Prism-hamiltonicity of triangulations

Daniel P. Biebighauser^{*} and M. N. Ellingham[†]

September 23, 2005

Abstract

The prism over a graph G is the Cartesian product $G \Box K_2$ of G with the complete graph K_2 . If the prism over G is hamiltonian, we say that G is prism-hamiltonian. We prove that triangulations of the plane, projective plane, torus, and Klein bottle are prism-hamiltonian. We additionally show that every 4-connected triangulation of a surface with sufficiently large representativity is prism-hamiltonian, and that every 3-connected planar bipartite graph is prism-hamiltonian.

1 Introduction

The prism over a graph G is the Cartesian product $G \Box K_2$ of G with the complete graph K_2 . If $G \Box K_2$ is hamiltonian, we say that G is prism-hamiltonian. Kaiser et al. [8] showed that the property of having a hamiltonian prism is stronger than that of having a 2-walk and weaker than that of having a hamilton path. (A 2-walk in a graph is a closed walk that visits every vertex at least once and at most twice.) Put another way,

hamilton path \Rightarrow prism-hamiltonian \Rightarrow 2-walk,

and there are examples in [8] showing that none of these implications can be reversed. The interesting question, then, is whether or not a graph fits in between the properties of having a hamilton path and having a 2-walk. In other words, which graphs are prismhamiltonian even though they may not have a hamilton path?

^{*}Department of Mathematics, 1326 Stevenson Center, Vanderbilt University, Nashville, Tennessee 37240. E-mail: dan.biebighauser@vanderbilt.edu. Supported by a Vanderbilt University Summer Research Award.

[†]Department of Mathematics, 1326 Stevenson Center, Vanderbilt University, Nashville, Tennessee 37240. E-mail: mne@math.vanderbilt.edu. Supported by National Security Agency grant H98230-04-1-0110. The United States Government is authorized to reproduce and distribute reprints notwithstanding any copyright notation herein.

Rosenfeld and Barnette [12] proved that 3-connected cubic planar graphs are prismhamiltonian, but their proof relied on the Four Color Theorem (which was still a conjecture at that time). Fleischner [5] proved the same fact without using the Four Color Theorem. Later, Paulraja [10] proved that all 3-connected cubic graphs (planar or not) are prismhamiltonian.

In [12], Rosenfeld and Barnette conjectured that every 3-connected planar graph is prism-hamiltonian. Kaiser et al. [8] proved that 3-connected planar chordal graphs (also known as *kleetopes*) are prism-hamiltonian, and stated that the conjecture is open for planar triangulations. Since there are planar triangulations that are not chordal, proving this conjecture for planar triangulations would extend the result in [8]. Our first main theorem, which we prove in Section 2, resolves this conjecture for triangulations of the plane, projective plane, torus, and Klein bottle.

Theorem 1.1. Let G be a triangulation of the plane, projective plane, torus, or Klein bottle. Then G is prism-hamiltonian.

Our general strategy for proving Theorem 1.1 is to modify the approach used by Gao and Richter [6] to show that every 3-connected planar graph has a 2-walk. Unlike in [6], we will be concerned about the parity of certain cycles, and so some care is needed to consider different cases, depending on whether or not the triangulations have certain chords. This concern about the parity of cycles also is what does not allow the proof of Theorem 1.1 to be generalized to all 3-connected planar graphs, as we discuss after the proof.

In Section 2 we also prove, using similar techniques, that every 3-connected planar bipartite graph is prism-hamiltonian. So we have resolved Rosenfeld and Barnette's conjecture for, in a sense, two extreme classes of 3-connected planar graphs: bipartite graphs, which have no odd cycles, and triangulations, which have many odd cycles.

In the final section of this paper we examine the prism-hamiltonicity of triangulations of higher surfaces. Let G be a graph embedded in a surface Σ . A simple closed curve in Σ is *noncontractible* if it is homotopically nontrivial in Σ . Otherwise, the curve is *contractible*. The *representativity* (or *face-width*) of G is the maximum integer $\rho(G, \Sigma)$ such that every noncontractible closed curve in Σ meets G at least $\rho(G, \Sigma)$ times, counting multiplicities. This number is finite, and we may assume (by homotopically shifting the curve) that a curve that meets G in $\rho(G, \Sigma)$ places meets the graph only in vertices.

In [16], Yu proved that every 4-connected graph embedded on a surface with large representativity has a 2-walk. We modify his approach to prove our second main theorem.

Theorem 1.2. Let G be a 4-connected triangulation of a surface Σ with $\rho(G, \Sigma)$ sufficiently large. Then G is prism-hamilitonian.

We conclude this section with some notation and terminology. Given distinct $x, y \in V(C)$ on a cycle C in a plane graph, we use xCy to denote the clockwise path in C from x to y.

A block is a 2-connected graph or a graph isomorphic to K_2 or K_1 . A block of a graph is a maximal subgraph which is a block. Every graph has a unique decomposition into edge-disjoint blocks. A vertex v of a graph is a *cut-vertex* if its deletion increases the number of components of the graph.

Let G be a connected graph. Suppose that there exist blocks B_1, B_2, \ldots, B_n (where $n \geq 1$) of G and vertices $b_1, b_2, \ldots, b_{n-1}$ in G such that $b_i \in V(B_i) \bigcap V(B_{i+1})$ for $i = 1, 2, \ldots, n-1$ and $G = \bigcup_{i=1}^n B_i$. We say that G is a *chain of blocks* (or *linear graph*) and that $(B_1, b_1, B_2, \ldots, b_{n-1}, B_n)$ is a *block-decomposition* of G. In this case, $b_1, b_2, \ldots, b_{n-1}$ are precisely the cut-vertices of G.

In the prism $G \Box K_2$, we may identify G with one of its two copies in the prism. Let v be a vertex in G. In the prism, we let v denote the copy of the vertex in the graph that is identified with G, and we let v^* denote the other copy. We use the same notation for edges. An edge of the form vv^* is called a *vertical* edge.

We can represent hamilton cycles in $G \Box K_2$ with certain edge colorings in G. This coloring scheme has been defined previously in [8]. Let C be a hamilton cycle in $G \Box K_2$. We color an edge $e \in E(G)$ blue (B) if $e \in E(C)$ and $e^* \notin E(C)$, yellow (Y) if $e \notin E(C)$ and $e^* \in E(C)$, and green (G) if $e, e^* \in E(C)$. If $e, e^* \notin E(C)$, then e does not get a color. The *type* of a vertex $v \in V(G)$ is the multiset of the colors of all the edges incident with v. It is easy to see that the only possible types of a vertex resulting from a hamilton cycle in $G \Box K_2$ are the types G, BY, GG, GBY, or BBYY. Conversely, any edge coloring of G in which each vertex has one of these types corresponds to a 2-factor in $G \Box K_2$ and is called an *admissible* coloring. If the 2-factor is a hamilton cycle, then we say that the coloring is a *hamilton* coloring.

2 Results for surfaces of low genus

In order to prove Theorem 1.1, we first demonstrate that near-triangulations are prismhamiltonian. A *near-triangulation* is a plane graph where every face is a triangle, except for possibly the outer face, which is bounded by a cycle. We also need some structural results concerning circuit graphs.

A *circuit graph* is an ordered pair (G, C) such that:

1. G is a 2-connected graph and C is a cycle in G,

2. there is an embedding of G in the plane such that C bounds a face, and

3. if (H, K) is a 2-separation of G, then $C \not\subseteq H$ and $C \not\subseteq K$.

(A 2-separation in a graph G is a pair (H, K) of edge-disjoint subgraphs H and K of G such that $G = H \cup K$, $|E(H)| \ge 2$, $|E(K)| \ge 2$, and $|V(H \cap K)| = 2$.) It follows that if G is a 3-connected plane graph and C is any facial cycle of G, then (G, C) is a circuit graph. Barnette [1] originally defined a circuit graph to be a graph obtained by deleting a vertex from a 3-connected planar graph — this definition is equivalent to the definition above.

The following lemmas are Lemma 2, Lemma 3, and Theorem 6, respectively, in [6]. The last sentence in each of Lemma 2.2 and Lemma 2.3 is not in the original statement of each lemma, but it is implicit in each proof.

Lemma 2.1. Let (G, C) be a circuit graph embedded in the plane with C being the outer cycle. If C' is any cycle in G and G' is the subgraph of G inside C', then (G', C') is a circuit graph.

Lemma 2.2. Let (G, C) be a circuit graph embedded in the plane with C being the outer cycle. Let $v \in V(C)$ and let v', v'' be the neighbors of v in the graph C. Then G - v is a plane chain of blocks $(B_1, b_1, B_2, \ldots, b_{k-1}, B_k)$ and, setting $b_0 = v'$ and $b_k = v''$, for $i = 1, 2, \ldots, k, B_i \cap C$ is a path in C with distinct ends b_{i-1} and b_i . Moreover, any face inside a B_i is a face of G.

Lemma 2.3. Let (G, C) be a circuit graph embedded in the plane with C being the outer cycle and let u, v be two distinct vertices in C. There is a partition of V(G) - V(C) into sets V_1, V_2, \ldots, V_m and there are distinct vertices v_1, v_2, \ldots, v_m from $V(C) - \{u, v\}$ such that, for $i = 1, 2, \ldots, m$, the subgraph induced by $V_i \cup \{v_i\}$ is a plane chain of blocks $(B_{i,1}, b_{i,1}, B_{i,2}, \ldots, b_{i,k_i-1}, B_{i,k_i})$ and $v_i \in V(B_{i,1}) - b_{i,1}$. Moreover, any face inside a $B_{i,j}$ is a face of G.

In order to illustrate how we use Lemma 2.3 in the proof of Theorem 1.1, we will show that bipartite circuit graphs (and hence 3-connected planar bipartite graphs) are prism-hamiltonian. We will speak of 'using the vertical edge at v' if v is of type BY or G (i.e., if the hamilton cycle in the prism uses the vertical edge vv^*). Vertical edges are important in the following proofs. Often, if two graphs share a common vertex v, we will find hamilton cycles in each prism that use the vertical edge at v and then join these cycles together, deleting the vertical edges at v, to form a larger cycle.

Theorem 2.4. Let (G, C) be a bipartite circuit graph embedded in the plane with C being the outer cycle and let u, v be two distinct vertices in C. Then there is a coloring of the edges in G that determines a hamilton cycle in $G \square K_2$ and such that the cycle uses the vertical edges at u and v.

Proof. We prove the theorem by induction on |V(G)|. The only bipartite circuit graph with $|V(G)| \leq 4$ is C_4 , and coloring the edges blue and yellow in an alternating fashion produces a hamilton coloring for C_4 .

So we may assume that |V(G)| > 4 and that the statement is true for any bipartite circuit graph with fewer vertices. Since G is bipartite, C is an even cycle and so we may color the edges of C alternately blue and yellow. By Lemma 2.3, we may partition V(G)-V(C) into sets V_1, V_2, \ldots, V_m with distinct vertices v_1, v_2, \ldots, v_m from $V(C)-\{u, v\}$ such that, for $i = 1, 2, \ldots, m$, the subgraph induced by $V_i \cup \{v_i\}$ is a plane chain of blocks $(B_{i,1}, b_{i,1}, B_{i,2}, \ldots, b_{i,k_i-1}, B_{i,k_i})$ with $v_i \in V(B_{i,1}) - b_{i,1}$. For $i = 1, 2, \ldots, m$, let $b_{i,0} = v_i$ and let b_{i,k_i} be any vertex in $V(B_{i,k_i}) - b_{i,k_i-1}$. For each block $B_{i,j}$, if the block is just an edge, we color this edge green. Otherwise, if $C_{i,j}$ is the outer cycle of $B_{i,j}$, then $(B_{i,j}, C_{i,j})$ is a bipartite circuit graph and we can apply the induction hypothesis to find a hamilton coloring in $B_{i,j}$ such that we use the vertical edges at $b_{i,j-1}$ and $b_{i,j}$.

After coloring each block, we have found a hamilton coloring for G. Notice that u, v are of type BY because $u, v \notin \{v_1, v_2, \ldots, v_m\}$, and so we have used the vertical edges at u and v.

The next lemma describes the connection between near-triangulations and circuit graphs.

Lemma 2.5. Let G be a near-triangulation with outer cycle C. Then (G, C) is a circuit graph.

Proof. Consider the graph G' obtained by adding an extra vertex v and joining v to every vertex of C in a planar fashion. Then G' is a triangulation, so G' is 3-connected. Since G can be obtained by deleting a vertex from a 3-connected planar graph, (G, C) is a circuit graph.

Lemma 2.6 describes some restrictions on hamilton cycles in $G \Box K_2$ if we know the cycle uses certain vertical edges.

Lemma 2.6. Let G be a 2-connected planar graph with outer cycle C. If $|V(G)| \ge 4$, a, b, c are consecutive vertices in C, a is adjacent to c, b is only adjacent to a and c, and there is a hamilton cycle in $G \square K_2$ that uses the vertical edges at a and c, then b must be of type BY. In addition, the edge ac must be uncolored.

Proof. Suppose the edge ab is colored green. Since we use the vertical edge at a, a must be of type G. In order for the hamilton cycle to contain c and c^* , b cannot be of type G and hence must be of type GG. Then c is of type G, since we use the vertical edge at c. But now we have described a cycle in $G \square K_2$ which does not contain the remaining vertices of G and their copies, a contradiction.

By symmetry, bc cannot be colored green. Since the hamilton cycle must contain b and b^* , the only remaining option is that b is of type BY. Thus a and c must also be of type BY, since we use the vertical edges at a and c. This implies that the edge ac is uncolored.

We now show that near-triangulations are prism-hamiltonian.

Theorem 2.7. Let G be a near-triangulation with C being the outer cycle and let u, v be two distinct vertices in C. Then there is a coloring of the edges in G that determines a hamilton cycle in $G \Box K_2$ such that the cycle uses the vertical edges at u and v. *Proof.* We prove the theorem by induction on |V(G)|. The only near-triangulation with $|V(G)| \leq 3$ is K_3 , and coloring the edges in the hamilton path from u to v in K_3 green produces a hamilton coloring for K_3 .

So we may assume that |V(G)| > 3 and that the statement is true for any neartriangulation with fewer vertices. If C is an even cycle, we proceed as in the proof of Theorem 2.4. We may color the edges of C alternately blue and yellow. By Lemma 2.3, we partition V(G) - V(C) into sets V_1, V_2, \ldots, V_m with distinct vertices v_1, v_2, \ldots, v_m from $V(C) - \{u, v\}$ such that, for $i = 1, 2, \ldots, m$, the subgraph induced by $V_i \cup \{v_i\}$ is a plane chain of blocks $(B_{i,1}, b_{i,1}, B_{i,2}, \ldots, b_{i,k_i-1}, B_{i,k_i})$ with $v_i \in V(B_{i,1}) - b_{i,1}$. For $i = 1, 2, \ldots, m$, let $b_{i,0} = v_i$ and let b_{i,k_i} be any vertex in $V(B_{i,k_i}) - b_{i,k_i-1}$. For each block $B_{i,j}$, if the block is just an edge, we color this edge green. Otherwise, $B_{i,j}$ is a near-triangulation and we can apply the induction hypothesis to find a hamilton coloring in $B_{i,j}$ such that we use the vertical edges at $b_{i,j-1}$ and $b_{i,j}$. After coloring each block, we have found a hamilton coloring for G. Notice that u and v are of type BY because $u, v \notin \{v_1, v_2, \ldots, v_m\}$, and so we have used the vertical edges at u and v.

Thus we may assume that C is an odd cycle. Since G is a near-triangulation and |V(G)| > 3, vertices in C of degree two in G must be separated from each other in C by vertices of degree at least three (because the edges incident with a degree two vertex must be two edges bounding a triangular face). Since C is an odd cycle, two vertices of degree at least three must be consecutive in C, i.e., there must be an edge $e = w_1w_2 \in E(C)$ such that $\deg(w_1), \deg(w_2) \geq 3$. Since e is incident with a triangular face, w_1 and w_2 must share at least one common neighbor. If there is a common neighbor that is not on C, we delete e to form a spanning subgraph H of G. H is a near-triangulation with an even outer cycle, so we can use the argument in the previous paragraph to find a hamilton cycle in $H \square K_2$ using the vertical edges at u and v, and this cycle is also a hamilton cycle in $G \square K_2$.

So we may assume that C is an odd cycle and for every choice of $e = w_1w_2$, the unique (by planarity) common neighbor x of w_1 and w_2 is also on C. We fix our choice of $e = w_1w_2$. See Figure 1. Note that there must be at least one internal vertex in both xCw_1 and w_2Cx . We now analyze different cases, depending on the locations of u and v. **Case 1**: $x \in \{u, v\}$

We may assume that v = x and that $u \in xCw_1$ (possibly $u = w_1$). Let L be the subgraph of G containing everything bounded by the cycle $C_L := xCw_1 \cup w_1 x$ (including C_L), and let R be the subgraph of G containing everything bounded by the cycle $C_R :=$ $w_1Cx \cup w_1x$ (including C_R). Let y be the neighbor of x in the graph C_R other than w_1 . Let L' be the graph obtained by adding a vertex l to L and joining l to w_1 and x. Let R'be the graph obtained by adding a vertex r to R and joining r to w_1 and x. See Figure 2. Then L, L', R, and R' are near-triangulations. L has at least three vertices (since $\deg(w_1) \geq 3$) and R has at least four vertices (since $\deg(w_2) \geq 3$ and R contains w_1 and x). Thus |V(L')| < |V(G)|, and $|V(R')| \leq |V(G)|$, where |V(R')| = |V(G)| if and only if L



Figure 1: The vertex x is on C

is a triangle. We wish to apply our induction hypothesis to L', R, and R', so we first deal with the case when L is a triangle. In this case, we let L'' be the subgraph of G containing everything bounded by the cycle $C_{L''} := xCw_2 \cup w_2 x$ (including $C_{L''}$), and we let R'' be the subgraph of G containing everything bounded by the cycle $C_{R''} := w_2 Cx \cup w_2 x$ (including $C_{R''}$). (L' and L'' are isomorphic, but we consider them to be distinct because $w_2 \in V(G)$ and $l \notin V(G)$.) $C_{L''}$ is a 4-cycle, so we color its edges blue and yellow in an alternating fashion. Then all four vertices in L'' are of type BY, including u and v = x. Recall that yis the neighbor of x in C_R other than w_1 . If we delete x in R'', we obtain a chain of blocks (since $(R'', C_{R''})$ is a circuit graph), where each nontrivial block is a near-triangulation. By applying the induction hypothesis to each nontrivial block (and coloring each trivial block green), we can find a hamilton cycle in the prism of this chain of blocks that uses the vertical edges at w_2 and y. Then the colorings in L'' and R'' determine a hamilton coloring for G, where u and v = x are still of type BY.

Now we may assume that L is not a triangle, so |V(R')| < |V(G)| and we can apply our induction hypothesis to L', R, and R'. Suppose first that $u = w_1$. By induction, we find hamilton cycles in $L' \Box K_2$ and $R' \Box K_2$ using the vertical edges at w_1 and x in both cycles. By Lemma 2.6, l and r must be of type BY, and the edge $w_1 x$ is uncolored in both L' and R'. When we delete l from L', we have a hamilton path in $L \Box K_2$ with ends at w_1 and x^* or w_1^* and x. The same is true for $R \Box K_2$ when we delete r from R', so after possibly reversing the colors 'blue' and 'yellow' in R' (in order to use the vertical edges at w_1 and x), the colorings in L and R describe a hamilton cycle in $G \Box K_2$, where w_1 and x are both of type BY.

So assume that $u \neq w_1$. By induction, we find a hamilton cycle in $L' \Box K_2$ that uses the vertical edges at u and x. Since $\deg(l) = 2$, l must be of type GG, BY, or G. If l is of type GG, then x is of type G. We find a hamilton cycle in $R \Box K_2$ using the vertical edges at w_1 and x. Then the colorings in L and R determine a hamilton coloring for G. If l is of type BY, then x is of type BY. We find a hamilton cycle in $R' \Box K_2$ using the vertical edges at w_1 and x. By Lemma 2.6, r is of type BY and $w_1 x$ is uncolored. After



Figure 2: The graphs L, L', L'', R, R', and R''

possibly reversing the colors 'blue' and 'yellow' in R', the colorings in L and R determine a hamilton coloring for G. (Since w_1x is not colored in R', we have not colored w_1x two different colors.) Finally, if l is of type G, then the edge w_1l must be green and the edge xl must be uncolored, since we use the vertical edge at x. Recall that y is the neighbor of x in the graph C_R other than w_1 . If we delete x in R, we obtain a chain of blocks (since (R, C_R) is a circuit graph), where each nontrivial block is a near-triangulation. By applying the induction hypothesis to each nontrivial block (and coloring each trivial block green), we can find a hamilton cycle in the prism of this chain of blocks that uses the vertical edges at w_1 and y. Then the colorings in L and R determine a hamilton coloring for G.

Case 2: $x \notin \{u, v\}$

Let L, L', L'', R, R', and R'' be as before. Suppose first that $u \in xCw_1$ and $v \in w_2Cx$. By induction, we find a hamilton cycle in $L \Box K_2$ using the vertical edges at x and u, and we find a hamilton cycle in $R'' \Box K_2$ using the vertical edges at x and v. Then the colorings in L and R'' determine a hamilton coloring for G.

So we may assume that $u, v \in xCw_1$. If L is a triangle, we find hamilton cycles in $L'' \square K_2$ and $R'' \square K_2$ in exactly the same manner as in Case 1. So we may assume that L is not a triangle and we can apply our induction hypothesis to L', R, R', and R''. By induction, we find a hamilton cycle in $L' \square K_2$ using the vertical edges at u and v. Since $\deg(l) = 2$, l must be of type BY, G, or GG. If l is of type BY, we find a hamilton cycle in $R' \square K_2$ using the vertical edges at w_1 and x. By Lemma 2.6, r is of type BY and the

edge $w_1 x$ is uncolored. After possibly reversing the colors 'blue' and 'yellow' in R', the colorings in L and R determine a hamilton coloring for G. (Since $w_1 x$ is uncolored in R', we have not colored $w_1 x$ with two different colors.) If l is of type G, we assume first that the edge $w_1 l$ is colored green and the edge xl is uncolored. In this case, w_1 is not equal to u or v, because the hamilton cycle in $L' \Box K_2$ uses the vertical edges at u and v. Recall that y is the neighbor of x in the graph C_R other than w_1 . If we delete x in R, we obtain a chain of blocks (since (R, C_R)) is a circuit graph), where each nontrivial block is a near-triangulation. By applying the induction hypothesis to each nontrivial block (and coloring each trivial block green), we can find a hamilton cycle in the prism of this chain of blocks that uses the vertical edges at w_1 and y. Then the colorings in L and R determine a hamilton coloring for G. If the edge xl is colored green and the edge w_1l is uncolored, then, by induction, we find a hamilton cycle in $R'' \Box K_2$ that uses the vertical edges at w_2 and x, and the colorings in L and R'' determine a hamilton coloring for G.

Finally, if l is of type GG, there are two possibilities. If we delete l and l^* from our hamilton cycle in $L' \square K_2$, we get two disjoint paths in $L \square K_2$. If one of these paths has endpoints w_1 and w_1^* and the other has endpoints x and x^* (it is possible that one of the paths could just be a single vertical edge), then we find a hamilton cycle in $R \square K_2$ using the vertical edges at w_1 and x. The colorings in L and R determine a hamilton coloring in G. Otherwise, we add the vertical edge xx^* to form a single hamilton path in $L \square K_2$ from w_1 to w_1^* . Then we proceed as in the case when l is of type G. We delete x in R and find a hamilton cycle in the prism of the resulting chain of blocks that uses the vertical edges at w_1 and y. After replacing the vertical edge at w_1 with the hamilton path in Lfrom w_1 to w_1^* , we have found a hamilton cycle in $G \square K_2$.

Note that, in general, the theorem is not true for circuit graphs that are not neartriangulations. In particular, the theorem is not true for any odd cycle of length at least five. The inability to find hamilton cycles that use any two arbitrary vertical edges corresponding to vertices on the outside of general circuit graphs is what prevents us from using this method to show that all 3-connected planar graphs are prism-hamiltonian.

In order to complete the proof of Theorem 1.1, we need the following two lemmas. Lemma 2.8 follows from [4, Proposition 1] and [6, Lemma 4], and Lemma 2.9 follows from [2, Theorems 2 and 3] and [9, Theorem 6.12].

Lemma 2.8. Let G be a 3-connected graph embedded in the projective plane. Then there is a cycle C in G that bounds a closed disk such that the subgraph H of G contained in the closed disk is a spanning subgraph, and (H, C) is a circuit graph.

Lemma 2.9. Let G be a 3-connected graph embedded in the torus or in the Klein bottle. Then there is a spanning subgraph H of G such that H is a plane chain of blocks $(B_1, b_1, B_2, \ldots, b_{k-1}, B_k)$, and, for each nontrivial block B_i with outer cycle C_i , (B_i, C_i) is a circuit graph. Moreover, any face inside a B_i is a face of G.

(Again, it is important to note that the last sentence of Lemma 2.9 is not in the original statements in [2] and [9], but it is implicit in the proofs.)

Proof of Theorem 1.1. If G is embedded in the plane the result follows from Theorem 2.7. If G is embedded in the projective plane, then by Lemma 2.8, G has a spanning near-triangulation and hence is prism-hamiltonian. Finally, if G is embedded in the torus or in the Klein bottle, then by Lemma 2.9, there is a spanning subgraph H such that H is a plane chain of blocks $(B_1, b_1, B_2, \ldots, b_{k-1}, B_k)$, where each nontrivial block is a near-triangulation. By applying Theorem 2.7 to each nontrivial block (and coloring each trivial block green), we see that H is prism-hamiltonian, and hence so is G.

3 Triangulations of higher surfaces

In this section, we give the proof of Theorem 1.2. As mentioned in the introduction, Yu [16] has proved that every 4-connected graph embedded on a surface with large representativity has a 2-walk, and we will modify his approach. In the first part of the proof, we will borrow some ideas and notation from [3], and so we begin with some definitions and preliminary lemmas.

A disk graph is a graph H embedded in a closed disk, such that a cycle Z of H bounds the disk. We will write $\partial H = Z$. An internally 4-connected disk graph (I4CD graph) is a disk graph H such that, from every $v \in V(H) - V(\partial H)$, there are four paths, pairwise disjoint except at v, from v to ∂H . If H is an I4CD graph, then $(H, \partial H)$ is a circuit graph.

A cylinder graph is a graph H embedded in a closed cylinder, such that two disjoint cycles Z_1 and Z_2 of H bound the cylinder. In this case, we will write $\partial H = Z_1 \cup Z_2$. An internally 4-connected cylinder graph (I4CC graph) is a cylinder graph H such that, from every $v \in V(H) - V(\partial H)$, there are four paths, pairwise disjoint except at v, from v to ∂H . An I4CC graph is not necessarily connected, as Z_1 and Z_2 may be in different components.

If G is an embedded graph and Z is a contractible cycle of G bounding a closed disk, then the embedded subgraph consisting of all vertices, edges, and faces in the closed disk is a disk subgraph of G. If Z_1 and Z_2 are disjoint homotopic cycles bounding a closed cylinder, then the embedded subgraph H consisting of all vertices, edges, and faces in the closed cylinder is a cylinder subgraph of G. We will write $H = Cyl_G[Z_1, Z_2]$ or just $H = Cyl[Z_1, Z_2]$. If the surface is a torus or Klein bottle and Z_1 and Z_2 are nonseparating, then this notation is ambiguous, but it will be clear from the context which of the two possible cylinders we mean. We define $Cyl(Z_1, Z_2]$ to be $Cyl[Z_1, Z_2] - V(Z_1)$, and we define $Cyl[Z_1, Z_2)$ and $Cyl(Z_1, Z_2)$ similarly.

The following lemma is Lemma 2.1 in [3].

Lemma 3.1. Suppose G is a 4-connected embedded graph. Any disk subgraph of G bounded by a cycle of length at least four is I4CD, and any cylinder subgraph of G is I4CC.

Let G be an embedded graph. If $\mathcal{R} = \{R_1, R_2, \ldots, R_m\}$ is a collection of pairwise disjoint homotopic cycles with $R_i \subseteq Cyl[R_1, R_m]$ for each $i, \mathcal{S} = \{S_1, S_2, \ldots, S_n\}$ is a

collection of disjoint paths with $S_j \subseteq Cyl[R_1, R_m]$ for each j, and $R_i \cap S_j$ is a nonempty path (possibly a single vertex) for each i and j, then we say that $(\mathcal{R}, \mathcal{S})$ is a *cylindrical* mesh in G.

The following lemma allows us to modify cylindrical meshes. It is Lemma 2.2 (i) in [3].

Lemma 3.2. Suppose H is an I4CC graph with $\partial H = R_1 \cup R_2$ that has a cylindrical mesh $(\{R_1, R_2\}, \{S_1, S_2, \ldots, S_n\})$. Then in H there are disjoint cycles R'_1 and R'_2 homotopic to R_1 (with R'_1 closer to R_1) and pairwise disjoint paths S'_1, S'_2, \ldots, S'_n , such that $Cyl(R'_1, R'_2)$ is empty, each S'_j has the same ends as S_j , and $R'_i \cap S'_j$ is a nonempty path for each i and j.

Let H be a subgraph of a graph G. A bridge of H (or H-bridge) in G is either (a) an edge of E(G) - E(H) with both ends in H, or (b) a component C of G - V(H) together with all of the edges with one end in C and the other in H. Type (a) bridges are called trivial. If J is an H-bridge in G, then J contains no edges of H, and the vertices of H contained in J are called the vertices of attachment of J on H.

Let H be a subgraph of a graph G, and let S be another subgraph of G (usually a path or a cycle). Then S is a *Tutte subgraph* (or *Tutte path* or *Tutte cycle*) with respect to H if:

1. every bridge of S in G has at most three vertices of attachment, and

2. every bridge of S in G that contains an edge of H has at most two vertices of attachment.

If $H = \emptyset$, we simply say that S is a Tutte subgraph.

Near the end of the proof, we will need to find certain Tutte paths in cylinder subgraphs of G. In order to do so, we use the following lemma, which is Lemma 3.3 in [16].

Lemma 3.3. Let H be a 2-connected cylinder graph with $\partial H = Z_1 \cup Z_2$. Let $x, y \in Z_1$ and $u, v \in Z_2$ be four distinct vertices of H. Then H has two disjoint paths P and Q with P from x to y and Q from u to v such that every $(P \cup Q)$ -bridge in H has at most three attachments.

Let G and H be graphs that are both embedded on the same closed surface Σ . H is a *surface minor* of G if the embedding of H can be obtained from the embedding of G by a sequence of contractions and deletions of edges. The following deep result is due to Robertson and Seymour [11].

Lemma 3.4. Let J be a fixed graph embedded on a closed surface Σ . There exists a positive integer R(J) such that if G is embedded on Σ with $\rho(G, \Sigma) \geq R(J)$, then G has J as a surface minor.

Finally, we note that if a surface Σ has Euler genus at least 3, there are triangulations of Σ with arbitrarily high representativity that do not contain spanning trees of maximum degree at most 3 [14] and hence do not contain 2-walks [7]. Since these graphs cannot be prism-hamiltonian, the assumption of 4-connectivity is essential.

Proof of Theorem 1.2. By Theorem 1.1 we may assume that Σ has Euler genus at least 3.

Suppose Σ has Euler genus 2g or 2g + 1, where $g \ge 1$. We can find a connected graph J embedded on Σ that contains g pairwise disjoint copies of $Q = P_6 \Box C_4$, in such a way that deleting the vertices of one C_4 in each of the g copies results in a planar or projectiveplanar graph, and such that J has a vertex at distance at least three from every copy of Q. Assume that $\rho(G, \Sigma)$ is at least max $\{4, R(J)\}$, where R(J) is provided by Lemma 3.4. Then G has J as a surface minor with pairwise disjoint subgraphs Q_1, Q_2, \ldots, Q_g of G contracting to copies of Q in J. Each Q_i has pairwise disjoint cycles $R_{i1}, R_{i2}, \ldots, R_{i6}$ (in that order) and paths $S_{i1}, S_{i2}, S_{i3}, S_{i4}$ (in that cyclic order) such that each R_{ij} contracts to one of the C_4 in a copy of Q, each S_{ik} contracts to one of the P_6 in a copy of Q, and $(\{R_{ij}|1 \le j \le 6\}, \{S_{ik}|1 \le k \le 4\})$ is a cylindrical mesh in G. When we delete one R_{ij} for each i from G, we obtain a planar or projective-planar graph. By Lemma 3.1 and Lemma 3.2, we may assume that $Cyl(R_{i2}, R_{i3})$ and $Cyl(R_{i4}, R_{i5})$ are empty for every i.

Let $H = G - \bigcup_{i=1}^{g} V(Cyl[R_{i3}, R_{i4}])$. *H* can be embedded in the plane or the projective plane where each cycle R_{i2} and R_{i5} , $1 \leq i \leq g$, bounds a face. The vertices of *G* are partitioned by *H* and $Cyl[R_{i3}, R_{i4}]$, $1 \leq i \leq g$. Each of these graphs is 2-connected, because if there was a cut-vertex in any of these graphs, either it would be a cut-vertex in *G*, or there would be a nonseparating simple closed curve intersecting *G* only at the cut-vertex, which contradicts the fact that *G* is 4-connected and $\rho(G, \Sigma) \geq 4$. Similarly, any 2-cut or 3-cut *S* in *H* must contain at least two vertices of some R_{i2} (or some R_{i5}), and H - S must have exactly two components, one of which is in $Cyl(R_{i1}, R_{i2}]$ ($Cyl[R_{i5}, R_{i6}$)).

Since, for every *i*, we have a cylindrical mesh in $Cyl[R_{i2}, R_{i3}]$ and $Cyl[R_{i4}, R_{i5}]$, and $Cyl(R_{i2}, R_{i3})$ and $Cyl(R_{i4}, R_{i5})$ are empty, we conclude that there is a matching M_i (M'_i) with at least 4 edges between R_{i2} and R_{i3} $(R_{i4} \text{ and } R_{i5})$. We form a new graph H' embedded in the plane or the projective-plane from H by, for each $1 \leq i \leq g$, adding a vertex v_i to the face of H bounded by R_{i2} and a vertex v'_i to the face of H bounded by R_{i2} and a vertex v'_i to the face of H bounded by R_{i2} and a vertex v'_i to the face of H bounded by R_{i2} and a vertex v'_i to the face of H bounded by R_{i5} , and joining v_i (v'_i) to every vertex in $R_{i2} \cap M_i$ $(R_{i5} \cap M'_i)$. H' is also 2-connected, and each v_i and v'_i has degree at least four. We wish to characterize the cutsets of H' that have size at most three. Let S' be any minimal cutset of H' with $|S'| \leq 3$. If S' contains no v_i or v'_i , then it is a cutset in H which uses at least two vertices of some R_{i2} (or R_{i5}). If S' contains some v_i or v'_i — we will assume it contains a v_i — then $S = S' - \{v_i\}$ is a cutset in H. Since H is 2-connected, |S| = 2 and both vertices of S belong to some R_{j2} or R_{j5} . In fact, both vertices of S belong to R_{i2} because the minimality of S' implies that v_i is adjacent to vertices in more than one component of H' - S'. So we have shown that every minimal cutset S' of H' with $|S'| \leq 3$ contains two vertices on some R_{i2} or R_{i5} , and that H' - S' has exactly two components, one of which is in $Cyl(R_{i1}, R_{i2}]$ $(Cyl[R_{i5}, R_{i6}))$.

Since J has a vertex at distance at least three from every copy of Q, there is a vertex $w \in V(G)$ at distance at least three from $\bigcup_{i=1}^{g} Cyl[R_{i1}, R_{i6}]$. We now wish to find a Tutte cycle C in H' - w. We need to make sure that C is not just a 3-cycle, that C is not contained in a disk subgraph of H' bounded by R_{i1} or R_{i6} for some i, that C contains

every v_i and v'_i , and that we can modify C to include w if necessary.

Let ww_1, ww_2, \ldots, ww_k be the edges around w in cyclic order, where $k \ge 4$. Since G is a triangulation and $\rho(G, \Sigma) \ge 4$, there is a cycle W in G, and hence in H', containing exactly the vertices w_1, w_2, \ldots, w_k in that order and bounding a closed disk containing all of the faces incident with w. W is a face of G - w and also of H' - w. H' - w is 2-connected, because if H' - w contained a cut-vertex, w and that cut-vertex would be a cutset of H' that does not contain two vertices from some R_{i2} or R_{i5} . Thus we can find a Tutte cycle C with respect to W in H' - w through the edge w_1w_2 , by [15, Theorem 1] if H' is planar or by [13, Theorem 4.1] if H' is projective-planar.

Suppose there is a vertex $w_j \in V(W)$ with $w_j \notin V(C)$. Let *B* be the bridge of *C* in H' - w containing w_j . *B* must have exactly two attachments, *a* and *b*, and we must have $a, b \in V(W)$. But then $\{a, b, w\}$ is a 3-cut in H' where H' - S' does not have a component in a $Cyl(R_{i1}, R_{i2}]$ or a $Cyl[R_{i5}, R_{i6})$, a contradiction. So *C* must contain every vertex of *W*. In particular, *C* is not a 3-cycle.

Let T be a component of H' - V(C). Since C is a Tutte cycle in H' - w and C contains every vertex of W, T has a set S' of exactly three neighbors on C. Since C is not a 3-cycle, S' must be a cutset of H', so S' contains two vertices of R_{i2} or two vertices of R_{i5} for some i. We may assume it contains two vertices of R_{i2} . We also know that H' - S' has exactly two components: T, and another component T' that contains C - S'. One of these two components is a subgraph of $Cyl(R_{i1}, R_{i2}]$, and we argue that T is this component. By our choice of w, w_1 is not adjacent to a vertex of S', so $w_1 \in V(C - S')$. Since $w_1 \notin \bigcup_{i=1}^g V(Cyl[R_{i1}, R_{i6}]), w_1$ and hence C - S' cannot be in the component inside $Cyl(R_{i1}, R_{i2}]$. Thus T is a subgraph of $Cyl(R_{i1}, R_{i2}]$, and hence T cannot contain any v_j or $v'_i, 1 \leq j \leq g$.

For every $i, 1 \leq i \leq g$, let $p_i, q_i \in V(R_{i2})$ be the two neighbors of v_i in C, and let $p'_i, q'_i \in V(R_{i5})$ be the two neighbors of v'_i in C. By the choices of v_i and v'_i , we may assume that p_i and q_i are adjacent to x_i and y_i in R_{i3} , respectively, that p'_i and q'_i are adjacent to x'_i and y'_i in R_{i4} , respectively, and that $x_i \neq y_i$ and $x'_i \neq y'_i$.

By Lemma 3.3, in each $Cyl[R_{i3}, R_{i4}]$ we can find two disjoint paths P_i from x_i to y_i and P'_i from x'_i to y'_i such that every $(P_i \cup P'_i)$ -bridge B of $Cyl[R_{i3}, R_{i4}]$ has at most three attachments. Since G is 4-connected, every such bridge B must contain at least one vertex of R_{i3} or R_{i4} (but not both) not as an attachment.

Let $C' = (C - \bigcup_{i=1}^{g} \{v_i, v'_i\}) \cup (\bigcup_{i=1}^{g} (P_i \cup P'_i \cup \{p_i x_i, q_i y_i, p'_i x'_i, q'_i y'_i\}))$. Then C' is a cycle in G. If C' is even, we color the edges of C' blue and yellow in an alternating fashion, and we also add the green edge ww_1 to form the graph C''. If C' is odd, we replace the edge w_1w_2 in C' with the edges ww_1 and ww_2 to form the graph C'', and we color the edges of C'' blue and yellow in an alternating fashion. Notice that we have used the vertical edge at every vertex of C'', except possibly at w_1 . We will detour into the bridges of C'', using these vertical edges, to find a hamilton coloring in G.

The remainder of this proof closely follows the proof of Theorem 6.3 in [16]. Every nontrivial bridge B of C'' in $H \cup (\bigcup_{i=1}^{g} (Cyl[R_{i3}, R_{i4}] \cup M_i \cup M'_i))$ (which is a spanning subgraph of G) is either (i) a C-bridge in H', and hence is in some $Cyl(R_{i1}, R_{i2}]$ or in some $Cyl[R_{i5}, R_{i6})$ with two attachments in R_{i2} or R_{i5} , respectively, and contains a vertex of R_{i2} or R_{i5} , respectively, not as an attachment or (ii) a $(P_i \cup P'_i)$ -bridge of some $Cyl[R_{i3}, R_{i4}]$ which contains a vertex of R_{i3} or R_{i4} (but not both) not as an attachment. It is possible that some of these bridges may be joined in G by edges in $E(Cyl[R_{i1}, R_{i2}]) - (E(R_{i1}) \cup E(R_{i2}))$ or in $E(Cyl[R_{i5}, R_{i6}]) - (E(R_{i5}) \cup E(R_{i6}))$, but we will not use these edges in our hamilton coloring.

For $i, 1 \leq i \leq g$, consider every type (i) bridge containing a vertex of R_{i2} not as an attachment. We may assume that T_1, T_2, \ldots, T_k are the *C*-bridges of H' appearing on R_{i2} in this order. We may also assume that the attachments of each T_j are $\{x_j, y_j, z_j\}$, where $x_j R_{i2} y_j \subseteq T_j$ and where either $y_j = z_j$ or $y_j \neq z_j$. We wish to show that $T_j - \{y_j, z_j\}$ is a chain of blocks with x_j in an endblock not as a cut-vertex. Suppose that $y_j \neq z_j$. If x_j is not adjacent to z_j or y_j is not adjacent to z_j in T_j , we add the edge $x_j z_j$ or $y_j z_j$ (or both) to T_j in order to form the graph T'_j . Then T'_j is a disk subgraph of a 4-connected graph embedded on Σ , so by Lemma 2.2, $T'_j - z_j = T_j - z_j$ is a chain of blocks $(B_1, b_1, B_2, \ldots, b_{n-1}, B_n)$ with $x_j \in V(B_1), y_j \in V(B_n), x_j \neq b_1$, and $y_j \neq b_{n-1}$. $B_n - y_j$ is also a chain of blocks. If this chain contains more than one block and b_{n-1} is a cut-vertex of this chain, then $\{b_{n-1}, y_j, z_j\}$ is a cutset in G. If the chain contains more than one block and b_{n-1} is not in an endblock not intersecting $x_j R_{i2} y_j$. In this case, $\{a, y_j, z_j\}$ is a cutset in G. Either situation contradicts the fact that G is 4-connected. Thus $T_j - \{y_j, z_j\}$ is a so desired. The proof is similar (simpler) when $y_j = z_j$.

We can apply Theorem 2.7 to each nontrivial block of $T_j - \{y_j, z_j\}$ (and color every trivial block green) to find a hamilton coloring in $T_j - \{y_j, z_j\}$ that uses the vertical edge at x_j . We find similar colorings in the type (i) bridges containing a vertex of R_{i5} not as an attachment, and in the type (ii) bridges containing a vertex of R_{i3} or R_{i4} not as an attachment. When we combine these colorings with the coloring of C'', since we have included every vertex of G and joined these colorings at vertical edges, we have described a hamilton coloring for G.

References

- [1] David Barnette, Trees in polyhedral graphs, Canad. J. Math. 18 (1966), 731–736.
- [2] Richard Brunet, Mark N. Ellingham, Zhicheng Gao, Alice Metzlar, and R. Bruce Richter, Spanning planar subgraphs of graphs in the torus and Klein bottle, J. Combin. Theory Ser. B 65 No. 1 (1995), 7–22.
- [3] M. N. Ellingham and Ken-ichi Kawarabayashi, 2-connected spanning subgraphs with low maximum degree in locally planar graphs, submitted.

- [4] J. R. Fiedler, J. P. Huneke, R. Bruce Richter, and N. Robertson, Computing the orientable genus of projective graphs, J. Graph Theory 20 No. 3 (1995), 297–308.
- [5] H. Fleischner, The prism of a 2-connected, planar, cubic graph is hamiltonian (a proof independent of the four colour theorem), Graph theory in memory of G. A. Dirac, Ann. Discrete Math., 41 (1989), 141–170.
- [6] Zhicheng Gao and R. Bruce Richter, 2-walks in circuit graphs, J. Combin. Theory Ser. B 62 No. 2 (1994), 259–267.
- [7] Bill Jackson and Nicholas C. Wormald, k-walks of graphs, Australas. J. Combin. 2 (1990), 135–146.
- [8] Tomáš Kaiser, Daniel Král', Moshe Rosenfeld, Zdeněk Ryjáček, and Heinz-Jürgen Voss, Hamilton cycles in prisms over graphs, submitted.
- [9] Ken-ichi Kawarabayashi, Atsuhiro Nakamoto, and Katsuhiro Ota, Subgraphs of graphs on surfaces with high representativity, J. Combin. Theory Ser. B 89 No. 2 (2003), 207–229.
- [10] P. Paulraja, A characterization of hamiltonian prisms, J. Graph Theory 17 No. 2 (1993), 161–171.
- [11] Neil Robertson and P. D. Seymour, Graph minors VII, Disjoint paths on a surface, J. Combin. Theory Ser. B 45 No. 2 (1988), 212–254.
- [12] Moshe Rosenfeld and David Barnette, Hamiltonian circuits in certain prisms, Discrete Math. 5 (1973), 389–394.
- [13] Robin Thomas and Xingxing Yu, 4-connected projective-planar graphs are hamiltonian, J. Combin. Theory Ser. B 62 No. 1 (1994), 114–132.
- [14] Carsten Thomassen, Trees in triangulations, J. Combin. Theory Ser. B 60 No. 1 (1994), 56–62.
- [15] W. T. Tutte, A theorem on planar graphs, Trans. Amer. Math. Soc. 82 (1956), 99–116.
- [16] Xingxing Yu, Disjoint paths, planarizing cycles, and spanning walks, Trans. Amer. Math. Soc. 349 No. 4 (1997), 1333–1358.