = 5n+1$, so $m \ge \lceil (5n+1)/2 \rceil$. By Observation 3.3, $\delta^*(\Phi_3) \ge 4$, so $2m \ge 4r$ and $r \le m/2$. Therefore, $-2 = \varepsilon \le n - m + r \le n - m + m/2 = n - m/2 \le n - \lceil (5n+1)/2 \rceil/2$. This fails if $n \ge 8$, so $n = 7$. Moreover, when $n = 7$ this is tight, so all steps in our reasoning are tight. In particular, G_3 has one vertex of degree 6 and $n - 1 = 6$ vertices of degree 5. But then $G_3 = K_7 - 3K_2$ (delete three independent edges from K_7), which is 4-colorable, a contradiction.

So no such Φ exists, and $\chi(G) \le 5$ for all G with an even-faced embedding in S_2. □

Before proving the main result of this section, we introduce two operations. By *adding a quadrangular handle* to an orientable quadrangulation Φ, we mean deleting two distinct faces, and inserting a handle (cylinder or annulus) with four new vertices as shown at left in Figure 2, identifying the inner and outer 4-cycles with the boundaries of the deleted faces, so that the resulting quadrangular embedding Φ' is still orientable. Note that if Φ is simple, so is Φ', and if Φ is face-simple, so is Φ'.

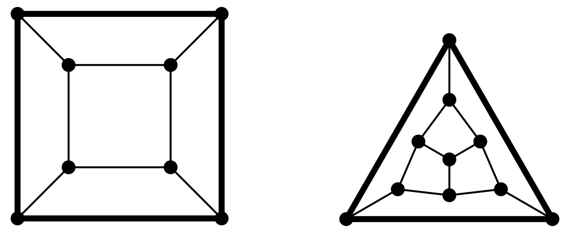


Figure 3: Adding new faces in the same surface.

Also, by *adding a quadrangular crosscap* to a quadrangular embedding Φ we mean inserting a crosscap and three new vertices in a face, as shown at right in Figure 2. The resulting quadrangular embedding Φ' is nonorientable. If Φ is simple, so is Φ', and if Φ is face-simple, so is Φ'.

In both cases, $\chi(\Phi') \geq \chi(\Phi)$.

Proof of Theorem 1.4, the Even Map Color Theorem. Consider first the vertex-coloring result (a). The upper bound on $\chi(\Phi)$ follows from Hutchinson's Theorem 5.1, our Proposition 5.6, and the well-known fact that graphs with even-faced embeddings in the plane are bipartite. Hutchinson provided sharpness examples for $\Sigma = N_1, N_2$ and S_1, but we now provide sharp quadrangular examples for all surfaces.

First we apply the results of Section 4. For $n \geq 7$, it follows from Corollaries 4.5 and 4.10 and addition of quadrangular handles or quadrangular crosscaps that there is a face-simple quadrangular embedding $\Omega_{n,\Sigma}$ in Σ of a simple graph with K_n as a subgraph, and hence with $\chi(\Omega_{n,\Sigma}) \geq n$, provided $\varepsilon(\Sigma) \leq n(5-n)/4$, i.e., $n(n-5) \leq -4\varepsilon(\Sigma)$. This also works for $n = 5$ if Σ is orientable and for $n = 6$ if Σ is nonorientable.

Now suppose that $\varepsilon(\Sigma) \leq -4$, and let $d = c(\Sigma) = H_{\text{even}}(\Sigma) \geq 7$. By Lemma 5.5, d is the smallest positive integer with $(d+1)(d-4) > -4\varepsilon(\Sigma)$, so this inequality fails with d replaced by $d-1$, giving $d(d-5) \leq -4\varepsilon(\Sigma)$. Hence, by the previous paragraph, there is an embedding $\Omega_{d,\Sigma}$ with $d \leq \chi(\Omega_{d,\Sigma}) \leq H_{\text{even}}(\Sigma) = d$, which provides the required sharpness example.

The remaining surfaces are S_h for $0 \leq h \leq 2$ and N_k for $1 \leq k \leq 5$. For S_0 take the standard planar (spherical) embedding of the cube. If $\Sigma = S_1$ or S_2 take $\Omega_{5,\Sigma}$, which has $5 \leq \chi(\Omega_{5,\Sigma}) \leq c(\Sigma) = 5$. For N_1 and N_2, take $\widetilde{\Theta}_4$ in N_1 (from Appendix A), add four new vertices inside each face as shown at left in Figure 3 to obtain face-simple $\widetilde{\Theta}'_4$ in N_1, and then add a quadrangular crosscap to give $\widetilde{\Theta}''_4$ in N_2. Then $4 = \chi(\widetilde{\Theta}_4) \leq \chi(\widetilde{\Theta}'_4) \leq \chi(\widetilde{\Theta}''_4) \leq c(N_1) = c(N_2) = 4$, so take $\widetilde{\Theta}'_4$ and $\widetilde{\Theta}''_4$ for N_1 and N_2, respectively. For N_3 add a quadrangular crosscap to Θ_5 (from Appendix A). If $\Sigma = N_4$ or N_5 use $\Omega_{6,\Sigma}$.

The face-coloring version (b) follows from the vertex coloring-version (a) by taking duals, after applying Lemma 5.2 if an embedding is noncellular. All of the vertex-coloring sharpness examples are quadrangular (implying closed-2-cell), face-simple and simple, so their duals provide face-coloring sharpness examples that are 4-regular, closed-2-cell, simple and face-simple. □

We cannot extend the Even Map Color Theorem to embeddings with all face degrees at least 4, because there is no counterpart to Observation 5.3 for such embeddings. They may realize the

Heawood bound of the original Map Color Theorem. Suppose Φ is a sharpness example (such as a triangular embedding of some K_n) for the original Map Color Theorem in a surface $\Sigma \neq S_0$, N_2, so that $\chi(\Phi) = H(\Sigma)$. If we add new vertices inside each triangular face as shown at right in Figure 3, we obtain an embedding Φ' in Σ with $\delta^*(\Phi') \geq 4$ and $\chi(\Phi') = H(\Sigma) > H_{\text{even}}(\Sigma)$.

6 Embeddings from graphical surfaces and voltage graphs

In this section, we use graphical surfaces and voltage graphs to construct both orientable and nonorientable quadrangular embeddings of certain graphs of the form $G[K_4]$, proving Theorem 1.5.

Theorem 6.1. *Let G be a connected simple graph with a perfect matching. Then $G[K_4]$ has a face-simple orientable quadrangular embedding.*

Proof. Let G have perfect matching M, and let $S(G)$ be the graphical surface derived from G.

First, construct a quadrangular embedding Θ of $H = G[\overline{K_2}\,]$ in $S(G)$ as in Lemma 3.6. For each vertex $v \in V(G)$ there are two vertices $v_\text{N}, v_\text{S} \in V(H)$. For each $uv \in E(G)$, there is a tube T_{uv} in $S(G)$, along which run the edges $u_\text{N}v_\text{N}, u_\text{N}v_\text{S}, u_\text{S}v_\text{S}$ and $u_\text{S}v_\text{N}$ of H. Each edge $u_P v_Q$ of H belongs to two quadrilaterals of the form $(u_\text{N}v_Q u_\text{S} t_X)$ and $(u_P v_\text{N} w_Y v_\text{S})$ where $tu, vw \in E(G)$ and $P, Q, X, Y \in \{\text{N}, \text{S}\}$.

Modify the embedding Θ by splitting each edge into a digon (2-cycle) bounding a face. Let Ψ be the new embedding, with underlying graph J. The other faces of Ψ are quadrilaterals, in one-to-one correspondence with the quadrilaterals of Θ. We now assign voltages from the group $\mathbb{Z}_2$; since all elements of $\mathbb{Z}_2$ are self-inverse, the designation of plus directions for edges does not matter. Choose a voltage assignment $\alpha : E(J) \to \mathbb{Z}_2$ so that the voltages of the edges of J around each tube T_{uv} alternate between 0 and 1. Then each digon of Ψ has one edge of voltage 0 and one edge of voltage 1. Each quadrilateral of Ψ, which uses edges from two tubes T_{uv} and T_{vw}, has an edge of voltage 0 and an edge of voltage 1 on T_{uv}, and similarly for T_{vw}. Therefore, every digon has total voltage 1 and every quadrilateral has total voltage 0 in $\langle J, \alpha\rangle$. Thus, the derived embedding Ψ^α, with underlying graph $J^\alpha = H[\overline{K_2}\,] = G[\overline{K_2}\,][\overline{K_2}\,] = G[\overline{K_4}\,]$, is an orientable quadrangulation.

We could also have obtained a quadrangular embedding of $G[\overline{K_4}\,] = G[\overline{K_2}\,][\overline{K_2}\,]$ directly from Lemma 3.6, but that would not have had the special structure which we now exploit to obtain an embedding of $G[K_4]$. We work with the vertices of G in pairs specified by the perfect matching M.

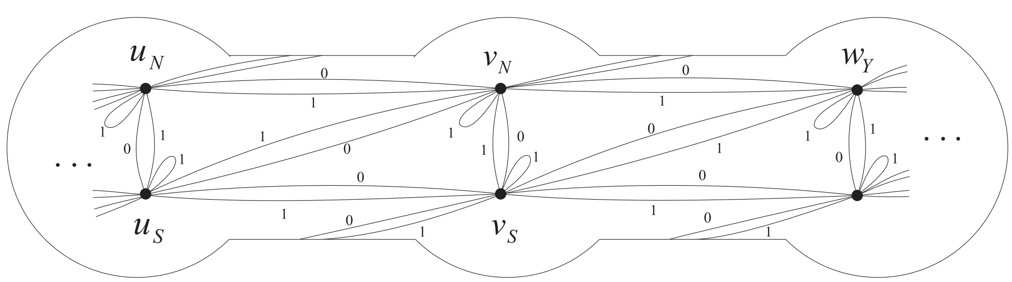


Figure 4: Voltage graph $\langle J_1, \alpha_1\rangle$ generated from graphical surface of $G[\overline{K_2}\,]$.

For each $uv \in M$, let e be one of the four edges of H on the tube T_{uv}, between $\{u_{\mathrm{N}}, u_{\mathrm{S}}\}$ and $\{v_{\mathrm{N}}, v_{\mathrm{S}}\}$. We choose $e = u_{\mathrm{S}}v_{\mathrm{N}}$, as this makes it easier to illustrate what is happening (see Figure 4). In Θ, $e = u_{\mathrm{S}}v_{\mathrm{N}}$ belongs to two quadrilaterals $Q_u = (t_X u_{\mathrm{N}} v_{\mathrm{N}} u_{\mathrm{S}})$ and $Q_v = (u_{\mathrm{S}} v_{\mathrm{N}} w_Y v_{\mathrm{S}})$ where $tu, vw \in E(G)$ and $X, Y \in \{\mathrm{N}, \mathrm{S}\}$. Let the two edges of the digon in J corresponding to e be e_1 and e_2, where e_1 belongs to the quadrilateral Q'_u of Ψ corresponding to Q_u and e_2 belongs to the quadrilateral Q'_v of Ψ corresponding to Q_v. Add a digon of two edges d_1 and d_2 in Q'_u between u_{N} and u_{S}, and a digon of two edges d_3 and d_4 in Q'_v between v_{N} and v_{S}, so we have four triangles $T_1(u) = (t_X u_{\mathrm{N}} u_{\mathrm{S}})$ using d_1, $T_2(u) = (u_{\mathrm{N}} v_{\mathrm{N}} u_{\mathrm{S}})$ using d_2, $T_1(v) = (u_{\mathrm{S}} v_{\mathrm{N}} v_{\mathrm{S}})$ using d_3 and $T_2(v) = (v_{\mathrm{S}} v_{\mathrm{N}} w_Y)$ using d_4. Assign voltage 1 to d_2 and d_3, and 0 to d_1 and d_4. Insert a loop with voltage 1 at each of $u_{\mathrm{N}}, u_{\mathrm{S}}, v_{\mathrm{N}}$ and v_{S} and put these loops in the four different triangles $T_1(u), T_2(u), T_1(v)$ and $T_2(v)$, respectively.

Everything up to this point could have been done using independently chosen quadrilaterals Q'_u containing u_{N} and u_{S} and Q'_v containing v_{N} and v_{S}. However, the total voltages for the 4-faces containing the loops at u_{S} and v_{N} are currently 1, so they will not generate quadrilaterals in the derived embedding. To fix this, swap the voltages on e_1 and e_2: this is where we use the pairing of vertices via M. Let Ψ_1, J_1 and α_1 be the final embedding, graph and voltage assignment, as shown in Figure 4.

In Ψ_1 there are four types of faces. Each 2-face has total voltage 1 in $\langle J_1, \alpha_1 \rangle$, and each quadrilateral and 4-face containing a loop has total voltage 0. These three types of faces all lift to quadrilaterals in $\Psi_1^{\alpha_1}$. The final type of face is bounded by a loop of total voltage 1. This lifts to a face in $\Psi_1^{\alpha_1}$ bounded by a digon between $(u_X, 0)$ and $(u_X, 1)$, where $u \in V(G)$ and $X \in \{\mathrm{N}, \mathrm{S}\}$. Replacing each such digon in $\Psi_1^{\alpha_1}$ by a single edge generates the required orientable quadrangular embedding of $G[K_4]$, which is automatically face-simple by Observation 3.4. □

As a special case of Theorem 6.1, by taking $G = K_{2k}$ for $k \geq 1$ we obtain a proof of Theorem 1.2 if the case where $n \equiv 0 \pmod 8$. We can also obtain a nonorientable version of Theorem 6.1, which provides a proof of Theorem 1.3 in the case where $n \geq 16$ and $n \equiv 0 \pmod 8$.

Theorem 6.2. *Let G be a connected simple graph with a perfect matching and a cycle. Then $G[K_4]$ has a face-simple nonorientable quadrangular embedding.*

Proof. Use Lemma 3.7 instead of Lemma 3.6 in the proof of Theorem 6.1. Replacing one or more tubes by twisted tubes does not affect the argument. Take the orientation-reversing cycle $C = (u_{\mathrm{N}} v_{\mathrm{N}} w_{\mathrm{N}} \ldots z_{\mathrm{N}})$ in $H = G[\,\overline{K_2}\,]$ from the proof of Lemma 3.7 and replace each edge of C by the edge of voltage 0 in the corresponding digon of J_1. This gives an orientation-reversing cycle of total voltage 0 in $\langle J_1, \alpha_1 \rangle$, so the final embedding is nonorientable.

In the nonorientable case we also need to verify that the embedding is face-simple. We can properly 2-face-color Ψ_1, coloring the 4-faces white and the other faces black; this lifts to a proper 2-face-coloring of $\Psi_1^{\alpha_1}$. In Ψ_1 each white face shares an edge with four distinct black faces, so this also holds in $\Psi_1^{\alpha_1}$, and thus $\Psi_1^{\alpha_1}$ is face-simple. Replacing digons makes each white face share at most one edge with another white face, and the final embedding is still face-simple. □

Theorems 6.1 and 6.2 together prove Theorem 1.5.

7 Minimal quadrangulations

In this section we apply our results to determine the order of some minimal quadrangulations.

Hartsfield and Ringel [14, 15] showed that an n-vertex simple quadrangulation of Σ must satisfy $n(n-5) \geq -4\varepsilon(\Sigma)$. They used this to investigate minimal quadrangulations of surfaces of small genus, and to show that quadrangular embeddings of complete graphs and generalized octahedra $O_{2k} = K_k[\overline{K_2}]$, $k \geq 4$ are minimal. Lawrencenko [21] showed that certain orientable quadrangular embeddings of a graph $G[\,\overline{K_2}\,]$, as described in Subsection 3.4, are minimal. The following lemma implies the minimality results of [14, 15, 21].

Lemma 7.1. *Suppose that L is obtained by deleting at most $n-4$ edges from the complete graph K_n, $n \geq 5$. Then any quadrangular embedding of L is minimal.*

Proof. Let $f(x) = x(x-5)/2$. Suppose that $x \geq 5$. If $2\frac{1}{2} \leq x' \leq x-1$, then because f is increasing on $[2\frac{1}{2}, \infty)$ we have $f(x) - f(x') \geq f(x) - f(x-1) = x-3$. If $1 \leq x' \leq 2\frac{1}{2}$, then because $2\frac{1}{2} \leq 5 - x' \leq x-1$ we have $f(x) - f(x') = f(x) - f(5-x') \geq x-3$. Thus, $f(x) - f(x') \geq x-3$ whenever $1 \leq x' \leq x-1$. If n is a nonnegative integer then $f(n) = \binom{n}{2} - 2n$.

Now suppose L has n vertices, m edges, and a quadrangular embedding in Σ. Since at most $n-4$ edges of K_n were deleted, L, and hence also Σ, is connected. If we have another quadrangulation of Σ with $n' \leq n-1$ vertices and m' edges, then, since $m' = 2n' - 2\varepsilon(\Sigma)$ from Observation 1.1,

$$\begin{aligned} m' - \binom{n'}{2} &= 2n' - 2\varepsilon(\Sigma) - \binom{n'}{2} = -f(n') - 2\varepsilon(\Sigma) = -f(n') + m - 2n \\ &\geq -f(n') + \binom{n}{2} - (n-4) - 2n = f(n) - f(n') - (n-4) \geq 1, \end{aligned}$$

proving that the other graph is not simple. □

Lemma 7.1 is sharp whenever K_{n-1} has a quadrangular embedding Φ of the appropriate orientability type (as in Theorems 1.2 and 1.3). Adding a new vertex of degree 2 adjacent to two opposite vertices of a face of Φ yields a quadrangular embedding of a graph obtained from K_n by deleting $n-3$ edges, but this is not minimal.

We can now apply Lemma 7.1 to Lemmas 3.6 and 3.7, and to Theorems 6.1 and 6.2. The orientable case of Corollary 7.2 is due to Lawrencenko [21, Theorem 2].

Corollary 7.2. *Let k and p be integers with $k \geq 4$ and $0 \leq p \leq k/4 - 1$. Suppose G is obtained from K_k by deleting p edges. Then $G[\,\overline{K_2}\,]$ has both orientable and nonorientable quadrangular embeddings that are minimal. Thus, minimal quadrangulations of the orientable surface of genus $k(k-3)/2 - p + 1$ and of the nonorientable surface of genus $k^2 - 3k - 2p + 2$ have order $2k$.*

Proof. Deleting p edges from K_k does not create isolated vertices or destroy all cycles. Thus, by Lemmas 3.6 and 3.7, $G[\,\overline{K_2}\,]$ has orientable and nonorientable quadrangular embeddings. These have order $2k$, and are minimal by Lemma 7.1 since we get $G[\,\overline{K_2}\,]$ by deleting $k + 4p \leq 2k - 4$ edges from K_{2k}. We compute the genera of the surfaces from $m = 2n - 2\varepsilon(\Sigma)$. □

Corollary 7.3. *Let ℓ and q be integers with $\ell \geq 1$ and $0 \leq q \leq (\ell-1)/2$. Suppose G is obtained from $K_{2\ell}$ by deleting q edges. Then $G[K_4]$ has both orientable and nonorientable quadrangular embeddings that are minimal. Thus, minimal quadrangulations of the orientable surface of genus $8\ell^2-5\ell-4q+1$ and of the nonorientable surface of genus $16\ell^2-10\ell-8q+2$ have order 8ℓ.*

Proof. If $\ell = 1$ then $q = 0$ and $G[K_4] = K_{8k}$, so orientable and nonorientable quadrangular embeddings exist by Theorems 1.2 and 1.3. If $\ell \geq 2$ then the q edges deleted from $K_{2\ell}$ are incident with at most $\ell - 1$ vertices, so G has $K_{2\ell} - E(K_{\ell-1})$ as a subgraph, and hence has a perfect matching and a cycle. Thus, by Theorems 6.1 and 6.2, $G[K_4]$ has the required embeddings.

For all ℓ these embeddings have order 8ℓ, and are minimal by Lemma 7.1 since we get $G[K_4]$ by deleting $16q < 8\ell - 4$ edges from $K_{8\ell}$. We compute the genera of the surfaces from $m = 2n - 2\varepsilon(\Sigma)$. □

The simple quadrangulations described in Corollaries 4.5 and 4.10 are also minimal. The simple quadrangulations with $\ell = n + 1$ vertices are embeddings of K_ℓ with $\ell - 4$, $\ell - 5$ or $\ell - 6$ edges deleted, and so are minimal by Lemma 7.1.

Corollary 7.4. *If $n \equiv 2$ or $3 \pmod 4$, $n \geq 6$ and $k = 2 + \lceil n(n-5)/4 \rceil$, then a minimal quadrangulation of N_k has $n+1$ vertices. If $n \equiv 1, 2, 3, 4, 6$ or $7 \pmod 8$, $n \geq 7$ and $h = 1 + \lceil n(n-5)/8 \rceil$, then a minimal quadrangulation of S_h has $n + 1$ vertices.*

There is some overlap here between the conclusions about the order of minimal quadrangulations. The case of Corollary 7.3 with $\ell/4 \leq q \leq (\ell - 1)/2$ is also covered by Corollary 7.2 with $k = 4\ell$ and $p = 4q - \ell$. Some (but not all) cases of Corollary 7.4 are also covered by Corollary 7.2.

8 Conclusion

We give some final remarks.

(1) Hartsfield and Ringel [14, 15] defined quadrangulations more strictly than we do: they insisted that two distinct faces share at most one edge and at most three vertices. For an embedding of a simple graph, this is equivalent to being face-simple. The reason for this restriction is unclear. Perhaps they wished to make the embedding "polyhedral". However, an embedding is now usually considered polyhedral if it is a 3-*representative* (every noncontractible simple closed curve in the surface intersects the graph in at least three points) embedding of a 3-connected graph. A quadrangular embedding of K_n is never polyhedral in this sense: given a face $(uvwx)$, the edge uw is part of the boundary of some other face, and using these two faces we can find a simple closed curve intersecting the graph at just u and w, which must be noncontractible. In any case, all our embeddings, with a few small exceptions, are face-simple and so satisfy Hartsfield and Ringel's definition.

(2) It may be possible to carry out the graphical surface/voltage graph construction from the proof of Theorem 6.1 with non-perfect matchings M of G as well as with perfect matchings, to give orientable and nonorientable quadrangular embeddings of some graphs L with $G[\,\overline{K_4}\,] \subseteq L \subseteq G[K_4]$. This could provide some further examples of minimal quadrangulations.

(3) Our constructions have a lot of flexibility, particularly the constructions from Subsection 3.4 and Section 6. The graphical surface embeddings of $G[\,\overline{K_2}\,]$ in $S(G)$ (with twisted tubes allowed) require a cyclic order of tubes around the equator of each sphere, and a designation of which tubes are to be twisted. (This corresponds to choosing an arbitrary embedding of G, described by a rotation system with edge signatures.) There are two ways to run the edges along each tube. For Theorem 6.1 or 6.2 we may choose an arbitrary perfect matching M of G, and for each edge uv of M we may choose one of four possible edges along the corresponding tube to determine Q_u and Q_v. We also have two ways to assign the voltages for the digons of J running along each tube.

It therefore seems natural to ask whether our techniques can be used to provide useful lower bounds on the number of nonisomorphic quadrangular embeddings of K_n.

A Appendix: Small cases

In this appendix we provide the embeddings for the bases of the inductive proofs in Section 4.

A.1 Nonorientable embeddings

At left in Figure 5 is a face-simple quadrangular embedding $\widetilde{\Psi}_6^-$ of $K_6 - e$ ($e = 01$) in N_3, which is used for constructing embeddings related to K_5. This is shown as a polygon with labeled vertices, indicating how edges are to be identified around the boundary. Nonorientability follows from the existence of edges used twice in the same direction around the outer boundary.

We give nonorientable embeddings $\widetilde{\Theta}_n$ for $n \in \{4, 6, 7, 8\}$ in which all faces are C_4-faces except for possibly one C_6-face. They are all closed-2-cell and all except $\widetilde{\Theta}_4$ are face-simple. The embedding $\widetilde{\Theta}_4$ of K_4 is obtained by taking each of the three hamilton 4-cycles in K_4 as a face boundary. The embedding $\widetilde{\Theta}_6$ of K_6 with six C_4-faces and one C_6-face is generated by the voltage graph shown at center in Figure 5. The loop of voltage 3 generates digons, which are replaced by single edges. A polygon representation of $\widetilde{\Theta}_6$ is also given at right in Figure 5. The embeddings $\widetilde{\Theta}_7$ of K_7 and $\widetilde{\Theta}_8$ of K_8 are shown at left and right, respectively, in Figure 6.

A.2 Orientable embeddings

Below are orientable embeddings in which all faces are C_4-faces except for at most two specified faces. These embeddings are represented as rotation systems using vertices 0, 1, 2, . . . , 9, a, b, c, d. All embeddings are even-faced except for Θ_6. All are face-simple and closed-2-cell except Θ_4.

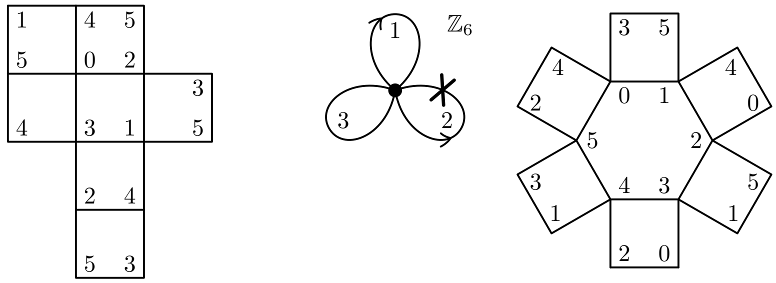


Figure 5: Nonorientable embeddings $\widetilde{\Psi}_6^-$ (left) and $\widetilde{\Theta}_6$ (center and right).

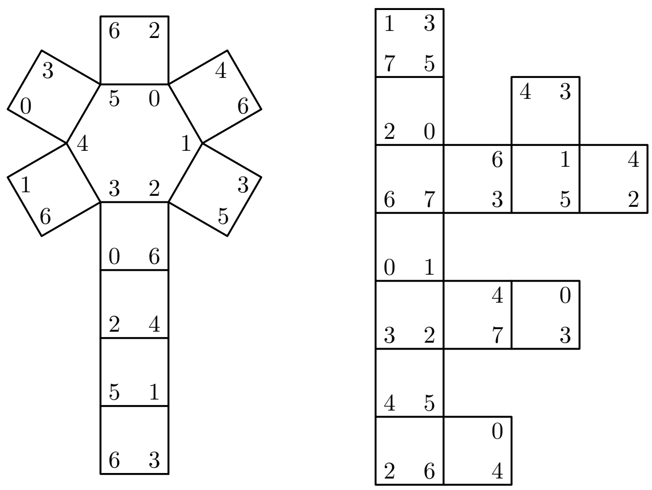


Figure 6: Nonorientable embeddings $\widetilde{\Theta}_7$ (left) and $\widetilde{\Theta}_8$ (right).

Θ_4 of K_4
with 8-face
(01231032):

0. 1 3 2
1. 0 2 3
2. 0 1 3
3. 0 2 1

Θ_5 of K_5:

0. 1 4 2 3
1. 0 4 3 2
2. 0 3 4 1
3. 0 1 2 4
4. 0 2 1 3

Θ_6 of K_6
with C_5-faces
$(01234), (03142)$:

0. 1 2 3 5 4
1. 0 2 3 4 5
2. 0 5 1 3 4
3. 0 1 5 2 4
4. 0 1 2 5 3
5. 0 4 3 2 1

Θ_7 of K_7
with C_6-face
(012345):

0. 1 4 3 2 6 5
1. 0 2 5 3 4 6
2. 0 6 1 3 5 4
3. 0 1 2 4 5 6
4. 0 6 2 3 5 1
5. 0 3 2 1 6 4
6. 0 4 1 5 2 3

Θ_8 of K_8:

0. 1 7 4 3 5 2 6
1. 0 7 2 5 6 4 3
2. 0 7 3 4 6 1 5
3. 0 7 1 6 5 2 4
4. 0 6 1 2 5 3 7
5. 0 4 3 1 6 2 7
6. 0 5 2 1 3 4 7
7. 0 3 4 2 5 1 6

Θ_9 of K_9 with C_8-face (01234567):

0. 1 4 6 2 8 5 3 7
1. 0 2 6 4 3 7 5 8
2. 0 8 5 7 6 4 1 3
3. 0 8 2 4 6 1 7 5
4. 0 8 6 2 3 5 7 1
5. 0 7 3 2 1 4 6 8
6. 0 1 4 5 7 3 2 8
7. 0 5 8 2 3 4 1 6
8. 0 3 5 4 1 2 6 7

Θ_{10} of K_{10} with C_{10}-face (0123456789):

0. 1 5 6 4 7 3 8 2 9
1. 0 2 4 3 6 7 5 8 9
2. 0 9 8 7 6 1 3 5 4
3. 0 9 1 8 2 4 5 7 6
4. 0 9 2 3 5 7 8 1 6
5. 0 9 4 6 7 1 3 2 8
6. 0 8 3 5 7 4 2 1 9
7. 0 6 8 5 3 1 2 4 9
8. 0 6 5 3 4 2 1 7 9
9. 0 4 6 3 7 5 1 2 8

Θ_{11} of K_{11} with C_{10}-face (0123456789):

0. 1 5 7 3 a 6 4 8 2 9
1. 0 2 7 9 6 4 5 8 3 a
2. 0 a 1 3 5 6 9 8 4 7
3. 0 a 7 2 4 1 9 8 6 5
4. 0 a 3 5 2 6 8 1 9 7
5. 0 a 3 2 7 4 6 1 8 9
6. 0 9 1 2 3 4 8 5 7 a
7. 0 9 2 5 3 6 8 4 1 a
8. 0 7 9 1 6 2 5 4 3 a
9. 0 7 5 2 1 4 6 a 3 8
a. 0 5 1 4 6 3 7 2 8 9

Θ_{12} of K_{12} with C_8-face (01234567):

0. 1 a 2 9 3 8 4 b 6 5 7
1. 0 2 b 6 5 8 4 7 3 a 9
2. 0 b 7 5 1 3 8 4 6 9 a
3. 0 b 7 8 5 9 6 2 4 1 a
4. 0 b 1 a 3 5 8 2 9 6 7
5. 0 b 2 8 7 3 4 6 9 1 a
6. 0 a 1 9 4 2 8 3 5 7 b
7. 0 a 1 b 4 9 5 3 8 2 6
8. 0 a 1 9 2 6 4 3 5 7 b
9. 0 a 1 2 4 8 5 3 7 6 b
a. 0 9 2 1 4 8 3 7 5 6 b
b. 0 7 4 8 3 9 1 5 6 2 a

Θ_{14} of K_{14} with C_6-face (012345):

0. 1 7 a 4 b 3 c 2 d 8 6 9 5
1. 0 2 6 7 a 3 b 9 4 8 5 c d
2. 0 d c 1 3 9 b a 4 8 5 7 6
3. 0 d 2 4 6 9 1 c 5 7 8 a b
4. 0 d 6 8 7 9 3 5 a 2 b 1 c
5. 0 d 7 a 8 3 1 c 2 b 9 6 4
6. 0 d 2 a 5 3 9 4 8 7 b 1 c
7. 0 d 6 9 1 b 4 3 8 5 a 2 c
8. 0 c 1 b 2 a 7 5 3 4 9 6 d
9. 0 c 1 a 7 8 6 4 b 3 5 2 d
a. 0 c 1 9 b 2 3 5 8 4 6 7 d
b. 0 c 3 2 5 8 4 a 1 9 7 6 d
c. 0 b 4 9 6 2 1 3 a 7 5 8 d
d. 0 6 8 4 a 5 9 3 b 7 1 2 c

B Appendix: Analysis of K_6 in S_2

In this appendix we prove Proposition 4.8, and show that K_6 has no even-faced general embedding in S_2.

In an embedding the faces around a vertex must form a *proper rotation*, a cyclic sequence closing up so that the vertex has a neighborhood homeomorphic to an open disk. If a potential set of faces incident with a vertex v close up in a cyclic sequence without including all edges incident with v, we say the rotation at v is *improperly closed.*

Lemma B.1. *K_6 has no orientable embedding in which every face is a C_4-face except for one C_6-face.*

Proof. Assume that such an embedding exists. Since the embedding is orientable we may assign consistent orientations to all the faces, and describe each face using a cyclic list of vertices following the orientation (it is not equivalent to its reverse).

We will label the vertices of K_6 by elements of $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$. Each edge has an obviously defined *length* of 1, 2 or 3 depending on $j-i$. Each arc (directed edge) from i to j has *length* $j-i \in \mathbb{Z}_6$

which we will write as an element of the set $\{-2,-1,1,2,3\}$ (where $-2=4$, $-1=5$). Each arc is used exactly once by our embedding. Without loss of generality we may label the vertices so that the C_6-face is $Z=(054321)$, using all the arcs of length -1. Therefore the remaining 6 C_4-faces use 6 arcs of each length 1, 2, -2 and 3. Since no C_4 in K_6 can use more than two arcs of length ± 2, and altogether they use 12 arcs of length ± 2, each of the 6 C_4 faces must use exactly two arcs of length ± 2.

Therefore the cyclic pattern of lengths in each C_4-face (following the arcs in their positive direction) must be one of 6 possibilities: $A=(1,1,2,2)$, $B=(1,2,1,2)$, $C=(1,-2,-2,3)$, $D=(1,3,-2,-2)$, $E=(1,-2,3,-2)$ or $F=(2,3,-2,3)$. For each pattern P let P_i be the potential face starting at vertex i and following pattern P; for example, $C_3=(3420)$ ($3+1=4$, $4-2=2$, $2-2=0$, $0+3=3$).

Any face A_i together with Z improperly closes the rotation at $i+1$, so there are no faces of pattern A. Let n_F be the number of faces of pattern F and n_{CDE} the number of faces of pattern C, D or E. Counting the arcs of length -2 we have $2n_{CDE}+n_F=6$. Counting the arcs of length 3 we have $n_{CDE}+2n_F=6$. Therefore $n_{CDE}=n_F=2$.

Consider the two faces of pattern F. They must share an edge of length 3, which we may assume is 03. The faces of pattern F using this edge are F_3 and F_4, which use arc 03, and F_0 and F_1, which use arc 30. We must have one face that uses arc 03 and one that uses arc 30. Moreover, we cannot have F_3 and F_0 because they are reverses of each other, giving improper rotations at all of their vertices if they occur together in an embedding. Similarly, we cannot have F_4 and F_1, because they are reverses of each other. So we must have F_3 and F_1, or F_4 and F_0. Without loss of generality we assume we have F_4 and F_0; if we have F_3 and F_1 we just add 3 to all the vertex labels and they become F_0 and F_4.

Now consider the arc 01, which must belong to some face. The possible faces are $B_0=B_3$, C_0, D_0 or E_0. If B_0 is a face then the arc 45 is used by both B_0 and F_4. If C_0 is a face then the arcs 53 and 30 are used by both C_0 and F_0, a contradiction. If D_0 is a face then the arc 14 is used by both D_0 and F_4, a contradiction. If E_0 is a face then E_0 and F_0 improperly close up the rotation at vertex 2, which is a contradiction.

Hence all situations lead to a contradiction, so, as claimed, there is no such embedding of K_6. □

Lemma B.2. *Every cellular orientable embedding of K_6 in which some vertex is incident with five C_4-faces must have five C_4-faces and two C_5-faces.*

Proof. Label the vertices of K_6 by ∞ and the elements of $\mathbb{Z}_5=\{0,1,2,3,4\}$. Without loss of generality we may suppose that ∞ has clockwise rotation $(0,1,2,3,4)$, and that for $i\in\mathbb{Z}_5$ there is a 4-cycle face $(\infty,i,a_{i+3},i+1)$ (writing faces also in clockwise order; this labelling makes a_i 'opposite' to i in the face neighborhood around ∞). For each $i\in\mathbb{Z}_5$ we must have $a_i\notin\{\infty,i-3,i-2\}$, so $a_i\in\{i-1,i,i+1\}$ for each i; let the rotation around i be $(\infty,a_{i+2},b_{i1},b_{i2},a_{i+3})$. Since the same vertex cannot occur twice in the rotation around $i-2$, $a_i\neq a_{i+1}$ for each i.

Suppose first that $a_i=a_j$ for some $i\neq j$. The only way this can happen is if $j=i\pm 2$ and a_i is the number between i and j. Without loss of generality suppose that $a_4=a_1=0$. Then the rotation around 0 contains the sequence $4,3$ since $a_1=0$ and the sequence $2,1$ since $a_4=0$. Since

$a_2 \neq 4$, the rotation around 0 must be $(\infty, a_2 = 2, 1, 4, a_3 = 3)$. Since $a_3 = 3$, the rotation around 3 contains the sequence $1, 0$; but we already know that the rotation around 3 contains the sequence $b_{32}, a_1 = 0$ and hence $b_{32} = 1$. This means that $a_0 \neq 1$. By similar reasoning, $b_{21} = 4$ and hence $a_0 \neq 4$. Also $a_0 \neq a_4 = 0$. But now there are no possible values for a_0, which is a contradiction. So we know that $a_i \neq a_j$ when $i \neq j$. Hence each $j \in \mathbb{Z}_5$ occurs exactly once as some a_i.

Suppose that $a_0 = 1$. The rotation around $a_0 = 1$ contains the sequence $3, 2$, so we have $(3, 2) = (a_3, b_{11})$, (b_{11}, b_{12}) or (b_{12}, a_4). We cannot have $a_4 = 2$ so $(3, 2)$ cannot be (b_{12}, a_4). If $(3, 2) = (a_3, b_{11})$ then both $1 = a_0$ and $3 = a_3$ occur as vertices a_i, so we must have $a_2 = 2$. Then the rotation around $a_2 = 2$ contains the sequence $0, 4$. Now $a_4 \neq a_3 = 3$ so $a_4 \in \{0, 4\}$ and the rotation around 2 contains the sequence ∞, a_4, so we cannot have $a_4 = 4$ and we must have $a_4 = 0$. But now $1, 2, 0$ have all been used as vertices a_i, so there is no valid value for a_1, a contradiction. Hence we must have $(3, 2) = (b_{11}, b_{12})$. But then a_3 cannot be equal to either 3 or 2, so $a_3 = 4$.

Generalizing the above, we have shown that $a_i = i+1$ implies that $a_{i+3} = i+4$, and $b_{i+1,1} = i+3$, $b_{i+1,2} = i + 2$. Repeating this reasoning determines the rotation around every vertex i as being $(\infty, i + 3, i + 2, i + 1, i - 1)$. This gives an embedding of K_6 with five C_4-faces and two C_5-faces (03142) and (01234), in clockwise order.

The case where $a_i = i - 1$ for some i is symmetric. So we need only deal with the case where $a_i = i$ for all i. However, this is impossible: for example, it leads to the rotation around vertex 0 containing $a_3, \infty = 3, \infty$ but also the rotation around $a_0 = 0$ containing $3, 2$.

Thus, the only possible situations lead to the embedding specified. □

Proof of Proposition 4.8. Assume there is a general embedding Φ of K_6 in S_2. As noted in Section 4, Φ must be cellular with six C_4-faces and one 6-face, or five C_4-faces and two C_5-faces.

Suppose there is a 6-face. By Lemma B.1 the 6-face is not a C_6-face. So the 6-face has fewer than 6 distinct vertices, and thus there is some vertex all of whose adjacent faces are C_4-faces. But then, by Lemma B.2, Φ does not have a 6-face, a contradiction.

Therefore the embedding has five C_4-faces and two C_5-faces. The embedding Θ_6 in Appendix A, or the embedding found in the proof of Lemma B.2, shows that such an embedding exists. □

Acknowledgement

The authors thank Joan Hutchinson for information regarding [13].

References

[1] A. Altshuler and U. Brehm, Neighborly maps with few vertices, *Discrete Comput. Geom.* **8** (1992) 93–104.

[2] C.P. Bonnington, M.J. Grannell, T.S. Griggs and J. Širáň, Exponential families of non-isomorphic triangulations of complete graphs, *J. Combin. Theory Ser. B* **78** (2000) 169–184.

[3] A. Bouchet, Orientable and nonorientable genus of the complete bipartite graph, *J. Combin. Theory Ser. B* **24** (1978) 24-33.

[4] Beifang Chen, Serge Lawrencenko and Hui Yang, Determination of the 4-genus of a complete graph, preprint, 1998; posted at `https://www.researchgate.net/publication/306255986`, 2016.

[5] D.L. Craft, *Surgical techniques for constructing minimal orientable imbeddings of joins and compositions of graphs*, Ph.D. thesis, Western Michigan University, 1991.

[6] David Craft, Quadrilateral embeddings of composition graphs, 1995, problem posted by Dan Archdeacon at `http://www.cems.uvm.edu/TopologicalGraphTheoryProblems/composit.htm`, downloaded 2 August 2018.

[7] D.L. Craft, On the genus of joins and compositions of graphs, *Discrete Math.* **178** (1998) 25–50.

[8] M.J. Grannell, T.S. Griggs and J. Širáň, A lower bound for the number of triangular embeddings of some complete graphs and complete regular tripartite graphs, *J. Combin. Theory Ser. B* **98** (2008) 637–650.

[9] M.J. Grannell and M. Knor, A lower bound for the number of orientable triangular embeddings of some complete graphs, *J. Combin. Theory Ser. B* **100** (2010) 216–225.

[10] M.J. Grannell and T.A. McCourt, Doubly even orientable closed 2-cell embeddings of the complete graph, *Electron. J. Combin.* **21** (2014) P1.22, 17pp.

[11] J. L. Gross and T. W. Tucker, *Topological Graph Theory*, Dover, Mineola, New York, 2001.

[12] Nora Hartsfield, Nonorientable quadrangular embeddings of complete multipartite graphs, Proc. 25th Southeastern Internat. Conf. on Combinatorics, Graph Theory and Computing (Boca Raton, FL, 1994), *Congr. Numer.* **103** (1994) 161–171.

[13] Nora Hartsfield, The quadrangular genus of complete graphs, preprint, 1994.

[14] N. Hartsfield and G. Ringel, Minimal quadrangulations of orientable surfaces, *J. Combin. Theory Ser. B* **46** (1989) 84–95.

[15] N. Hartsfield and G. Ringel, Minimal quadrangulations of nonorientable surfaces, *J. Combin. Theory Ser. A* **50** (1989) 186–195.

[16] Joan P. Hutchinson, On coloring maps made from Eulerian graphs, Proc. Fifth British Combinatorial Conf. (Univ. Aberdeen, Aberdeen, 1975), *Congr. Numer.* **15** (1976) 343–354.

[17] Joan P. Hutchinson, Three-coloring graphs embedded on surfaces with all faces even-sided, *J. Combin. Theory Ser. B* **65** (1995) 139–155.

[18] K. Kawarabayashi, C. Stephens and X. Zha, Orientable and nonorientable genera of some complete tripartite graphs, *SIAM J. Discrete Math.* **18** (2004) 479–487.

[19] V.P. Korzhik, Generating nonisomorphic quadrangular embeddings of a complete graph, *J. Graph Theory* **74** (2013) 133–142.

[20] V.P. Korzhik and H.-J. Voss, On the number of nonisomorphic orientable regular embeddings of complete graphs, *J. Combin. Theory Ser. B* **81** (1) (2001) 58–76.

[21] Serge Lawrencenko, Realizing the chromatic numbers and orders of spinal quadrangulations of surfaces, *J. Combin. Math. Combin. Comput.* **87** (2013) 303–308.

[22] Serge Lawrencenko, Beifang Chen and Hui Yang, Determination of the 4-genus of a complete graph (with an appendix), `arXiv:1803.03855v1`, 2018.

[23] Serge Lawrencenko, Beifang Chen, Hui Yang and Nora Hartsfield, The orientable 4-Genus Formula for the complete graph, in preparation.

[24] S. Lawrencenko, S. Negami and A.T. White, Three nonisomorphic triangulations of an orientable surface with the same complete graph, *Discrete Math.* **135** (1994) 367–369.

[25] Wenzhong Liu, M.N. Ellingham, Dong Ye and Xiaoya Zha, Quadrangular embeddings of complete graphs, `arXiv:1606.00948v1`, 2016.

[26] Z. Magajna, B. Mohar and T. Pisanski, Minimal ordered triangulations of surfaces, *J. Graph Theory* **10** (1986) 451–460.

[27] B. Mohar, T.D. Parsons, and T. Pisanski, The genus of nearly complete bipartite graphs, *Ars Combin.* **20** (1985) 173–183.

[28] G. Ringel, Das Geschlecht des vollständigen paaren Graphen, *Abh. Math. Sem. Univ. Hamburg.* **28** (1965) 139–150.

[29] G. Ringel, Der vollständige paare Graph auf nichtorientierbaren Flächen, *J. Reine Angew. Math.* **220** (1965) 88–93.

[30] G. Ringel, *Map Color Theorem*, Springer, Berlin, 1974.

[31] Yusuke Suzuki, Triangulations on closed surfaces which quadrangulate other surfaces II, *Discrete Math.* **303** (2005) 235–243.

[32] Arthur T. White, On the genus of the composition of two graphs, *Pacific J. Math.* **41** (1972) 275–279.

[33] J.W.T. Youngs, The mystery of the Heawood conjecture, in 'Graph Theory and its Applications,' Proc. Advanced Sem., Math. Research Center, Univ. of Wisconsin, Madison, Wis., 1969, Academic Press, New York, 1970, pp. 17–50.