Toughness and prism-hamiltonicity of P_4-free graphs

M. N. Ellingham*† Pouria Salehi Nowbandegani† Songling Shan‡

† Department of Mathematics, 1326 Stevenson Center,
Vanderbilt University, Nashville, TN 37240

‡ Department of Mathematics, Illinois State University,
Normal, IL 61790

mark.ellingham@vanderbilt.edu
pouria.salehi.nowbandegani@vanderbilt.edu
sshan12@ilstu.edu

6 January 2019

Abstract

The *prism* over a graph G is the product $G \square K_2$, i.e., the graph obtained by taking two copies of G and adding a perfect matching joining the two copies of each vertex by an edge. The graph G is called *prism-hamiltonian* if it has a hamiltonian prism. Jung showed that every 1-tough P_4-free graph with at least three vertices is hamiltonian. In this paper, we extend this to observe that for $k \geq 1$ a P_4-free graph has a spanning k-*walk* (closed walk using each vertex at most k times) if and only if it is $\frac{1}{k}$-tough. As our main result, we show that for the class of P_4-free graphs, the three properties of being prism-hamiltonian, having a spanning 2-walk, and being $\frac{1}{2}$-tough are all equivalent.

Keywords: Toughness, Prism-hamiltonicity, P_4-free graph.

1 Introduction

All graphs considered are simple and finite. Let G be a graph. For $S \subseteq V(G)$ the subgraph induced on $V(G) - S$ is denoted by $G - S$; we abbreviate $G - \{v\}$ to $G - v$. The number of components of G is denoted by $c(G)$. The graph is said to be t-*tough* for a real number $t \geq 0$ if $|S| \geq t \cdot c(G - S)$ for each $S \subseteq V(G)$ with $c(G - S) \geq 2$. The *toughness* $\tau(G)$ is the largest real number t for which G is t-tough, or ∞ if G is complete. Positive toughness implies that G is connected. If G has a hamiltonian cycle it is well known that G is 1-tough.

In 1973, Chvátal [3] conjectured that for some constant t_0, every t_0-tough graph is hamiltonian. Thomassen (see [2, p. 132]) showed that there are nonhamiltonian graphs

*Supported by Simons Foundation award no. 429625.

with toughness greater than $\frac{3}{2}$. Enomoto, Jackson, Katerinis and Saito [6] showed that every 2-tough graph has a 2-factor (2-regular spanning subgraph), but also for every $\varepsilon > 0$ constructed $(2-\varepsilon)$-tough graphs with no 2-factor, and hence no hamiltonian cycle. Bauer, Broersma and Veldman [1] constructed $(\frac{9}{4}-\varepsilon)$-tough nonhamiltonian graphs for every $\varepsilon > 0$. Thus, any such t_0 is at least $\frac{9}{4}$.

There have been a number of papers on toughness conditions that guarantee the existence of more general spanning structures in a graph. A *k-tree* is a tree with maximum degree at most k, and a *k-walk* is a closed walk with each vertex repeated at most k times. A k-walk can be obtained from a k-tree by visiting each edge of the tree twice. Note that a spanning 2-tree is a hamiltonian path and if a graph has at least three vertices then a spanning 1-walk is a hamiltonian cycle. Win [12] showed that for $k \geq 3$, every $\frac{1}{k-2}$-tough graph has a spanning k-tree, and hence a spanning k-walk. In 1990, Jackson and Wormald made the following conjecture.

Conjecture 1.1 (Jackson and Wormald [8])**.** *For each integer $k \geq 2$, every connected $\frac{1}{k-1}$-tough graph has a spanning k-walk.*

The *prism* over a graph G is the Cartesian product $G \square K_2$. If $G \square K_2$ is hamiltonian, we say that G is *prism-hamiltonian.* Kaiser et al. [10] showed that existence of a hamiltonian path implies prism-hamiltonicity, which in turn implies existence of a spanning 2-walk. They gave examples showing that none of these implications can be reversed. They also made the following conjecture, which is analogous to those of Chvátal and of Jackson and Wormald.

Conjecture 1.2 (Kaiser et al. [10])**.** *There exists a constant t_1 such that the prism over any t_1-tough graph is hamiltonian.*

Kaiser et al. also showed that t_1 must be at least $\frac{9}{8}$.

Our goal is to investigate the conjectures above for P_4-free graphs, which have no induced subgraph isomorphic to a 4-vertex path. P_4-free graphs are also known as *cographs.* Connected P_4-free graphs can have arbitrarily low or high toughness: $K_m + nK_1$ (where '+' denotes join) with $m, n \geq 1$ is P_4-free and has toughness m/n if $n \geq 2$, and ∞ if $n = 1$. The following result of Jung shows that Chvátal's conjecture holds for P_4-free graphs.

Theorem 1.3 (Jung [9, Theorem 4.4(2)])**.** *Every P_4-free graph with at least three vertices is hamiltonian if and only if it is 1-tough.*

The following corollary of Theorem 1.3 shows that a stronger version of Conjecture 1.1 holds for P_4-free graphs. The *composition* or *lexicographic product* of graphs H and K, denoted by $H[K]$, is defined as the graph with vertex set $V(H) \times V(K)$ and edge set $\{(u_1, v_1)(u_2, v_2) : u_1u_2 \in E(H) \text{ or } u_1 = u_2 \text{ and } v_1v_2 \in E(K)\}$.

Corollary 1.4. *Let $k \geq 1$ be a positive integer. Then a P_4-free graph has a spanning k-walk if and only if it is $\frac{1}{k}$-tough.*

Proof. For necessity, Jackson and Wormald [8, Lemma 2.1(i)] showed that every graph with a spanning k-walk is $\frac{1}{k}$-tough. So we just show sufficiency.

The statement is true for graphs on one or two vertices (note that in those cases a spanning 1-walk is not a hamiltonian cycle). Hence, we may assume that G has at least three vertices. Also, we may assume that $k \geq 2$, since the statement is true for $k = 1$ by Theorem 1.3.

Jackson and Wormald [8] showed that G has a spanning k-walk if and only if $G[K_k]$ has a hamiltonian cycle. Now suppose G is a $\frac{1}{k}$-tough P_4-graph. It is an easy observation that $G[K_k]$ is P_4-free. Goddard and Swart [7, Theorem 6.1(b)] showed that $\tau(G[K_k]) = k\tau(G)$, so $\tau(G[K_k]) \geq 1$, and hence $G[K_k]$ is hamiltonian by Theorem 1.3. Therefore, G has a spanning k-walk using Jackson and Wormald's result. □

Theorem 1.5. *A P_4-free graph with at least two vertices is prism-hamiltonian if and only if it is $\frac{1}{2}$-tough.*

Jung's result, Theorem 1.3, also confirms that sufficiently tough P_4-free graphs are prism-hamiltonian. However, we show that a weaker toughness condition is both necessary and sufficient, and it is the same toughness condition required for P_4-free graphs to have a spanning 2-walk. In a similar way, two of the authors (Ellingham and Salehi Nowbandegani) [5] showed that for general graphs having a spanning 2-walk and being prism-hamiltonian require the same Chvátal-Erdős condition. Note that if G is P_4-free, $G \square K_2$ is not in general P_4-free, so Theorem 1.3 cannot directly provide a necessary and sufficient condition for a P_4-free graph to be prism-hamiltonian.

The following is a simple corollary of Theorem 1.5 and Corollary 1.4.

Corollary 1.6. *In the class of P_4-free graphs with at least two vertices, the properties of being prism-hamiltonian, having a spanning 2-walk, and being $\frac{1}{2}$-tough are equivalent.*

To confirm the above result we just need to note that the subgraph corresponding to any 2-walk is $\frac{1}{2}$-tough, and prism-hamiltonicity implies the existence of a spanning 2-walk.

The proof of Theorem 1.5 uses an inductive approach, which in general is hard to do for showing results based on toughness. In Section 2, we develop tools for proving Theorem 1.5, which is then proved in the last section.

We conclude this section with a remark on algorithms. Corneil, Lerchs and Stewart Burlingham [4] showed that hamiltonicity can be determined in polynomial time for a P_4-free graph G. Determining whether G has a spanning k-walk amounts to determining whether the P_4-free graph $G[K_k]$ is hamiltonian. Every connected n-vertex graph has a spanning $(n-1)$-tree and hence a spanning $(n-1)$-walk, so we only need to check $G[K_k]$ if $k \leq n-2$, and this can be done in time polynomial in n. Therefore, determining, for a given P_4-free graph G and positive integer k, whether G has a spanning k-walk can be done in polynomial time. By Corollary 1.6, determining whether G is prism-hamiltonian can also be done in polynomial time.

2 Preliminary results

In this section, we provide some lemmas for proving Theorem 1.5. We define a class of graphs which (when they occur as spanning subgraphs) form a subclass of the SEEP-subgraphs introduced by Paulraja [11] for finding hamiltonian cycles in prisms.

Definition 2.1. A *simple block EP (SBEP)* graph H is a connected graph with the following properties:

(i) each block of H is either an even cycle or an edge, and

(ii) each vertex of H is contained in at most two blocks.

The edges of an SBEP graph are partitioned into cutedges and cycle edges, and the vertices of an SBEP graph are partitioned into cutvertices and single-block vertices. Note that any SBEP graph has at least two single-block vertices (at least one in each leaf block, if there are two or more blocks). The following lemma lets us build a new SBEP subgraph from two given SBEP subgraphs.

Lemma 2.2. *Suppose H_1 and H_2 are disjoint SBEP subgraphs of a graph G, with $x_1y_1 \in E(H_1)$, $x_2y_2 \in E(H_2)$, and $x_1y_2, x_2y_1 \in E(G)$. Then there is an SBEP subgraph H of G with $V(H) = V(H_1) \cup V(H_2)$.*

Proof. Each edge x_1y_1 or x_2y_2 is either a cycle edge or a cutedge. By symmetry, we consider three cases.

If x_1y_1 and x_2y_2 are cycle edges, then define $H = H_1 \cup H_2 \cup \{x_1y_2, x_2y_1\} - \{x_1y_1, x_2y_2\}$. If x_1y_1 is a cutedge and x_2y_2 is a cycle edge, then define $H = H_1 \cup H_2 \cup \{x_1y_2, x_2y_1\} - \{x_2y_2\}$. If x_1y_1 and x_2y_2 are cutedges, then define $H = H_1 \cup H_2 \cup \{x_1y_2, x_2y_1\}$.

In each case the two blocks containing x_1y_1 and x_2y_2 are replaced by a new block that is an even cycle, without changing the number of blocks to which any vertex belongs. Therefore, the result H is also an SBEP subgraph. □

Theorem 2.3. *Every SBEP graph is prism-hamiltonian.*

Proof. Let G be an SBEP graph and let $H = G \square K_2$, consisting of G and a copy G' of G, with each $v \in V(G)$ joined to its copy $v' \in V(G')$ by a *vertical edge*. We show a stronger statement, that H has a hamiltonian cycle C such that each single-block vertex v of G and its copy v' are joined by a vertical edge of H in C. We show this stronger statement inductively on the number of blocks in G. The statement holds if G has a single block, i.e., G is an edge or even cycle. So we assume that G has a cutvertex x.

By Definition 2.1(ii), x is contained in exactly two blocks B_1, B_2 of G. Hence, G is the union of two connected subgraphs G_1 (containing B_1) and G_2 (containing B_2) that have only x in common. Each of G_1 and G_2 is an SBEP graph in which x is a single-block vertex.

By induction $G_1 \square K_2$ and $G_2 \square K_2$ have hamiltonian cycles C_1 and C_2, respectively, using vertical edges corresponding to all single-block vertices, including xx'. Now $(C_1 - xx') \cup (C_2 - xx')$ is the required hamiltonian cycle in $G \square K_2$. □

Let G be a graph and $S \subseteq V(G)$. The set S is called a *tough-set* of G if S is a cutset of G and $\frac{|S|}{c(G-S)} = \tau(G)$. Let S be a cutset of G and $X \subseteq S$. Define $c(G, S, X)$ to be the number of components of $G - S$ that are adjacent in G to vertices of X. If $X_1, X_2, \ldots, X_k$ are disjoint nonempty subsets of $V(G)$ then by $G[X_1, X_2, \ldots, X_k]$ we mean the k-partite subgraph of G with vertex set $X_1 \cup X_2 \cup \cdots \cup X_k$ and edge set $\{uv \in E(G) | u \in X_i, v \in X_j, 1 \le i < j \le k\}$.

Lemma 2.4. *Let G be a connected P_4-free graph and let S be a cutset of G such that each vertex in S is adjacent to at least two distinct components of $G - S$. Then the following statements are true.*

(i) For each $u \in S$ and each component $R \subseteq G - S$, if u is adjacent to one vertex in R then u is adjacent to every vertex in R.

(ii) Let R be a component of $G - S$, and let G' be obtained from G by contracting R into a single vertex. Then G' is P_4-free.

(iii) If S is a minimal cutset of G, then $G[S, V(G) - S]$ is a complete bipartite subgraph of G.

(iv) Suppose that S is not a minimal cutset of G. There exist a cutset $U \subseteq S$ of G, nonempty $X \subseteq S - U$ and nonempty $Y \subseteq V(G) - S$ such that each of the following holds.

(a) $G[X \cup Y]$ is a component of $G - U$.

(b) $G[U, X, Y]$ is a complete tripartite subgraph of G.

Proof. For (i), suppose u is adjacent to some but not all vertices of R. Since R is connected there must be $v_1 v_2 \in E(R)$ where v_1 is adjacent to u but v_2 is not. We know u is also adjacent to w in another component of $G - S$. Then $v_2 v_1 u w$ is an induced P_4, a contradiction.

The statement (ii) follows easily by noting that any induced P_4 of G' corresponds to an induced P_4 of G (using (i) if the contracted vertex is contained in the P_4). For (iii), if S is a minimal cutset then each $u \in S$ is adjacent to every component of $G - S$, and hence, by (i), to every vertex of every component of $G - S$.

We now show (iv) by induction on $|V(G)|$. Let U_0 be a minimal cutset of G that is contained in S. Every vertex in U_0 is adjacent to every vertex in $V(G) - U_0$ by (iii); call this ($\star$). As $S - U_0 \neq \emptyset$, $G - U_0$ has a nontrivial component G_1 such that $S \cap V(G_1) \neq \emptyset$. Let $S_1 = S \cap V(G_1)$. Then G_1 consists of $G[S_1]$, the components of $G - S$ adjacent to S_1, and the edges of G between S_1 and these components. Hence, each vertex in S_1 is adjacent to at least two components of $G_1 - S_1$ (thus, S_1 is a cutset of G_1). If S_1 is a minimal cutset of G_1, then let $U = U_0$, $X = S_1$ and $Y = V(G_1) - S_1$. Then (a) holds by definition of G_1 and (b) holds by ($\star$) and because $G[X, Y] = G_1[S_1, V(G_1) - S_1]$ is complete bipartite by (iii).

Otherwise, by induction, with G_1 taking the role of G and S_1 taking the role of S, we find a cutset $U_1 \subseteq S_1$ of G_1, $X_1 \subseteq S_1 - U_1$ and $Y_1 \subseteq V(G_1) - S_1$ such that $G_1[X_1 \cup Y_1]$ is a component of $G_1 - U_1$ and $G_1[U_1, X_1, Y_1]$ is a complete tripartite subgraph of G_1. Let $U = U_0 \cup U_1$, $X = X_1$, and $Y = Y_1$. Clearly, $U \subseteq S$, $X \subseteq S - U$ and $Y \subseteq V(G_1) - S_1 \subseteq V(G) - S$. We claim that U, X and Y satisfy (a) and (b). Since G_1 is a component of $G - U_0$, every component of $G_1 - U_1$ is a component of $G - U_0 - U_1 = G - U$, so U is a cutset of G and $G_1[X_1 \cup Y_1] = G[X \cup Y]$ is a component of $G - U$. Because $G_1[U_1, X_1, Y_1] = G[U_1, X, Y]$ is a complete tripartite graph and by ($\star$), we see that $G[U, X, Y]$ is a complete tripartite subgraph of G. $\square$

Lemma 2.5. *Let G be a connected graph and let S be a tough-set of G. Suppose $\tau(G) = t \leq 1$. Then the following statements hold.*

(i) For any nonempty $S' \subseteq S$ with $S' \neq S$, S' is adjacent in G to at least $|S'|/t + 1$ components of $G - S$.

(ii) For any nonempty $S' \subseteq S$, S' is adjacent in G to at least $|S'|/t$ components of $G - S$.

(iii) Every vertex of S is adjacent to at least two components in $G - S$.

(iv) Let R be a component of $G - S$. If S is a maximal tough-set of G, k is a positive integer, and $t \geq \frac{1}{k}$, then R is $\frac{1}{k}$-tough.

(v) Suppose G is P_4-free. Let R be a component of $G - S$, and let G' be obtained from G by contracting R into a single vertex. Then G' is t-tough.

An equivalent way to state the conclusion of (iv) is that R is $(1/\lceil 1/t \rceil)$-tough. We cannot in general strengthen this to say that R is t-tough. For example, suppose that $p \geq 2$ and $G = ((2p-2)K_1 \cup K_{1,2}) + K_p$. It is not difficult to show that $\tau(G) = \frac{p}{2p-1}$, with maximal tough-set $S = V(K_p)$, but the component $R = K_{1,2}$ of $G - S$ is only $\frac{1}{2}$-tough, not $\frac{p}{2p-1}$-tough.

Proof. For (i), let $S^* = S - S' \neq \emptyset$. Note that $|S^*| \geq t\, c(G - S^*)$, by toughness if $c(G - S^*) \geq 2$, and because $t \leq 1$ if $c(G - S^*) = 1$. Also, $c(G - S^*) \geq c(G - S) - c(G, S, S') + 1$. Then

$$\begin{aligned} |S'| &= |S| - |S^*| \leq |S| - t\, c(G - S^*) = t\, c(G - S) - t\, c(G - S^*) \\ &\leq t\, c(G - S) - t(c(G - S) - c(G, S, S') + 1) = t\, c(G, S, S') - t. \end{aligned}$$

implying that $c(G, S, S') \geq |S'|/t + 1$. For (ii), use (i) if $S' \neq S$, and if $S' = S$ we have $c(G - S) = |S|/t$ since S is a tough-set.

For (iii), if $|S| \geq 2$, it follows directly from (i) by taking S' as singletons. If $|S| = 1$, then the single vertex of S is adjacent to every component of $G - S$.

For (iv), we may assume R is not complete. Let $Q \subseteq V(R)$ be a tough-set of R. Since S is a maximal tough-set of G, $S \cup Q$ is not a tough-set of G, but it is a cutset of G. Then

$$|S| + |Q| = |S \cup Q| > t\, c(G - (S \cup Q)) = t(c(G - S) - 1 + c(R - Q)).$$

Since $|S| = t\, c(G - S)$, we see that $|Q| > t(c(R - Q) - 1)$, and since $t \geq \frac{1}{k}$ we have $k|Q| > c(R-Q) - 1$. Because both sides are integers, $k|Q| \geq c(R-Q)$, and so R is $\frac{1}{k}$-tough.

Now we prove (v). By (iii), Lemma 2.4 applies to G and S. By Lemma 2.4(ii), G' is P_4-free. Let Q be a tough-set of G' and $\tau(G') = t'$. We may assume that $t' \leq 1$; otherwise, $t \leq 1 < t'$. Then by (iii), Lemma 2.4 also applies to G' and Q. Let v_R be the vertex to which R is contracted. If $v_R \notin Q$ then Q is also a cutset of G with $c(G - Q) = c(G' - Q)$. Then

$$t' = \frac{|Q|}{c(G' - Q)} = \frac{|Q|}{c(G - Q)} \geq t.$$

So we may assume $v_R \in Q$. Let $A_1, A_2, \ldots, A_a$ be the components of $G' - Q$ adjacent in G' to v_R, where $a \geq 2$ by (iii). By Lemma 2.4(i) for G' and Q, v_R is adjacent in G' to every vertex of A_i for all i with $1 \leq i \leq a$, i.e., v_R is adjacent in G' to every vertex of $X = \bigcup_{i=1}^{a} V(A_i)$. On the other hand, all neighbors of v_R in G' lie in S, and hence $X \subseteq S$.

Let $B_1 = v_R, B_2, \ldots, B_b$ be the components of $G' - S$ adjacent in G' to vertices of X, and $Y = \bigcup_{i=1}^{b} V(B_i)$. The components of $G - S$ adjacent in G to X are just R and $B_2, \ldots, B_b$, i.e., $c(G, S, X) = b$. Now by (ii) for G and S, we have

$$|Y| \geq b = c(G, S, X) \geq |X|/t. \tag{1}$$

Suppose $2 \leq i \leq b$. By Lemma 2.4(i) for G and S, if $u \in X$ is adjacent in G to some vertex of B_i, then u is adjacent to all vertices of B_i. Thus, every vertex of B_i is adjacent in G, and hence in G', to some vertex of X. Since X is the union of components of $G' - Q$, all edges leaving X go to Q, so $V(B_i) \subseteq Q$. Moreover, $V(B_1) = \{v_R\} \subseteq Q$ and hence $Y \subseteq Q$.

Let Z be the set of vertices in all components of $G' - Q$ other than $A_1, A_2, \ldots, A_a$. Then $Z = V(G') - Q - X$, and there are no edges of G' from $\{v_R\} \cup X$ to Z. Thus, there is no edge in G' from Y to Z; otherwise, there is an induced P_4 starting at v_R then visiting a vertex of X, a vertex of $Y - \{v_R\}$ (which is nonempty because $|Y| \geq |X|$ by (**1**), and $|X| \geq a \geq 2$) and a vertex of Z. Therefore, $c(G', Q, Y) = a$, and by (ii) for G' and Q we have

$$|X| \geq a = c(G', Q, Y) \geq |Y|/t'. \tag{2}$$

By (**1**) and (**2**), $tt' \geq 1$, but $t \leq 1$ by hypothesis and $t' \leq 1$ by assumption, so $t' = t = 1$, and $t' \geq t$ as required. □

3 Proof of Theorem 1.5

In this section, we prove Theorem 1.5. We actually prove a stronger result, of which the following lemma is a special case.

Lemma 3.1. *If $G = G[X, Y]$ is a complete bipartite graph with $|X| \le |Y| \le 2|X|$, then G has a spanning SBEP subgraph in which every element of Y is a single-block vertex.*

Proof. If $|X| = 1$ then G itself is the required subgraph, so suppose that $|X| \ge 2$. Since $|X| \le |Y|$ there is a cycle C using X and $|X|$ vertices of Y. Since $|Y| \le 2|X|$, the vertices not in C form a subset of Y of size at most $|X|$, so we can add an edge joining each such vertex to a distinct vertex of X to obtain the required subgraph. □

The theorem we prove is the following.

Theorem 3.2. *Let G be a connected P_4-free graph with at least two vertices. Then G has a spanning SBEP subgraph if and only if $\tau(G) \ge \frac{1}{2}$.*

Proof. The necessity is clear, as any SBEP subgraph contains a spanning 2-walk and the subgraph corresponding to a 2-walk is $\frac{1}{2}$-tough. We show sufficiency. We may assume that $t = \tau(G) < 1$, otherwise Theorem 1.3 implies that G has a hamiltonian cycle, which is a spanning SBEP subgraph. We prove Theorem 3.2 by induction on $|V(G)|$. The result holds if $|V(G)| \le 3$. So we assume that $|V(G)| \ge 4$. Let $S \subseteq V(G)$ be a maximal tough-set of G. By Lemma 2.5(iii), Lemma 2.4 applies to G and S. We consider two cases.

Case 1. Suppose $G - S$ has a nontrivial component. Let R be a nontrivial component of $G - S$, and let G' be the graph obtained from G by contracting R into a single vertex, which has at least two vertices. By Lemma 2.5(v), the graph G' is $\frac{1}{2}$-tough, and by Lemma 2.5(iv), the component R is $\frac{1}{2}$-tough.

By induction, G' has a spanning SBEP subgraph T' and R has a spanning SBEP subgraph T_R. Let v_R be the corresponding contracted vertex in G', and let x, y be two single-block vertices in T_R (any SBEP graph has at least two single-block vertices). By Lemma 2.4(i), the neighbors of v_R in T' are all adjacent in G to the vertices x, y. Therefore, any subgraph of G', or T', can be embedded in G by replacing v_R by either x or y.

If v_R is a single-block vertex in T', we embed T' in G with x replacing v_R. Then $T' \cup T_R$ is a spanning SBEP subgraph of G. Now suppose v_R is a cutvertex. Then v_R is contained in exactly two blocks B_1, B_2 of T'. Hence, T' is the union of two connected subgraphs T'_1 (containing B_1) and T'_2 (containing B_2) that have only v_R in common. Each of T'_1 and T'_2 is an SBEP graph in which v_R is a single-block vertex. Embed T'_1 in G with x replacing v_R, and embed T'_2 in G with y replacing v_R. Then $T'_1 \cup T'_2 \cup T_R$ is a spanning SBEP subgraph of G.

Case 2. Suppose each component of $G-S$ is a single vertex. We may assume that S is not a minimal cutset of G. For otherwise, $G[S, V(G)-S]$ is complete bipartite by Lemma 2.4(i). Since G is $\frac{1}{2}$-tough and less than 1-tough, $|S| < |V(G)-S| \leq 2|S|$ and so $G[S, V(G)-S]$, and hence G, has a spanning SBEP subgraph by Lemma 3.1.

Applying Lemma 2.4(iv), we find a cutset $U \subseteq S$ of G, $X \subseteq S-U$ and $Y \subseteq V(G)-S$ such that $G[X \cup Y]$ is a component of $G-U$, and $G[U, X, Y]$ is a complete tripartite subgraph of G. Consequently, $G[X, Y]$ is a spanning complete bipartite subgraph of the component $G[X \cup Y]$ of $G-U$.

By Lemma 2.5(i), $|Y| = c(G, S, X) \geq \lceil |X|/t \rceil + 1$. Let Y_1 be a subset of Y of size $\lceil |X|/t \rceil$, and $Y_2 = Y - Y_1 \neq \emptyset$. Let R be the complete bipartite subgraph $G[X, Y_1]$ of G, and let $G' = G - V(R)$.

We now show that G' is $\frac{1}{2}$-tough. Assume to the contrary that $t' = \tau(G') < \frac{1}{2}$, so that $t' < t$. Let $Q \subseteq V(G')$ be a tough-set of G', and let

$$Q_1 = Q \cap S \quad \text{and} \quad Q_2 = Q \cap (V(G)-S).$$

By Lemma 2.5(iii), Lemma 2.4 applies to G' and Q. We consider three cases below.

Case 2.1. Suppose that $U - Q \neq \emptyset$ and $Y_2 - Q \neq \emptyset$. Then there is one component of $G'-Q$ containing all of $U-Q$ and all of $Y_2 - Q$, since $G'[U-Q, Y_2-Q]$ is a complete bipartite subgraph of $G[U, X, Y]$. Adding back X and Y_1 to G' just adds X and Y_1 to this component without changing any of the other components of $G'-Q$, so

$$2 \leq c(G'-Q) = c(G-Q) \leq |Q|/t$$

by toughness of G, contradicting $c(G'-Q) = |Q|/t' > |Q|/t$.

Case 2.2. Suppose that $U - Q = \emptyset$. Since $G[X \cup Y]$ is a component of $G-U$ and $U \subseteq Q$, there are no edges of G from $X \cup Y$, or in particular from Y_1, to $V(G')-Q$. Thus, if $Q^* = Q \cup X$, then $G - Q^* = G[(V(G')-Q) \cup Y_1]$ is $G'-Q$ together with isolated vertices from Y_1. Hence, $c(G-Q^*) = c(G'-Q) + |Y_1|$. Then because $|Q| = t'\, c(G'-Q)$, $|Y_1| \geq |X|/t$ and $t' < t$ we have that

$$\begin{aligned} |Q^*| &= |Q| + |X| \leq t'\, c(G'-Q) + t|Y_1| \\ &< t(c(G'-Q) + |Y_1|) = t\, c(G-Q^*), \end{aligned}$$

contradicting G being t-tough.

Case 2.3. Suppose that $Y_2 - Q = \emptyset$. Then $Y_2 \subseteq Q_2$, so $Q_2 \neq \emptyset$. Let $A_1, A_2, \ldots, A_a$ be the components of $G'-Q$ adjacent in G' to vertices of Q_2. Given A_i, $1 \leq i \leq a$, there is $w \in V(Q_2)$ adjacent to some vertex of A_i. By Lemma 2.4(i) for G' and Q, w is adacent to every vertex of A_i, and hence $V(A_i) \subseteq S$. Let $S_1 = \bigcup_{i=1}^{a} V(A_i) \subseteq S$. Vertices of S_1 can only be adjacent in G' to vertices of $S \cup Q = S \cup Q_2$. Now by Lemma 2.5(ii) for G' and Q, and because $t' < \frac{1}{2}$, $|S_1| \geq a = c(G', Q, Q_2) \geq |Q_2|/t' > 2|Q_2|$.

Since $X \subseteq S$, and $G[X \cup Y]$ is a component of $G - U$, we see that all vertices in $S_1 \cup X$ together are adjacent in G to at most $|Q_2 \cup Y| = |Q_2| + |Y_1|$ components of $G - S$. Therefore, by Lemma 2.5(ii), we have $|Q_2| + |Y_1| \geq c(G, S, S_1 \cup X) \geq (|S_1| + |X|)/t$. But $|X|/t + 1 > \lceil |X|/t \rceil = |Y_1|$ and $|S_1| > 2|Q_2|$, so we get $|Q_2| + |X|/t + 1 > 2|Q_2|/t + |X|/t$, giving $|Q_2| + 1 > 2|Q_2|/t \geq 2|Q_2|$, from which $|Q_2| < 1$, which is a contradiction.

This concludes the proof that G' is $\frac{1}{2}$-tough.

Since $\frac{1}{2} \leq t < 1$, we have $|X| < |Y_1| = \lceil |X|/t \rceil \leq 2|X|$. Thus, by Lemma 3.1, the complete bipartite subgraph R has a spanning SBEP subgraph T_R. By induction, G' has a spanning SBEP subgraph T'. Let $xy_1 \in E(T_R)$ with $x \in X$ and $y_1 \in Y_1$. Let $zy_2 \in E(T')$ with $y_2 \in Y_2$; then $z \in U$. Then T_R and T' are two disjoint SBEP subgraphs, and $zy_1, xy_2 \in E(G)$ because $G[U, X, Y]$ is complete tripartite. Hence, by Lemma 2.2 we obtain a spanning SBEP subgraph of G. □

Now combining Theorems 2.3 and 3.2 gives Theorem 1.5.

References

[1] D. Bauer, H. J. Broersma, and H. J. Veldman, Not every 2-tough graph is Hamiltonian, *Discrete Appl. Math.* 99 (2000) 317–321.

[2] J.-C. Bermond, *Hamiltonian graphs*, in *Selected topics in graph theory*, Academic Press, London-New York, 1978.

[3] V. Chvátal, Tough graphs and Hamiltonian circuits, *Discrete Math.* 5 (1973) 215–228.

[4] D. G Corneil, H. Lerchs and L. Stewart Burlingham, Complement reducible graphs, *Discrete Appl. Math.* 3(3) (1981) 163–174.

[5] M. N. Ellingham and P. Salehi Nowbandegani. The Chvátal-Erdős condition for prism-hamiltonicity, submitted, available as `arXiv:1812.02894`, 2018.

[6] Hikoe Enomoto, Bill Jackson, P. Katerinis, and Akira Saito, Toughness and the existence of k-factors, *J. Graph Theory* 9(1) (1985) 87–95.

[7] W. D. Goddard and Henda C. Swart, On the toughness of a graph, *Quaestiones Math.* 13(2) (1990) 217–232.

[8] B. Jackson and N. C. Wormald, k-walks of graphs, *Australas. J. Combin.* 2 (1990) 135–146.

[9] H. A. Jung, On a class of posets and the corresponding comparability graphs, *J. Combin. Theory Ser. B* 24(2) (1978) 125–133.

[10] Tomáš Kaiser, Zdeněk Ryjáček, Daniel Král, Moshe Rosenfeld, and Heinz-Jürgen Voss, Hamilton cycles in prisms, *J. Graph Theory* 56(4) (2007) 249–269.

[11] P. Paulraja, A characterization of Hamiltonian prisms, *J. Graph Theory* 17(2) (1993) 161–171.

[12] Sein Win, On a connection between the existence of k-trees and the toughness of a graph, *Graphs Combin.* 5(2) (1989) 201–205.