Petersen subdivisions in some regular graphs

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ABSTRACT

Tutte has conjectured that every 2-connected cubic graph not containing a subdivision of the Petersen graph is 3-edge-colourable, a result which would be a strong form of the Four Colour Theorem. This is shown to be true for graphs which have a 2-factor consisting of two chordless cycles. General r-regular graphs with a 2-factor of this kind are also discussed; if they have no Petersen subdivision they are hamiltonian and r-edge-colourable.

1. Introduction

A graph is *k-edge-colourable* if the edges can be coloured using k colours so that no two edges with a common vertex have the same colour. A *snark* is a non-3-edge-colourable 2-connected cubic (i.e. 3-regular) graph. Interest in snarks derives from their connection with the Four Colour Problem. In 1880 Tait [5] showed that the Four Colour Theorem is true if and only if every planar 2-connected cubic graph was 3-edge-colourable, or, equivalently, every snark is nonplanar.

The unique snark with fewest vertices is the Petersen graph, which will henceforth be denoted P. This graph is shown in Figure 1.1. It is nonplanar. A *subdivision* of a graph G is obtained by replacing each edge by a path of length ≥ 1. In [6] Tutte made the following conjecture:

Conjecture 1.1: Every snark has a subgraph which is a subdivision of the Petersen graph.

If this is true it implies that every snark is nonplanar, since P is nonplanar. This conjecture, if valid, therefore implies the Four Colour Theorem.

Recent interest in snarks was stimulated by a paper by Isaacs [3] and a

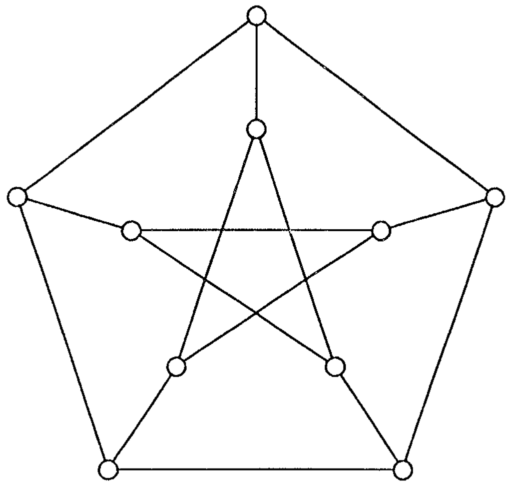

Figure 1.1

subsequent article by Martin Gardner [2], where the term 'snark' was coined. Most of the work done since seems to have focussed on constructing families of snarks; not much progress appears to have been made on Conjecture 1.1.

A 2-*factor* of a graph is a 2-regular spanning subgraph. Such a subgraph must be a union of vertex-disjoint cycles. Petersen, after whom the Petersen graph is named, proved that every 2-connected cubic graph has a 2-factor [4]. It is well-known that a 2-connected cubic graph is 3-edge-colourable if and only if it has an *even* 2-factor, i.e. one in which all cycles have even length. Thus all 2-factors of a snark contain at least one odd cycle. Since every cubic graph has an even number of vertices, every 2-factor must contain an even number of odd cycles. Therefore the simplest type of snark, in terms of 2-factors, is a 2-*odd* snark: one with a 2-factor containing exactly two odd cycles. It can be shown that it is sufficient to prove Conjecture 1.1 for 2-odd snarks alone.

This paper was motivated by a desire to find a subclass of 2-odd snarks for which Conjecture 1.1 could be proved. This was achieved as a corollary of a more general result.

2. The main theorem

In this section the main result of the paper is presented along with its application to a subclass of 2-odd snarks.

Let G be a cubic graph, and $e = uv \in EG$. Let the neighbours of u other than v be u_1 and u_2, and the neighbours of v other than u be v_1 and v_2. Then the *edge-reduction of G by e* or *e-reduction of G* is the graph $(G - \{u, v\}) \cup \{u_1u_2, v_1v_2\}$. This is illustrated in Figure 2.1.

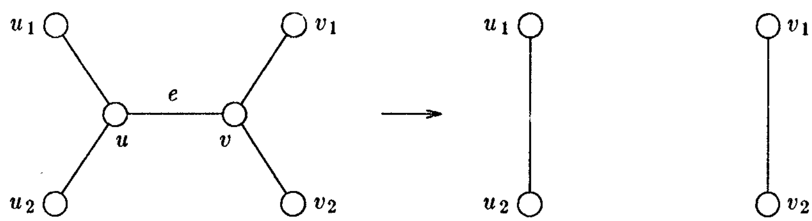

Figure 2.1

Theorem 2.1: Let G be a cubic graph with a 2-factor consisting of two cycles A and B, both of which are chordless in G. (G is sometimes called a *cycle permutation graph*.) Then if G contains no subdivision of P the following hold:

(a) G has a 4-cycle;

(b) G is hamiltonian;

(c) G is 3-edge-colourable.

Proof: Part (a) will be proved by induction on $|VG|$, and then (b) and (c) will be deduced from (a). The result (a) is true for $|VG| \leq 10$ (this is not difficult to check). Therefore, assume that $|VG| \geq 12$ and that (a) is true for all graphs G' satisfying the hypothesis of the theorem and with $|VG'| < |VG|$.

Part (a) will be shown to be true for G by contradiction: assume that G has no 4-cycle. Let a *crossedge* of G be an edge not in EA or EB. If some crossedge-reduction (edge-reduction by a crossedge) of G is a graph G' with no 4-cycle, then by induction G', and thus G, contains a subdivision of P. This is a contradiction. Hence every crossedge-reduction contains a 4-cycle.

Since G itself contains no 4-cycles, a 4-cycle in a graph G' obtained by

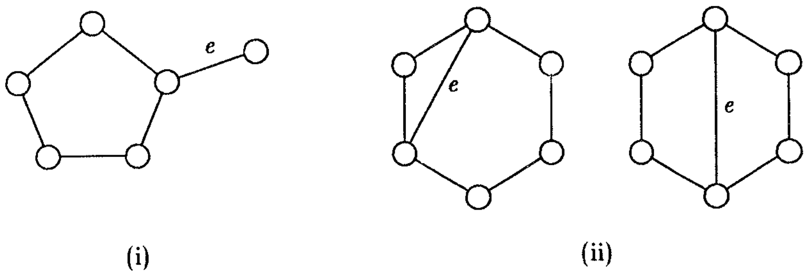

Figure 2.2

reducing a crossedge e must therefore correspond in G to (i) a 5-cycle to which e is incident once, or (ii) a 6-cycle to which e is incident twice (see Figure 2.2). But in case (ii) G contains either a 3-cycle or a 4-cycle. 4-cycles in G do not exist by assumption. 3-cycles in G do not exist because neither A nor B is a 3-cycle ($|VG| \geq 12$) and any other 3-cycle would have to contain either a chord of A or B, or two crossedges incident to a common vertex, in which case this vertex would have degree ≥ 4. Thus case (i) must hold: every crossedge is incident to a 5-cycle.

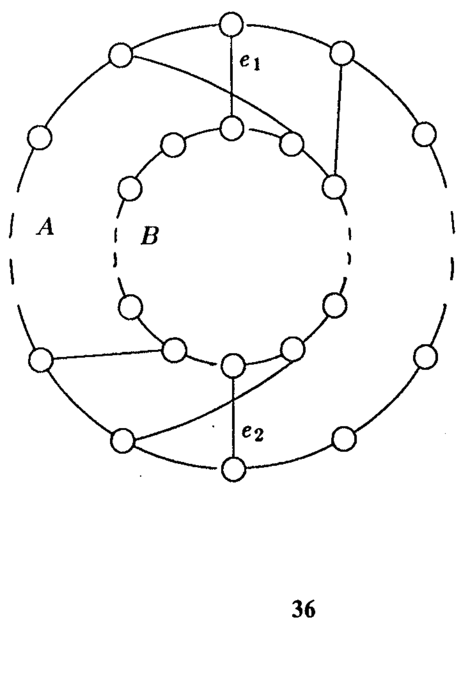

Figure 2.3

If a crossedge e is incident to a 5-cycle at its A- (B-)end, it will be called an A-(B-)*crossedge*. This is illustrated by Figure 2.3, in which e_1 is an A-crossedge and e_2 is a B-crossedge. The edge e must be either an A-crossedge or a B-crossedge; possibly it is both. Since A and B are chordless the crossedges give a 1-1 correspondence between the vertices of A and the vertices of B. Thus $|VA| = |VB| = n$. Let $A = a_1 a_2 ... a_n a_1$ and $B = b_1 b_2 ... b_n b_1$; without loss of generality it may be assumed that $a_1 b_1 \in EG$. Let f be the permutation of $\{1,2,...,n\}$ such that $a_k b_{f(k)} \in EG$ for all k (note that $f(1)=1$); let $g = f^{-1}$, so that $a_{g(l)} b_l \in EG$ for all l.

Let $[i, j]$ denote the closed interval *in the integers* between i and j.

Claim: If $a_k b_{f(k)}$ is an A-crossedge then $a_{k+1} b_{f(k+1)}$ is also an A-crossedge.

Proof of claim: Without loss of generality k may be taken to be 1. Suppose that $f(2) = i$. Since $a_1 b_1$ is an A-crossedge, it is incident to a 5-cycle at a_1. To form this 5-cycle one of the two edges $a_n b_{i+1}$ or $a_n b_{i-1}$ must occur. Without loss of generality assume that $a_n b_{i+1} \in EG$. Then the situation is as shown in Figure 2.4(i).

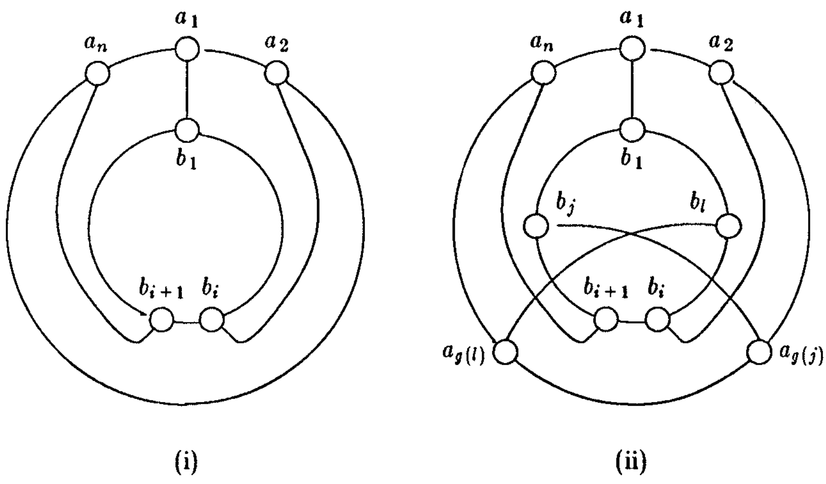

Figure 2.4

Now if there exist $j \in [i+2, n]$ and $l \in [2, i-1]$ such that $g(j) < g(l)$ then G contains a subdivision of P, as illustrated in Figure 2.4(ii). This is impossible. Therefore

$$g(j)>g(l) \text{ for all } j\in[i+2,\ n],\ l\in[2,\ i-1]$$

which implies that

$$g([2,\ i-1])=[3,\ i] \quad \text{and} \quad g([i+2,\ n])=[i+1,\ n-1]\ . \tag{2.1}$$

(Note that $3\leq i\leq n-2$: if $i=2$ then $a_1a_2b_2b_1a_1$ is a 4-cycle, and if $i=n-1$ then $a_1a_nb_nb_1a_1$ is a 4-cycle.)

Now a_2b_i is incident to a 5-cycle. If a_2b_i is a B-crossedge then $a_{n-1}b_{i-1}\in EG$, i.e. $g(i-1)=n-1$, which contradicts (2.1). Therefore a_2b_i is an A-crossedge.

This concludes the proof of the claim. □

Without loss of generality a_1b_1 can be assumed to be an A-crossedge. Applying the claim $n-1$ times then shows that every crossedge is an A-crossedge.

Now a_2b_i must be adjacent to some 5-cycle at its A-end. Using (2.1) it is clear that this 5-cycle must be $a_1a_2a_3b_2b_1a_1$; therefore $a_3b_2\in EG$, i.e. $f(3)=2$. Consider next the 5-cycle to which a_3b_2 must be incident at its A-end; using (2.1) once again, this must be $a_2a_3a_4b_{i-1}b_ia_2$. It is possible to continue in this manner to obtain

$$\begin{array}{ll} f(1)=1 & f(2)=i \\ f(3)=2 & f(4)=i-1 \\ f(5)=3 & f(6)=i-2 \\ \vdots & \vdots \\ f(2k-1)=k & f(2k)=i+1-k \\ \vdots & \vdots \end{array}$$

(where $2k-1, 2k\in[1,\ i]$). But then if i is even,

$$f(i-1)=\frac{i}{2} \quad \text{and} \quad f(i)=\frac{i}{2}+1\ ;$$

therefore there is a 4-cycle $a_{\frac{i}{2}}a_{\frac{i}{2}+1}b_ib_{i-1}a_{\frac{i}{2}}$ in G. And if i is odd,

$$f(i-1)=\frac{i-1}{2} \quad \text{and} \quad f(i)=\frac{i+1}{2}\ ;$$

therefore there is a 4-cycle $a_{\frac{i+1}{2}}a_{\frac{i-1}{2}}b_{i-1}b_ia_{\frac{i+1}{2}}$ (note that this is true even when $i=3$).

Thus a 4-cycle is present, which is a contradiction. This completes the proof of (a).

To prove (b) from (a): any 4-cycle in G must have the form $a_k b_l b_{l\pm 1} a_{k+1} a_k$ (working modulo n), for some k and l. Then $(A-\{a_k a_{k+1}\}) \cup (B-\{b_l b_{l\pm 1}\}) \cup \{a_k b_l, a_{k+1} b_{l\pm 1}\}$ is a hamiltonian cycle in G. A hamiltonian cycle of a cubic graph is an even 2-factor and therefore, using a result mentioned earlier, G is 3-edge-colourable, proving (c). □

Now it is possible to state the promised corollary concerning a subclass of 2-odd snarks:

Corollary 2.2: Let G be a snark which has a 2-factor consisting of exactly two odd cycles, both chordless in G. Then G has a subgraph which is a subdivision of the Petersen graph.

This follows immediately from Theorem 2.1(c). At present it is not known whether the snarks satisfying the conditions of this corollary form a significant subclass of the 2-odd snarks; further investigation is required. The three smallest known 'nontrivial' snarks, namely the Petersen graph and the two Blanuša snarks (see [3]), do belong to this subclass.

3. Extension to regular graphs of arbitrary degree

Theorem 2.1 can be extended very simply to r-regular graphs for any $r \geq 3$, as follows:

Theorem 3.1: Suppose that $r \geq 3$. Let G be an r-regular graph with a 2-factor consisting of two cycles A and B, both chordless in G. Then if G contains no subdivision of P the following hold:

(a) G has a 4-cycle;

(b) G is hamiltonian;

(c) G is r-edge-colourable.

Proof: Consider $G-(EA \cup EB)$. This is an $(r-2)$-regular bipartite graph. Since every regular bipartite graph has a perfect matching (see [1], page 73), it has a perfect matching M. Let $C=A \cup B \cup M$. C is a cubic spanning subgraph of G, for which Theorem 2.1 is applicable. Since G contains no subdivision of P, neither does C. Thus C, and hence G, has a 4-cycle and a hamiltonian cycle. C is 3-edge-colourable, and $G-EC$ is an $(r-3)$-regular bipartite graph and is thus (since $\chi'=\Delta$ for bipartite graphs – see [1], page 93) $(r-3)$-edge-colourable. The colourings of C and $G-EC$ can be combined to give an r-edge-colouring of G. □

Corollary 2.2 can be similarly extended. Let a *supersnark* be a non-r-edge-colourable r-regular graph, for any $r \geq 3$. (Some graphs of this type are non-r-edge-colourable for trivial reasons and are usually excluded from the definition of a supersnark, but this is irrelevant here.)

Corollary 3.2: Let G be a supersnark which has a 2-factor consisting of exactly two cycles, both chordless in G. Then G has a subgraph which is a subdivision of the Petersen graph.

This is a direct consequence of Theorem 3.1(c). Note that if a supersnark has a 2-factor as above, then both its cycles must be odd. The word 'odd', which was included in the statement of Corollary 2.2 to emphasise that a subclass of the 2-odd snarks was being considered, is therefore unnecessary.

References

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