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Spanning paths, cycles, trees and walks for graphs on surfaces

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Abstract

This paper is a survey of results about hamilton paths and cycles for graphs on surfaces, and their generalizations to spanning trees and walks with bounded degree. We focus particularly on results based on connectivity alone or connectivity and local planarity conditions. Other areas, especially the vast literature on the existence or nonexistence of hamilton cycles in planar graphs, are dealt with comprehensively but not in great detail.

1. Introduction

1a. General introduction

This paper is a survey of results about hamilton paths and cycles for graphs on surfaces, and their generalizations to spanning trees and walks with bounded degree. We focus particularly on results based on connectivity alone or connectivity and local planarity conditions. The seminal results in this area are those by Whitney [W11] and Tutte [T12, T15] on hamilton cycles in 4-connected planar graphs, and that of Barnette [B4] on spanning trees in 3-connected planar graphs. Other areas, especially the vast literature on the existence or nonexistence of hamilton cycles in planar graphs, are dealt with comprehensively but not in great detail, and the reader should be aware that our coverage of those areas is limited mostly to papers already in print, as opposed to preprints or papers accepted but not yet published.

There are many other survey papers or book sections which discuss some of the topics of this one. In particular there are many places where the reader may find information on hamilton paths and cycles in planar graphs. Those we have found useful include (in chronological order from earliest to latest) Klee [K4], Grünbaum [G12], Grünbaum and Walther [G16], Grünbaum [G13], Nash-Williams [N3], Bermond [B14], Capobianco and Molluzzo [C1] or Molluzzo [M7], Berge [B13, Section 10.6], Saaty and Kainen [S1, Section 4-4] and Malkevitch [M3]. For a survey of representativity (facewidth) in graphs which discusses some of the results on paths, cycles, trees and walks also covered here, see Mohar [M6, Section 9].

1b. Definitions for graphs

We begin with some introductory definitions. We assume that the reader is familiar with the basic terminology of graph theory, and refer her or him to [B19] for any terms not defined here. All our graphs are finite and simple (no loops or multiple edges) unless we explicitly state otherwise.

A subgraph or walk in a graph is said to be *spanning* if it uses every vertex of the graph. Spanning paths and cycles are known as *hamilton* paths and cycles. A graph is *hamiltonian* if it has a hamilton cycle, and *traceable* if it has a hamilton path. It is *hamilton-connected* if for every given pair of vertices there is a hamilton path with that pair as its ends, and k-*hamiltonian* if every subgraph obtained by deleting at most k vertices from the original graph is hamiltonian.

Define a k-*walk* in a graph to be a spanning closed walk in which each vertex is passed through at most k times, and an m-*tree* to be a spanning tree in which each vertex has degree at most m. Note that by our definition, an m-tree need not have any vertex of degree m. A hamilton cycle is just a 1-walk, and a hamilton path is just a 2-tree. Thus, if a hamilton cycle or path cannot be found, it is natural to ask for the smallest k or m such that there is a k-walk or m-tree. The following relates the existence of k-walks and m-trees in a graph.

Lemma 1.1 (Jackson and Wormald, 1990 [J1])**.**
(i) *If a graph has a k-tree, it has a k-walk.*
(ii) *If a graph has a k-walk, it has a $(k+1)$-tree.*

Proof. Part (i) follows by traversing the k-tree using depth-first search to obtain a k-walk. Part (ii) follows by orienting the k-walk, then taking the first incoming edge for every vertex; since every vertex then has one incoming edge and at most k outgoing edges, the result is a $(k+1)$-tree. ∎

Therefore, the strongest result we can obtain is the existence of a 1-walk (hamilton cycle), the next strongest a 2-tree (hamilton path), then a 2-walk, then a 3-tree, and so on.

Let H be a subgraph of a graph G. A *bridge* of H in G is either (a) an edge of $E(G) - E(H)$ with both ends in H, or (b) a component C of $G - V(H)$ together with all edges with one end in C and the other in H. Bridges of type (a) are called *trivial*. If J is a bridge of H in G, then J contains no edges of H, and the vertices of H contained in J are called the *vertices of attachment* of J on H.

A *cutset* in a graph G is a set $S \subseteq V(G)$ such that $G - S$ has more components than G. A graph is k-*connected* if it is connected, has at least $k+1$ vertices, and has no cutset of cardinality less than k. The *connectivity* of a graph is the smallest k for which it is k-connected. A *block* of a graph is a maximal 2-connected subgraph or an edge contained in no cycle; every graph has a unique decomposition into edge-disjoint blocks. A *cyclic-edge-cutset* in a connected graph G is a set $T \subseteq E(G)$ such that $G - E(T)$ is disconnected and each component contains at least one cycle. A graph is *cyclically-k-edge-connected* if it is connected and has no cyclic-edge-cutset of cardinality less than k; note that a cyclically-k-edge-connected *cubic* graph is 3-connected if $k \geq 3$. A k-separation (H, K) of a graph G is a decomposition of G into two edge-disjoint subgraphs H and K which intersect in exactly k vertices and which both have at least k edges.

1c. Definitions for surface embeddings

By a *surface* we mean a compact continuous 2-manifold without boundary. The Classification Theorem for Surfaces, sometimes attributed to Brahana [B22], states that every such surface is homeomorphic to either S_h, a sphere with h added handles, or N_k, a sphere with k added crosscaps (Möbius strips attached by identifying their boundaries with the boundaries of cut-out discs). An *embedding* of a graph onto a surface maps vertices to points and edges to internally-disjoint non-self-intersecting curves, while preserving the incidence properties of the graph. The components of the surface after removing the image of the graph are called the *faces* of the embedding; if every face is homeomorphic to an open disc we say the embedding is a *2-cell embedding*. A face is a *k-gon* if its boundary is a cycle of length k in the graph; we refer to 3-gons as *triangles* and 4-gons as *quadrangles*. A *triangulation* or *quadrangulation* is an embedding in which all faces are triangles or quadrangles respectively. A *C-near-triangulation* is an embedding in which all faces are triangles with the possible exception of the one bounded by C, which is a cycle.

It is well-known that a graph is embeddable on the sphere if and only if it can be embedded in the plane; such graphs are called *planar*. If we wish to discuss a graph with a specific embedding on the plane, we call it a *plane graph*. Steinitz [S20] showed that a graph is 3-connected and planar if and only if it is a *3-polytope*, i.e. it reflects the incidence structure of the vertices and edges of the convex hull of a set of points in 3-dimensional Euclidean space. A *simple 3-polytope* is a 3-connected cubic planar graph, and a *simplicial 3-polytope* is a plane triangulation. Plane triangulations are always 3-connected.

The *Euler characteristic* of a surface Σ, denoted $\chi(\Sigma)$, is equal to $2-2h$ for S_h and $2-k$ for N_k. For any graph 2-cell-embedded on a surface Σ, with v vertices, e edges and f faces, we have that $v-e+f=\chi(\Sigma)$. We will use *Euler-negative* and *Euler-nonnegative* to denote surfaces Σ for which $\chi(\Sigma)<0$ and $\chi(\Sigma)\geq 0$ respectively. The only Euler-nonnegative surfaces are the sphere S_0, projective plane N_1, torus S_1 and Klein bottle N_2.

A simple closed curve on a surface is *contractible* if it freely homotopic to a point. The *edgewidth* of an embedded graph is the length of the shortest noncontractible cycle of the graph. The *representativity* or *facewidth* of an embedded graph is the minimum over all noncontractible closed curves in the surface of the number of points at which the curve intersects the graph; it is always finite. In rough terms, if the edgewidth or representativity of an embedded graph is high, the graph is in some sense 'locally planar' and may have properties close to those of a planar graph. For the basic theory of and results related to representativity, the reader should consult Robertson and Vitray [R5] and Mohar [M6].

Given a surface Π with boundary (such as a closed disc, annulus or Möbius strip), and a graph embedded in a surface Σ, we say that the graph *has a spanning* Π if there is a topological subspace of Σ homeomorphic to Π containing the images of every vertex and edge of the graph.

2. Necessary conditions

Before trying to find spanning paths, cycles, trees or walks in a graph, it is prudent to see if there are any obvious obstacles. Thus, in this section we discuss some necessary conditions for the existence of such spanning subgraphs.

2a. Toughness-related conditions

The first conditions we discuss apply not only to graphs on particular surfaces, but to graphs in general. Let $c(H)$ denote the number of components of a graph H.

Lemma 2.1 (Jackson and Wormald [J1])**.** *If a graph G has a k-walk, $k \geq 1$, then $c(G - S) \leq k|S|$ for every nonempty $S \subseteq V(G)$.*

Proof. A k-walk must enter each component of $c(G - S)$ from a vertex of S at least once, and can leave each vertex of S at most k times. ∎

Chvátal [C6] introduced the idea of toughness: a connected graph G is *t-tough* if $c(G - S) \leq |S|/t$ for all cutsets $S \subseteq V(G)$, and the *toughness* of G is the smallest t for which G is t-tough. Thus, Lemma 2.1 may be restated as saying that graphs with k-walks are $(1/k)$-tough, generalizing Chvátal's observation that hamiltonian graphs are 1-tough. Lemma 2.1 means that for each m, $K_{m,mk+1}$ is an m-connected graph without a k-walk, so no connectivity condition by itself can guarantee the existence of a k-walk.

If we seek k-trees, rather than k-walks, then a condition similar but not identical to Lemma 2.1 can be stated. It was used implicitly by Thomassen [T9].

Lemma 2.2. *If a graph G has a k-tree, $k \geq 2$, then $c(G - S) \leq (k-1)|S| + 1$ for every $|S| \subseteq V(G)$.*

Proof. Let T be a k-tree of G; then $c(G - S) \leq c(T - S)$. Since each vertex of T has degree at most k, deleting each vertex of S in turn creates at most k new components from one old component. Thus, in $T - S$ the number of components is increased from one by at most $k - 1$ for each vertex of S. ∎

For each m, $K_{m,m(k-1)+2}$ has no k-tree by Lemma 2.2.

2b. Grinberg's condition for plane graphs

Now we discuss a more specialized condition due to Grinberg [G11] and presented by Sachs [S3], who heard of it from Kozyrev.

Theorem 2.3: Grinberg's Condition (Grinberg, 1968 [G11, S3])**.** *Suppose G is a plane graph with a hamilton cycle C. For each $k \geq 3$, let p'_k and p''_k denote the number of k-gonal faces inside and outside C, respectively. Then*

$$\sum_{k \geq 3} (k-2)(p'_k - p''_k) = 0.$$

Proof. Let v be the number of vertices of G, and e' the number of edges inside C. Consider the subgraph G' of G induced by the edges on and inside C. The number of faces of G' is clearly $e' + 2$, but is also $(\sum_{k \geq 3} p'_k) + 1$. The total number of edges of G' is $e' + v$, but is also $[(\sum_{k \geq 3} kp'_k) + v]/2$. Combining these to eliminate e', we get $\sum_{k \geq 3}(k-2)p'_k = v - 2$. But similarly $\sum_{k \geq 3}(k-2)p''_k = v - 2$, and the result follows. ∎

A common way to use Grinberg's Condition is to construct a plane graph in which all faces but one have length congruent to 2 modulo 3; then Grinberg's Condition cannot be satisfied, and so the graph must be nonhamiltonian. Grinberg used this argument to construct a 46-vertex nonhamiltonian cyclically-5-edge-connected cubic planar graph. Grinberg's Condition has also been used in other ways. For example, the cyclically-5-edge-connected cubic planar graph, with 44 vertices, of Figure 1 was shown to be nonhamiltonian by Tutte [T14] using the following argument. All faces are pentagons and octagons, except for three mutually adjacent hexagons. Grinberg's Condition reduced modulo 3 requires that all three hexagons lie on one side of any hamilton cycle; but this is impossible, as one of the hexagons must be divided from the other two when the cycle goes through the vertex common to all three hexagons. See subsection 5a for more about this graph.

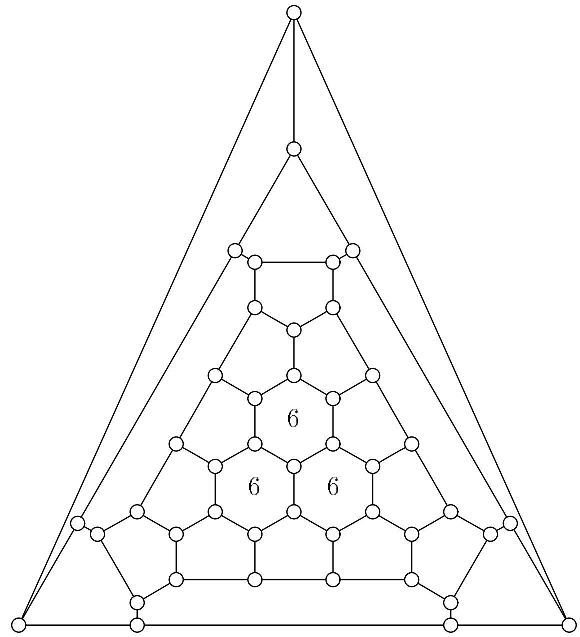

Figure 1: Tutte's 44-vertex cyclically-5-edge-connected nonhamiltonian cubic planar graph

3. Connectivity and hamiltonicity on Euler-nonnegative surfaces

3a. Whitney and 4-connected triangulations

The first major result on the existence of hamilton paths and cycles in graphs embedded in surfaces was by Whitney [W11] in 1931. Most of the work on hamilton cycles and paths for graphs with given connectivity on surfaces is based on the techniques developed by Whitney and extended by Tutte [T12] in 1956. We illustrate these techniques later in this section by outlining the proof of a result by Thomassen [T7] with corrections by Chiba and Nishizeki [C2].

As is usual in this area, Whitney's proof was by induction, using a somewhat stronger result than the one usually quoted, in order to make the induction proof succeed.

Lemma 3.1 (Whitney, 1931 [W11]). *Let G be a plane C-near-triangulation with no separating triangles. Suppose u, v and w are vertices of C in clockwise order, with $u \neq v$ and such that the clockwise subpaths of C from u to v, v to w and w to u are all chordless. Then G has a hamilton path from u to v.*

Theorem 3.2: Whitney's Theorem. *Every 4-connected plane triangulation has a hamilton cycle.*

Proof. Apply the lemma with C any face and u and v two adjacent vertices of C. ∎

Whitney mentions an example, provided by C. N. Reynolds, to show that 4-connectivity is necessary in Theorem 3.2. A cubic plane graph is given whose dual is a nonhamiltonian 11-vertex triangulation which is 3-connected but not 4-connected. Reynolds [R1] examined small 3-connected planar graphs, and found ways to reduce the hamiltonicity of a plane graph to properties of smaller graphs.

A recent application of Whitney's Theorem to constructing Venn diagrams is given by Chilakamarri, Hamburger and Pippert [C4].

3b. Tutte and 4-connected planar graphs

In 1956 Tutte [T12] generalized Theorem 3.2 from triangulations to arbitrary 4-connected graphs. In 1977 he published another paper [T15] giving the same proof in a way which is easier to follow. Useful expositions of Tutte's theorem were also given in English by Ore [O4] and in German by Sachs [S4].

The important new idea in Tutte's proof was that of a *Tutte cycle*. Let H be a subgraph of a graph G, and let S be another subgraph of G (usually a cycle or path). Then we say S is a *Tutte subgraph* (or *Tutte cycle* or *Tutte path*, if appropriate) with respect to H if

(i) every bridge of S has at most three vertices of attachment, and
(ii) every bridge of S that contains an edge of H has at most two vertices of attachment.

Tutte cycles and paths have also been called *slings* and *snakes* respectively. Tutte stated his result in terms of arbitrary plane graphs, but it loses nothing and is simpler to state it for 2-connected graphs.

Lemma 3.3 (Tutte, 1956 [T12, T15])**.** *Let G be a 2-connected plane graph with facial cycles C_1 and C_2 which have a common edge e. Let e' be another edge of C_1. Then G has a Tutte cycle C with respect to $C_1 \cup C_2$ which contains both e and e'.*

Theorem 3.4: Tutte's Theorem. *Every 4-connected planar graph has a hamilton cycle.*

Proof. Let C be the Tutte cycle which Lemma 3.3 provides for the plane embedding of such a graph, with any suitable C_1, C_2, e and e'. If there is a nontrivial bridge of C, its three or fewer vertices of attachment give a vertex cut contradicting the 4-connectedness of G. Hence there are no nontrivial bridges and C is a hamilton cycle. ∎

The following can then be deduced.

Corollary 3.5 (Nelson, see [P5])**.** *Every graph obtained by deleting a vertex from a 4-connected planar graph has a hamilton cycle.*

Proof. Suppose G is 4-connected and we delete v. We proceed as in Tutte's Theorem, choosing C_1 to be the boundary cycle of the face whence v was deleted. If there is a nontrivial bridge of C without an edge of C_1, its three or fewer vertices of attachment give a vertex cut in G as well as in $G - v$, and if there is a nontrivial bridge containing an edge of C_1, we get a vertex cut in G from its two or fewer vertices of attachment together with v. Thus, since G is 4-connected there cannot be any nontrivial bridges, and C is a hamilton cycle. ∎

Examples of nonhamiltonian 3-connected planar graphs are not difficult to construct. Any bipartite 3-connected planar graph with more vertices in one part of the bipartition than the other will do; such examples were given, for example, by Coxeter [C11, C12 page 8]. These examples are not hamiltonian because they are not 1-tough (see subsection 2a). However, there also exist 1-tough nonhamiltonian 3-connected planar graphs. Nishizeki [N5] constructed an infinite family of 1-tough nonhamiltonian plane triangulations, and Dillencourt [D6] constructed a 31-vertex plane near-triangulation without separating triangles which is 1-tough and nonhamiltonian. More recently Dillencourt [D8] has conducted a computer survey to determine the smallest nonhamiltonian planar graphs and triangulations which are 1-tough or satisfy a certain stronger toughness condition. Dillencourt has also [D7, D8] investigated the shortness exponent (see section 5) of 1-tough triangulations. M. Tkáč (see [D8]) independently obtained the smallest 1-tough nonhamiltonian triangulation, and used it to obtain a shortness exponent bound.

As mentioned above, the smallest triangulation with no hamilton cycle was found by Reynolds and mentioned by Whitney; however, no proof of minimality was provided. Barnette and Jucovič [B10] proved that the smallest 3-connected planar graph with no hamilton cycle has 11 vertices, 18 edges and 9 faces and that the smallest plane triangulation has 11 vertices as in Reynolds's example. Goodey [G7] proved that the smallest 3-connected planar graph with no hamilton path has 14 vertices, 24 edges and 12 faces, and that the smallest plane triangulation has 14 vertices. Examples of graphs (not triangulations) meeting the bounds of Barnette and Jucovič and of Goodey were given by Coxeter [C12].

At this point, we mention two conjectures related to Theorem 3.4. For the first, note that a graph on v vertices is *pancyclic* if it contains cycles of every length l with $3 \leq l \leq v$.

Conjecture 3.6 (Malkevitch, 1988 [M3])**.** *Every 4-connected planar graph containing a 4-cycle is pancyclic.*

For the second conjecture, which was provided by the referee, note that a *minor* of a graph G is a graph obtained from G by deleting and/or contracting edges.

Conjecture 3.7 (attributed to Robin Thomas)**.** *If $k \geq 1$, then every k-connected graph with no minor isomorphic to K_{k+1} is hamiltonian.*

For $k \leq 3$ there are no graphs fulfilling the condition of the conjecture, so it is vacuously true. For $k = 4$, it follows from Theorem 3.4 and Wagner's characterization of graphs with no K_5-minor [W1, W2 section 17]: all 4-connected graphs with no K_5-minor are planar.

3c. Thomassen and paths in planar graphs

Plummer [P5] asked whether every 4-connected planar graph is hamilton-connected. This was resolved in 1983 by Thomassen; his proof contained an error that was rectified by Chiba and Nishizeki.

Theorem 3.8: Thomassen's Path Theorem (Thomassen, 1983 [T7] and Chiba and Nishizeki, 1986 [C2])**.** *Let G be a 2-connected plane graph with outer cycle C. Let v be a vertex of C, e an edge of C, and u any vertex of G. Then there is a path P from v to u containing e which is a Tutte path with respect to C.*

Thomassen showed that Tutte's Theorem can be deduced from this, as can the fact that every 4-connected planar graph is hamilton-connected, and the fact that each pair of vertices in a cyclically 4-edge-connected cubic planar graph G is connected by a path P such that $G - V(P)$ consists of isolated vertices. We outline the proof of Theorem 3.8 because it provides a good illustration of proofs based on Whitney and Tutte's ideas. We use Thomassen's original notation wherever possible so that the correspondence between his proof and our outline is clear; we have, however, reordered the argument somewhat. Two definitions are needed. Let P be a path in a graph; if x and y are vertices of P, then $P[x, y]$ denotes the subpath of P from x to y. If H is a subgraph containing vertices of P, then the *subpath of P spanned by H* is the minimal subpath of P containing all vertices of $V(H) \cap V(P)$.

Proof outline. The proof is by induction on the number of vertices in the graph. First, suppose there is a 2-separation (H, K) of G with $V(H \cap K) \subseteq C$, $v \in V(H)$ and $e \in E(K)$. We add an edge or possibly a subdivided edge to each of H and K, apply induction, and splice the paths in the two parts together. We may thus assume that no such 2-separation exists.

The remainder of the proof is illustrated in Figure 2. Letting $e = v_0v_1$, we may assume that v, v_0, v_1 occur in clockwise order on C. Let P_1 be the path clockwise from v to v_0. Form $H = G - V(P_1)$, and let $P_2 = C - V(P_1)$, whose ends are v_1 and the vertex immediately anticlockwise from v on C, which we call v_2. It can be shown that there is a single block B of H containing P_2. We now consider the bridges of $B \cup C$; in other words, we are interested in how the vertices of $H - V(B)$ are connected to B and to P_1.

Each bridge of $B \cup C$ has (since G is 2-connected) at least two vertices of attachment, at most one of which may be on B because otherwise B would violate the maximality property of a block. We may assume that u belongs to B or to a bridge with at least one vertex of attachment on B. Otherwise, we can make this so by exchanging the rôles of v_0 and v_1 in our proof, making P_1 the path anticlockwise from v to v_1, and so on. We let u' be the vertex of B closest to u in H: if $u \in V(B)$ then $u' = u$, otherwise u' is the vertex of attachment on B of the bridge of $B \cup C$ containing u.

By induction, we can find a path P' in B from v_1 via v_2 to u' which is a Tutte path with respect to the outer cycle of B. The path $P'' = P_1 \cup e \cup P'$ is then a path from v to u' via e. To complete the proof we must show how to modify P'' so that we have a Tutte path with respect to C, and so that the path ends at u if $u \neq u'$. To attain the first objective we construct detours of P_1; to attain the second we cut and splice our path.

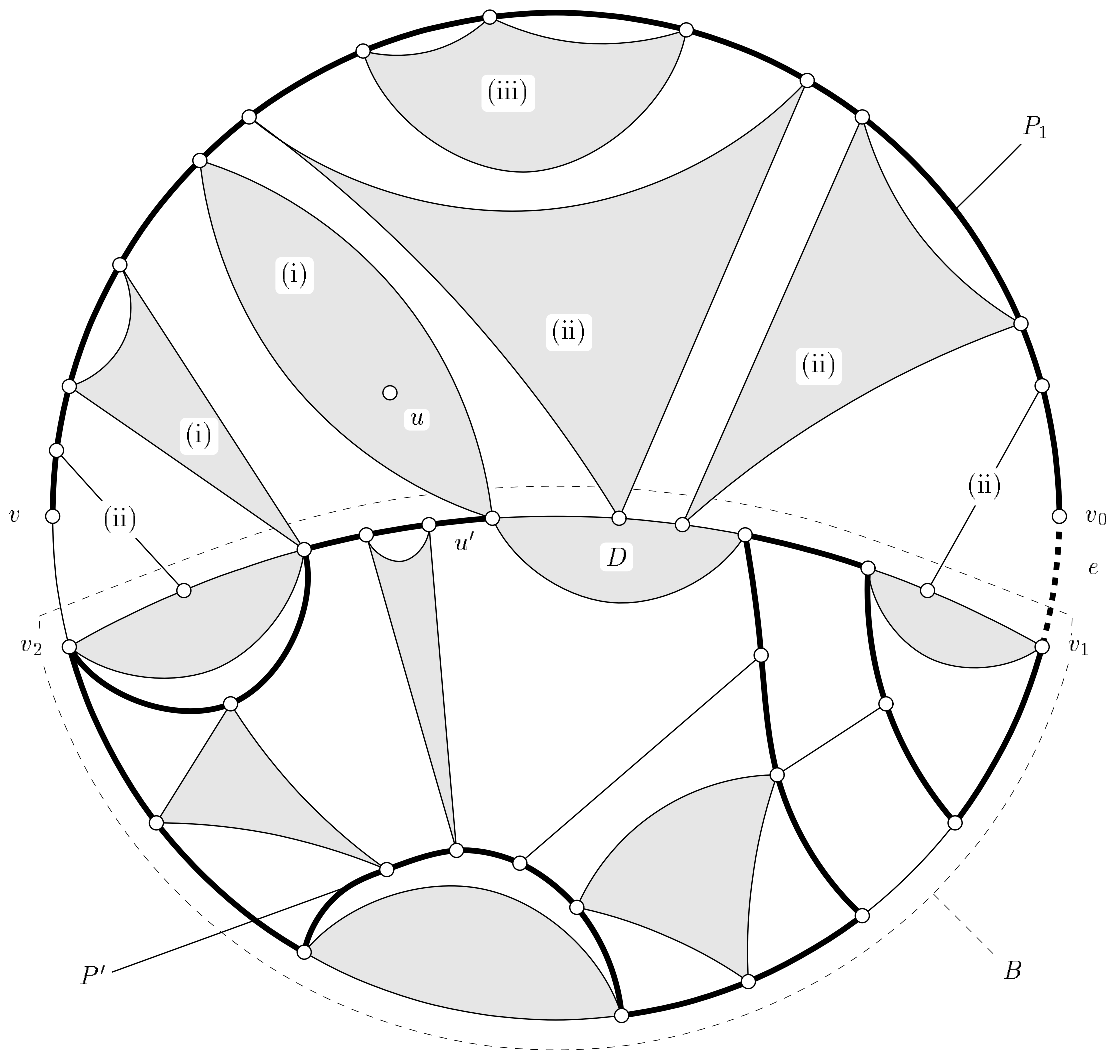

Figure 2: Proof of Theorem 3.8

The bridges of $B \cup C$ may be divided into three types: (i) those attached to B at a vertex in $V(P')$, (ii) those attached to B at a vertex of $V(B) - V(P')$, and (iii) those with all vertices of attachment on P_1.

Each bridge J of type (i) spans a subpath $P_1[x, y]$ of P_1. We use our induction hypothesis on $J \cup P_1[x, y]$ plus an extra edge to detour $P_1[x, y]$ through this subgraph. If $u \notin V(B)$, then u is in some such type (i) bridge, which we treat specially, with a cut and splice operation on our path. We replace $P_1[x, y]$ by not one, but two, paths in our subgraph: either an xy- and a $u'u$-path, or an xu'- and a yu-path, which we again obtain by induction on the subgraph plus an extra edge.

Each bridge of type (ii) has a vertex of attachment on B which is itself a vertex of a bridge D of P' in B. Thomassen dealt with bridges of type (ii) by grouping them into 'superbridges' according to their vertex of attachment on B, and constructing a detour of P_1 through this superbridge using the same ideas as for type (i) bridges. There is, however, a problem here. If some D has two or more vertices that serve as vertices of attachment for type (ii) bridges, this can leave D, or a subgraph of G containing D, as a bridge of our final path with four or more vertices of attachment, and our final path will not be a Tutte path. The solution, as pointed out by Chiba and Nishizeki, is essentially to form 'superbridges' by grouping together all bridges of type (ii) with the same D, not just those with the same vertex of attachment in B. These superbridges are then just bridges of $P_1 \cup P'$ in G.

After making all of the above detours, the subpath of P_1 spanned by each bridge of type (iii) may or may not be still in our path. If it is still in our path we detour the part of P_1 spanned by the type (iii) bridge in a way similar to, but even simpler than, what we did for type (i) bridges; if it is not still in our path we do nothing. Bridges of type (iii) may be 'nested' along P_1; we deal with the 'outermost' ones first and apply the above recursively.

This completes the construction of the path P from P'' by detouring and possibly cutting and splicing. ∎

As a historical note, our type (i) bridges, type (ii) superbridges, and type (iii) bridges correspond to Ore's δ_1-, ϵ- and δ_0-bridges respectively: hence Chiba and Nishizeki's cryptic description of the problem in Thomassen's argument as 'an argument on an "ϵ-bridge" ... is somehow missing'.

3d. Further results for the plane and projective plane

Plummer's second question [P5] asked whether every graph obtained by deleting *two* vertices of a 4-connected planar graph is always hamiltonian, extending Nelson's result in Corollary 3.5. The resolution of this is connected to results for graphs on the projective plane. Before discussing this we mention some other results for planar graphs.

Baybars [B12] investigated the relationship between the connectivity of a plane triangulation and the largest k for which every path of k edges is contained in a hamilton cycle. Dillencourt [D6] extended the result of Whitney in Lemma 3.1 by easing the restrictions on the chords of C in a plane C-near-triangulation without separating triangles. Sanders [S5] was able to generalize this from near-triangulations to arbitrary planar graphs. Sanders [S6] was also able to generalize Thomassen's Path Theorem by removing the restriction that v be a vertex of C. Brunet and Richter [B25] found general sufficient conditions for a set of edges in a planar graph to all belong to a path with properties similar to a Tutte path, and used this to find conditions for hamilton cycles through sets of edges.

Plummer's second question [P5] has been answered in the affirmative, as follows.

Lemma 3.9 (Thomas and Yu, 1994 [T3])**.** *Let G be a 2-connected plane graph with distinct facial cycles C_1 and C_2, and let $e \in E(C_1)$. Then there exists a Tutte cycle C with respect to $C_1 \cup C_2$, such that $e \in E(C)$ and no bridge of C contains edges of both C_1 and C_2.*

Theorem 3.10. *Every graph obtained by deleting two vertices from a 4-connected planar graph has a hamilton cycle.*

In the same paper Thomas and Yu also established the following important results for projective planar graphs, resolving part of a conjecture of Grünbaum [G12]. For a face F of an embedded graph, define the *F-width* to be the minimum over all noncontractible closed curves intersecting F of the number of points at which the curve intersects the graph.

Lemma 3.11 (Thomas and Yu, 1994 [T3])**.** *Let G be a 2-connected graph in the projective plane with a face F_1 having boundary subgraph C_1, and let $e \in E(C_1)$. Then there exists a Tutte cycle C with respect to C_1, such that $e \in E(C_1)$, every bridge of C containing a noncontractible cycle is edge disjoint from C, and if the F_1-width of G is 1 then C is noncontractible.*

Theorem 3.12. *For each edge of every 4-connected graph in the projective plane, the graph has a hamilton cycle through that edge.*

Note that Dean (see [D2]) conjectures that all 4-connected projective-planar graphs are in fact hamilton-connected.

The proofs of Lemmas 3.9 and 3.11 more or less follow the same outline as the proof of Theorem 3.8. First we deal with special cases (certain 2-separations in Lemma 3.9, cases where the F_1-width is small in Lemma 3.11). Then in both lemmas we consider $H = G - V(C_1)$, and concentrate on one of its blocks B; for Lemma 3.9, B is the block containing C_2, and for Lemma 3.11, B is the block containing all the noncontractible cycles of H. By induction we find a path P' in B which we join to a subpath P_1 of C_1 and then modify by detours through the subgraphs obtained from the bridges of $B \cup C_1$ to get the required Tutte cycle.

3e. Graphs on the torus and Klein bottle

For graphs embedded on the torus, Grünbaum and Nash-Williams independently proposed the following.

Conjecture 3.13 (Grünbaum, 1970 [G12], Nash-Williams, 1973 [N2])**.** *Every 4-connected graph embeddable on the torus is hamiltonian.*

As noted by Thomassen [T7], it is not true that every edge in a 4-connected toroidal graph is in a hamilton cycle, as may be seen by taking the Cartesian product of two even cycles, which embeds in an obvious way as a 4-connected quadrangulation of the torus, and adding one edge diagonally across one of the quadrangles. This edge is not on any hamilton cycle. This suggests that more subtle proof techniques will be required to prove this conjecture than for Theorems 3.4, 3.8 or 3.12.

Dean and Ota [D2] have proved a weakening of Conjecture 3.13. They showed that every member of a certain family of graphs, including all 4-connected graphs embeddable on the torus and Klein bottle, has a 2-factor. Unfortunately, their method yields no bound on the number of components of the 2-factor.

If we consider connectivity more than 4, positive results are known. Altshuler [A4] showed that all 6-connected graphs on the torus are hamiltonian. Euler's formula implies that all such graphs are in fact 6-regular triangulations, and this restricted structure is the basis of Altschuler's method of constructing a hamilton cycle. Ewald [E5] proved that, more generally, a triangulation of any surface with maximum degree at most 6 (the surface must thus be Euler-nonnegative) is hamiltonian. Altschuler also showed that 4-regular quadrangulations of the torus are hamiltonian. Moreover, he obtained partial results for 3-regular graphs embedded on the torus with all faces hexagons; such graphs were proved hamiltonian independently by Bouwer and Chernoff [B21] and Alspach and Zhang [A3].

For connectivity 5, results were obtained by several authors. Barnette [B7] showed that 3-connected graphs embedded with representativity at least 3 on the torus have a spanning annulus, and used this and Thomassen's Path Theorem to show that every 5-connected toroidal triangulation has a hamilton path. Brunet and Richter [B25] showed that every 5-connected triangulation of the torus has a spanning disc with special properties, and used this and a result on hamilton cycles in planar graphs to show that every 5-connected toroidal triangulation has a hamilton cycle. Sanders [S6] improved Thomassen's Path Theorem (by removing the restriction that v be a vertex of C) and used this to show that all 5-connected graphs embedded on the torus with representativity exactly 1 are hamiltonian. The best result so far is the following.

Theorem 3.14 (Thomas and Yu, 1994 [T2]). *Every edge of a 5-connected graph embeddable on the torus is contained in a hamilton cycle.*

The most general case of the proof (when the representativity is at least 4) follows the outline we have seen before: delete a cycle, focus on one of the remaining blocks, then construct detours to pick up everything that is left.

For the Klein bottle, which we have mentioned above only once, there are so far no positive results on the existence of hamilton paths or cycles. An obvious conjecture is that every 4-connected graph embeddable on the Klein bottle is hamiltonian.

4. Algorithms and complexity for hamiltonicity on surfaces

Given that 4-connected planar graphs are hamiltonian, it is natural to ask whether there are efficient algorithms for finding hamilton cycles in these graphs. Using Ore's proof of Tutte's Theorem, Gouyou-Beauchamps [G10] showed that there is an $O(n^3)$ algorithm for this. In the case of triangulations, this was improved, by using a modified version of Whitney's proof of Theorem 3.2, to an $O(n)$ algorithm by Asano, Kikuchi and Saito [A6]. An $O(n)$ algorithm for all 4-connected planar graphs was described by Chiba and Nishizeki [C3, N6] based on the corrected proof of Thomassen's Path Theorem.

The problem of determining whether a graph is hamiltonian was one of the first problems shown to be NP-complete. For graphs embedded in the plane, Garey, Johnson and Tarjan [G5] showed that it remains NP-complete even when restricted to 3-connected cubic planar graphs of girth 5; their proof uses a construction based on Tutte's nonhamiltonian 3-connected cubic planar graph [T11]. It was also shown to be NP-complete when restricted to cubic bipartite planar graphs, by Akiyama, Nishizeki and Saito [A1], and independently by Plesník [P4]. Chvátal and Wigderson showed that it is NP-complete when restricted to plane triangulations (see [C7, problem 31, page 427]).

Problems for graphs embedded on an arbitrary surface are at least as difficult as for planar graphs (because every planar graph can be embedded on any surface, albeit possibly only with representativity 0). Therefore, the hamilton cycle problem for graphs embeddable on a given surface will be NP-complete. However, Fellows, Hickling and Sysło [F3] have shown, using the proof of Wagner's conjecture by Robertson and Seymour (which is slowly appearing, in many parts, in the *Journal of Combinatorial Theory, Series B*), that the hamilton cycle problem can be solved in polynomial time for graphs embedded on a fixed surface with fixed disc dimension; the *disc dimension* is the smallest number of open discs in the surface not intersecting the graph such that every vertex lies on the boundary of a disc.

5. Hamiltonicity for special classes of planar graphs

In this section we examine the hamiltonian properties of special classes of planar graphs. Many of the results are negative. To quantify how far a family of graphs is from being hamiltonian or traceable, Grünbaum and Walther [G16] introduced *shortness parameters*. We modify their terminology slightly. Let $l_C(G)$ denote the length of the longest cycle in a graph G, and $v(G)$ the number of vertices of G. Let $\mathcal{G}$ be an infinite family of graphs. Then the *shortness index* τ, *shortness coefficient* ρ and *shortness exponent* σ of $\mathcal{G}$ are defined as

$$\begin{aligned}\tau &= \max_{G\in\mathcal{G}}\{v(G) - l_C(G)\}\\ \rho &= \liminf_{n\to\infty} \min\{l_C(G)/n : G \in \mathcal{G}, v(G) = n\}\\ \sigma &= \liminf_{n\to\infty} \min\{\log l_C(G)/\log n : G \in \mathcal{G}, v(G) = n\}\end{aligned}$$

(where the minimum of the empty set is $+\infty$). We can define similar shortness parameters using the length of a longest path instead of a longest cycle.

5a. Regular graphs with connectivity conditions

Tait [T1] conjectured in the 1880's that all 3-connected cubic planar graphs are hamiltonian. A positive answer would have verified the Four Colour Conjecture; for some of the history, see Biggs, Lloyd and Wilson [B15]. Purported proofs were produced in the 1930's by Schoblik [S13] and Chuard [C5] (criticized by Pannwitz [P1]). However, in 1946 Tutte [T11] found a 46-vertex nonhamiltonian 3-connected cubic planar graph. Using arguments based on Tutte's graphs, Walther [W3, W6] showed that the shortness exponent of 3-connected cubic graphs is less than 1, and Richmond, Robinson and Wormald [R2] showed that almost all 3-connected cubic planar graphs are nonhamiltonian.

Tutte's 46-vertex graph is not the smallest possible. A nonhamiltonian 3-connected cubic planar graph with 38 vertices was found by Bosák [B20], Lederberg [L1], and Barnette (mentioned by Lederberg). In fact, as Bosák pointed out a little later in a private communication (see [G16, H10]), it is possible to construct six different 38-vertex graphs. Lederberg [L1] also showed that all 3-connected cubic planar graphs on at most 18 vertices are hamiltonian. This was improved in a series of papers by Goodey [G6], Butler [B27, B28], Barnette and Wegner [B11], Faulkner and Younger [F2], Okamura [O2, O3], Mohar [M5] and Barnette [B6]. Finally, Holton and McKay [H10, H11] demonstrated that all 3-connected cubic graphs on at most 36 vertices are hamiltonian, and moreover the six known 38-vertex graphs are the only non-cyclically-4-edge-connected ones of this order. These results were deduced by the use of a computer and suitable reduction arguments.

Note that Exoo and Harary [E7, E8] give the smallest nonhamiltonian cubic planar graph of connectivity 2.

What if we consider cyclically-k-edge-connected cubic planar graphs for $k \leq 5$ (the only possible values)? Tutte [T13] found a 224-vertex nonhamiltonian cyclically-4-edge-connected cubic planar graph; later graphs of this type were found by Hunter [H13] (58 vertices), Grinberg [G11] (44 vertices), Grünbaum [G12], Faulkner and Younger [F2] and Holton and McKay [H10] (all 42 vertices). Whether 42 vertices is the minimum here appears to be still open. Walther [W4, W5, W7] found a 162-vertex nonhamiltonian cyclically-5-edge-connected cubic planar graph; later graphs of this type were found by Sachs [S2] and Walther [W8] (114 vertices), Grinberg [G11] (46 vertices, presented by Sachs [S3]), and Tutte [G12, T14] (44 vertices) – see subsection 2b and Figure 1. Faulkner and Younger [F2] showed that there are none on 42 or fewer vertices. McKay (personal communication) has generated all nonhamiltonian cyclically-5-edge-connected cubic graphs on up to 52 vertices; there are 1 on 44 vertices (Tutte's example), 1 on 46 vertices (Grinberg's example), none on 48 vertices, 3 on 50 vertices and 6 on 52 vertices. Zaks [Z8] showed that cyclically-5-edge-connected cubic planar graphs have shortness exponent less than 1.

For hamilton paths, Balinski [B1] conjectured that every 3-connected cubic planar graph has a hamilton path. Counterexamples were given by Grünbaum and Motzkin [G15], who in fact showed that 3-connected cubic planar graphs have a path shortness exponent less than 1. A smaller counterexample was later found by Brown (90 vertices, see [K4]). The smallest counterexample known seems to be one with 88 vertices by Zamfirescu [G12, Z9]. Zamfirescu [Z9] has also found a 44-vertex 3-connected cubic planar graph which is traceable but does not have a hamilton path starting at every vertex.

For graphs which are 4- and 5-regular, Sachs [S2] and Walther [W5, W8] were able to use nonhamiltonian 3-connected cubic planar graphs to construct nonhamiltonian 4- and 5-regular 3-connected planar graphs which were cyclically-6-edge-connected (the highest possible cyclic edge connectivity). Examples of 3-connected 4- and 5-regular planar graphs without hamilton cycles and paths were also constructed by Zaks [Z1] and Owens [O5] (who mentions corrections to some of Zaks' results).

5b. Restricted vertex and face degrees

In this subsection we examine the existence of hamilton cycles for planar graphs with restrictions on both the degrees of the vertices and the degrees of the faces. Because of time and space limitations, we cannot summarize all of the results in this area, although we try to give all the references of which we are aware. We focus on the few positive results, proving the existence of hamilton cycles or paths, at the expense of the many negative results, mostly giving bounds on shortness parameters.

Early results include those by Moon and Moser [M8], Brown [B23, B24], Jucovič [J5], Jucovič and Walther [J6], and Ewald [E4]. Grünbaum and Walther [G16] introduced shortness parameters and used them to restate these early results and give new results. Each of the results mentioned in [G16], excepting three by Ewald [E4], proves that the family of graphs under investigation contains infinitely many nonhamiltonian members. At the time of Grünbaum and Walther's paper, the only known positive result involving vertex and face degrees of plane graphs was by Ewald [E5], who showed that a triangulation of any surface with maximum degree at most 6 (the surface must thus be Euler-nonnegative) is hamiltonian. According to Ewald [E6], Harald Streubel later extended this for plane triangulations, showing that those with maximum degree at most 7 are hamiltonian, resolving a question by Barnette [B5].

Later results on 3-connected planar graphs in which all faces have the same degree (which must be 3, 4 or 5) were given by Jucovič [J5], Jucovič and Walther [J6], Owens [O11, O12], Fanelli and Ghiraldini [F1] and Jendrol' and Kekeňák [J2]. All of these provide bounds on path or cycle shortness parameters for classes of plane graphs with all faces of the same degree, except for Fanelli and Ghiraldini, who give sufficient conditions on vertex degrees for plane triangulations of maximum degree at most 8 to be hamiltonian.

Grünbaum and Zaks [G17] proposed the investigation of 3-connected planar graphs with exactly two types of face. Again, most of the results have been negative, in the form of bounds on shortness parameters. Negative results may be found in papers by Harant and Walther [H4], Owens [O6, O7, O8, O9, O10, O13], Tkáč [T10], Walther [W9], and Zaks [Z3, Z4, Z5, Z6, Z7]. There are, however, three positive results. Goodey showed that 3-connected cubic plane graphs with all faces quadrangles and hexagons [G8] or all faces triangles and hexagons [G9] are hamiltonian. Jendrol' and Mihók [J3] showed that all 3-connected cubic planar graphs whose faces are all pentagons and k-gons, with no two k-gons sharing an edge, are hamiltonian for $k \leq 12$ (for $k \leq 11$ there are only finitely many such graphs, and for $k = 12$ there is a sparse infinite family of graphs). The following important case seems to be open.

Question 5.1. *Is a 3-connected cubic plane graph hamiltonian if each face is either a pentagon or a hexagon? (Such graphs must have exactly* 12 *pentagonal faces; an infinite number of such graphs exist – see Etourneau* [E3] *and Zaks* [Z2]*.)*

Even more generally, we may ask:

Question 5.2 (Barnette, see [M3])**.** *Is a 3-connected cubic plane graph hamiltonian if every face has degree at most* 6*?*

Other shortness results were given by Owens [O8] for 4- and 5-regular 3-connected plane graphs with at most three types of faces, by Zaks [Z7] for 3-connected plane graphs with every face of degree at most 7, and by Walther [W10] for graphs where both vertex and face degrees are 3 or 8.

The following conjecture is well-known.

Conjecture 5.3: Barnette's Conjecture (see [G12]). *Every cubic bipartite* 3-*connected planar graph is hamiltonian.*

This is not true if '3-connected' is replaced by '2-connected'. Peterson [P3] and Asano et al. [A7] independently found and proved minimality for the smallest possible nonhamiltonian cubic bipartite 2-connected planar graph, which has 26 vertices. Peterson [P2, P3] and Kelmans [K2] showed that Barnette's Conjecture is equivalent to stronger conjectures about the existence of hamilton cycles behaving in particular ways with respect to edges. As mentioned above, Goodey [G8] has verified Conjecture 5.3 when the graph is embeddable with all faces quadrangles or hexagons. Sarvanov [S8] has shown that a cubic plane graph is hamiltonian if it can be 3-face coloured so that the faces of colour 1 together with the quadrangles of colour 2 form a connected subgraph of the dual; any 3-face-colourable cubic graph must be bipartite. Finally, Holton, Manvel and McKay [H9] verified by computer results and reduction procedures that Barnette's Conjecture is valid for graphs with 64 or fewer vertices.

Generalizing the idea behind Barnette's Conjecture, Schmidt and Zamfirescu [S11] examined 3-connected cubic plane graphs in which all faces are k-gonal modulo n, for some k and n. They found bounds for the shortness exponent of several such families of graphs.

5c. Geometric graphs

Computer scientists have been interested in finding hamilton cycles in planar graphs obtained from geometric constructions; problems of this type arise in various applications.

Given a finite set of points S in the plane, for each $s \in S$ we define its *Voronoï region* to be the set of points at least as close to s as to any other member of S. The *Delaunay triangulation* of S is the graph with vertex set S, such that two points of S are adjacent if their Voronoï regions intersect in more than one point. In general a Delaunay triangulation will be a plane near-triangulation, with all interior faces triangles, but in certain *degenerate* cases there may be non-triangular interior faces. Shamos [S14] asked whether the Delaunay triangulation of S always contains the shortest (in Euclidean distance) possible hamilton cycle in the complete graph on S, which can only be true if the Delaunay triangulation of S is hamiltonian. However, Kantabutra [K1] found an example of a nonhamiltonian Delaunay triangulation, which was degenerate. Dillencourt constructed examples which were nondegenerate [D4] or 3-connected [D5], and determined the smallest possible nonhamiltonian Delaunay triangulations satisfying various conditions [D8]. Dillencourt has also shown [D9] that determining whether a Delaunay triangulation is hamiltonian is NP-complete.

An *inscribable polytope* is a graph which can be realized as the set of edges of the convex hull of a set of points on a sphere. Inscribable polytopes are related to Delaunay triangulations. Dillencourt [D5, D8] constructed nonhamiltonian inscribable polytopes and bounded the shortness exponent of this class of graphs, answering a question attributed to R. Seidel (see [O1]). He has also shown that it is NP-complete to determine whether an inscribable polytope is hamiltonian [D9].

Cimikowski [C8, C9] investigated another family of graphs arising from geometric considerations in applications. He examined plane near-triangulations in which the vertices can be partitioned into nested levels, each level inducing a cycle, except for the innermost level which may induce a cycle or a tree. By finding a hamilton cycle on the innermost one or two levels and extending it outwards, such graphs can be shown to be hamiltonian.

5d. Halin graphs

One special class of planar graphs, which has been extensively studied with respect to its hamiltonian properties, is the class of Halin graphs. A *skirted tree* is obtained by embedding a tree in the plane and then connecting its endvertices with a cycle so as to maintain planarity. If the tree is *homeomorphically irreducible* (has no vertices of degree 2), then the result is a *Halin graph*, also called an *r-tope* by Hill and his colleagues (see below). Halin graphs have been studied since Kirkman [K3] tried to enumerate the cubic ones. Halin [H2] gave them as examples of minimally 3-connected graphs. That all Halin graphs are hamiltonian was established by Bondy [B16] and also, apparently, by Hill and Rogers in an unpublished manuscript mentioned in [H6, H7, H8, S15]. Bondy reported that he and Lovász could show that Halin graphs were pancyclic except for one even cycle length; that they are not necessarily pancyclic was shown by Malkevitch [M1].

The proof of Bondy and Lovász's result did not appear for over ten years. Meanwhile, Hill [H6] reported that cubic Halin graphs have at most three hamilton cycles, which by a result of Smith and Tutte [T11] means that they have exactly three hamiltonian cycles; this can be proved using a technique of Hill and Singmaster [H8] for generating all hamilton cycles of a Halin graph. It also follows directly from the fact that cubic Halin graphs can be reduced to K_4 by contracting triangles, as noted by Malkevitch [M1], who used this to show that cubic Halin graphs have a hamilton cycle including any given pair of edges. Singmaster and Hill [S15] showed that a broader class of graphs called *q-topes*, those 3-connected plane graphs in which there is one face intersecting every other face in at least a vertex, have a hamilton cycle through any edge. This was reproved for Halin graphs by Sysło and Proskurowski [S21] who also showed that certain pairs of adjacent edges in a Halin graph lie on a hamilton cycle. Skowrońska [S17] proved that the graphs studied by Singmaster and Hill are 1-hamiltonian. Cornuejols, Naddef and Pulleyblank [C10] showed that the travelling salesman problem is polynomial time solvable for Halin graphs.

Finally, Bondy and Lovász published their result that Halin graphs are almost pancyclic in [B18], and it was verified independently by Skowrońska [S18]. Both of these papers also verified and strengthened a conjecture of Malkevitch [M1] by showing that that Halin graphs with no interior vertices of degree 3 are pancyclic.

In continuing work on Halin and related graphs, Sarvanov [S9] investigated the existence of hamilton paths in plane graphs where every face touches one of two designated faces (as opposed to a single designated face for Halin graphs). Skowrońska and Sysło [S16] gave an algorithm to determine which skirted trees are hamiltonian. Barefoot showed that Halin graphs are hamilton-connected [B2] and that a closely related family of minimally-4-connected planar graphs also constructed by Halin [H2] are 2-hamilton-connected (i.e. the graph and all its one-vertex-deleted subgraphs are hamilton-connected) [B3]. Skupień [S19] showed that q-topes have hamilton paths and cycles satisfying some strong properties.

As we have seen, Halin graphs have very nice hamiltonian properties, prompting the following.

Conjecture 5.4 (Plummer, 1975 [P5]). *Every 4-connected plane triangulation has a spanning subgraph which is a Halin graph.*

This conjecture cannot be extended to all 4-connected planar graphs. A Halin graph with v vertices contains at least $(v+2)/4$ triangles. While Euler's formula implies that every 4-connected planar graph contains triangles, there may not be enough of them. For example, it is not difficult to construct arbitrarily large 4-connected 4-regular plane graphs with all faces quadrangles except for eight triangles, which is not enough if there are more than 34 vertices in the graph. However, it may be possible to extend Conjecture 5.4 slightly, to plane near-triangulations. An affirmative answer to this extension would strengthen a result (see subsection 10d) on the existence of homeomorphically irreducible spanning trees.

6. Special hamiltonicity results for planar graphs

6a. More than one hamilton cycle

If a graph has a hamilton cycle, it is natural to ask whether it has more than one. Smith and Tutte [T11] showed that every cubic graph has an even number of hamilton cycles through any given edge, and hence if it has any hamilton cycle it must have at least three; these results were extended to cubic multigraphs by Ninčak [N4]. Smith and Tutte's result, restricted to planar graphs, was shown by Price [P6] to be a corollary of an algebraic method which generates all the hamilton cycles of a cubic planar graph.

For graphs on surfaces, Hakimi, Schmeichel and Thomassen [H1] showed that every 4-connected plane triangulation with v vertices has at least $v/\log_2 v$ hamilton cycles. They also showed that there exist infinitely many plane triangulations (not 4-connected) with exactly 4 hamilton cycles. Kratochvíl and Zeps [K5] showed that any hamiltonian graph with every edge belonging to at least two triangles must have at least 4 hamilton cycles; this condition is satisfied by a triangulation of any surface, so every triangulation has either no hamilton cycles or at least 4.

It is also desirable for a graph to have not just more than one hamilton cycle, but more than one edge-disjoint hamilton cycle. Unfortunately, this cannot be guaranteed by connectivity conditions, at least for planar graphs. Examples of 4-connected 4-regular planar graphs without two edge-disjoint hamilton cycles (i.e. a hamiltonian decomposition) were given by Grünbaum and Malkevitch [G14] and independently by Martin [M4]; both papers show that line graphs of nonhamiltonian cyclically-4-edge-connected cubic planar graphs provide the required examples.

Bondy and Häggkvist [B17] found a necessary condition, similar to Grinberg's Condition, for a 4-regular plane graph to have a hamiltonian decomposition. Rosenfeld [R6] found another construction for 4-connected 4-regular planar graphs with no hamiltonian decomposition.

Even if the connectivity is raised to 5, a planar graph need not contain two edge disjoint hamilton cycles. Examples of 5-connected 5-regular planar graphs without two edge-disjoint hamilton cycles were given by Zaks [Z1] and Owens [O5]. Rosenfeld [R7, R8] also provided constructions for 5-connected planar graphs, both regular and nonregular, with no two edge disjoint hamilton cycles.

6b. Hypohamiltonian planar graphs

A graph is *hypohamiltonian (hypotraceable)* if it has no hamilton cycle (path) but all its single-vertex-deleted subgraphs do have a hamilton cycle (path). Thomassen [T4] found infinite families of hypohamiltonian and hypotraceable planar graphs with given girth (3, 4 or 5), and of hypotraceable cubic planar graphs. He showed [T5] that all planar hypohamiltonian graphs have a vertex of degree 3. Hatzel [H5] found a small hypohamiltonian planar graph, with only 57 vertices. Thomassen [T6] constructed infinite families of hypohamiltonian and hypotraceable cubic planar graphs, the smallest in each family having 94 vertices and 460 vertices respectively.

7. Connectivity, walks and trees on Euler-nonnegative surfaces

In this section we discuss spanning trees and walks of bounded degree, i.e. k-trees and k-walks, for graphs on the plane, projective plane, torus and Klein bottle. Recall the results of Lemma 1.1, that a k-tree implies a k-walk and a k-walk implies a $(k+1)$-tree.

Even for planar graphs, the complexity of determining whether there is a k-tree is NP-complete for each fixed $k \geq 2$; this was shown by Douglas [D11], who actually proved a more general result. Thus, it is NP-hard to determine what we might call the *tree number* of a graph, the smallest k for which a k-tree exists. On the other hand, Fürer and Raghavachari [F5] have shown that there is a polynomial time algorithm to produce a spanning tree whose maximum degree exceeds the tree number by at most 1. Thus, the tree number is similar to the chromatic index, in that its value can be narrowed down to one of two easily-calculated consecutive values, but choosing the correct one of those two is then NP-hard.

Before discussing existence results for k-trees and k-walks, we need to introduce two types of graph which fall somewhere in between 2-connected and 3-connected planar graphs. A pair (G, C), where G is a 2-connected plane graph and C is a facial cycle of G, is called a *circuit graph* if there is no 2-separation (H, K) of G with $C \subseteq H$. A triple (G, C_1, C_2), where G is a 2-connected plane graph and C_1 and C_2 are (not necessarily distinct or disjoint) facial cycles of G, is called an *annulus graph* if there is no 2-separation (H, K) with $C_1 \cup C_2 \subseteq H$. A circuit graph is just an annulus graph with $C_1 = C_2$. Circuit graphs were introduced by Barnette [B4] and annulus graphs by Brunet et al. [B26]. Every 3-connected graph is a circuit and annulus graph.

The following property of circuit and annulus graphs is very useful in the induction arguments used in many of the results listed below.

Lemma 7.1 (Brunet et al., 1992 [B26]). *Suppose (G, C_1, C_2) is an annulus graph, and C_3 is a cycle bounding a closed disc D (which may include the point at infinity) such that $(C_1 \cup C_2) \cap D \subseteq C_3$. Then $(G \cap D, C_3)$ is a circuit graph.*

The special case of this lemma where $C_1 = C_2$, i.e. for circuit graphs, is due to Gao and Richter [G2]

The first result concerning the existence of k-trees for graphs on surfaces was by Barnette [B4], who proved that all circuit graphs, and therefore all 3-connected planar graphs, have 3-trees. Barnette strengthened this in [B8], showing that any vertex could be chosen to be an endvertex of the 3-tree. With this strengthened version, and by proving [B7] the existence of a spanning disc (for the projective plane), annulus (for the torus) or Möbius strip (for the Klein bottle) he was able to show that polyhedral maps (3-connected graphs embedded with representativity at least 3) on the projective plane, torus and Klein bottle all had 3-trees also.

Jackson and Wormald [J1] conjectured that Barnette's result for planar graphs could be improved to show that all 3-connected planar graphs have 2-walks. This was proved, in fact for circuit graphs, by Gao and Richter [G2]. Their argument used decomposition theorems for circuit graphs based on Lemma 7.1, describing the structure of the graph obtained by deleting a vertex or path in C from the circuit graph (G, C). Using a result of Fiedler et al. [F4] showing that every nonplanar 3-connected graph embedded in the projective plane has a spanning disc (somewhat stronger than Barnette's result [B7] which requires representativity at least 3), Gao and Richter deduced that 3-connected projective planar graphs also have 2-walks.

Brunet et al. [B26] showed that every 3-connected graph embedded in the torus or Klein bottle has a spanning annulus graph (which had been shown for toroidal embeddings with representativity at least 3 by Barnette [B7]), and that every annulus graph has a 2-walk; it therefore follows that every 3-connected graph embedded on the torus or Klein bottle has a 2-walk.

R. Thomas (personal communication) suggested that it might be possible to generalize both Tutte's Theorem and Gao and Richter's theorem on the existence of 2-walks in 3-connected planar graphs. He conjectured that a 3-connected planar graph has a 2-walk W such that every vertex used twice by W lies in a 3-cut. This was proved by Gao, Richter, and Yu [G3] in a slightly more general form. The proof involves finding in a circuit graph (G, C) a Tutte cycle C' with respect to C, with the property that there is an injection taking each nontrivial bridge of C' to one of its vertices of attachment. This injection allows C' to be extended to a 2-walk by detouring into each bridge, and since the only vertices used twice are vertices of attachment of a bridge of a Tutte cycle, they are part of a 2-cut or 3-cut.

We briefly mention two applications of the above results. Barnette used his results on the existence of 3-trees for graphs on Euler-nonnegative surfaces to obtain lower bounds on the length of the longest path and cycle in 3-connected graphs embedded on those surfaces with representativity at least 3 [B4, B8]. The representativity condition may be dropped using the work of Gao and Richter [G2] and

Brunet et al. [B26]. Hong Yuan [H12] has used the same results of Barnette to obtain improved bounds on the spectral radius of graphs embedded in Euler-nonnegative surfaces.

8. Euler-negative surfaces

8a. Limiting examples

Having shown the existence of 2-walks for 3-connected graphs on Euler-nonnegative surfaces, it may seem reasonable to ask if in fact all 3-connected graphs have 2-walks. An easy counterexample is provided by $K_{3,n}$, which has no 2-walk for $n \geq 7$. In fact, Thomassen [T9] showed that on every Euler-negative surface there are many triangulations which do not have 3-trees, and hence do not have 2-walks. The construction is simply to take any triangulation G of the surface and form a new triangulation $S(G)$ by inserting a new vertex into every face adjacent to all vertices of that face. If G has n vertices, then by applying Euler's formula it has $2n - 2\chi$ faces, and so by deleting the n vertices of $V(G)$ from $S(G)$ we obtain $2n - 2\chi \geq 2n + 2$ components, i.e. the new vertices of $S(G)$. By Lemma 2.2, $S(G)$ therefore has no 3-tree. Taking G to be a quadrangulation instead of a triangulation yields a triangulation $S(G)$ with no 2-tree (hamilton path), by the same argument. By making the representativity of G large, the representativity of $S(G)$ can also be made large in both cases.

Therefore, on any Euler-negative surface there are triangulations of arbitrarily large representativity with connectivity 3 and no 3-tree or 2-walk, and with connectivity 4 and no hamilton path or cycle. In fact, it has been shown by Archdeacon, Hartsfield and Little [A5] and also by Thomassen (personal communication) that for each k there are triangulations of both some orientable surface and some nonorientable surface which are k-connected, have representativity (or edgewidth) at least k, and have no k-tree. Thus, we cannot hope to guarantee the existence of a k-walk or k-tree (or, as a special case, a hamilton cycle or path) for a graph on an Euler-negative surface by merely using a sufficiently large, but surface-independent, connectivity. Connectivity must be surface-dependent, or if we wish to examine graphs of a fixed connectivity, we must include some extra conditions to guarantee the existence of k-walks or k-trees.

8b. Connectivity depending on the surface

First let us consider how we may guarantee the existence of k-walks or k-trees on a surface of Euler characteristic $\chi < 0$ by requiring a connectivity dependent on χ. This question was first examined by Duke [D12] in the context of hamilton cycles. Duke considered only orientable surfaces, but his reasoning also applies to nonorientable surfaces.

Theorem 8.1 (Duke, 1972 [D12]). *Suppose that Σ is a surface of Euler characteristic $\chi < 0$. Let m be a positive integer.*

(i) *If $m \geq 3 + \sqrt{9 - 3\chi}$, then every m-connected graph embeddable on Σ is hamiltonian.*

(ii) *If $m \leq \frac{1}{2}(3 + \sqrt{17 - 8\chi})$ then $K_{m,m+1}$ is a nonhamiltonian m-connected graph embeddable on Σ.*

Proof outline. (i) Consider a graph G on Σ with v vertices and minimum degree δ. If G is m-connected, then $\delta \geq m$ and thus $\delta \geq 3 + \sqrt{9 - 3\chi}$, which implies (after some arithmetic) that $-6\chi/(\delta - 6) \leq 2\delta$. But from Euler's formula and the fact that every face has degree at least 3, $v \leq -6\chi/(\delta - 6)$, and so $v \leq 2\delta$. By a well-known theorem of Dirac [D10] G is therefore hamiltonian.

(ii) $K_{m,m+1}$ is nonhamiltonian and m-connected, and if the given condition is satisfied it follows from results of Ringel [R3, R4] that $K_{m,m+1}$ embeds on Σ. ∎

Duke also gives a stronger version of (i) for graphs of a given girth $g > 3$ embeddable on a surface with $\chi < 0$; his result is sharp for $g = 4$. The above argument for (i) does not work when $\chi \geq 0$, although (i) is vacuously true for $\chi > 0$ (no graphs satisfy the bound) and is true for the torus (as shown by Altshuler [A4]). It does not seem to be known whether (i) holds for the Klein bottle.

We could try to generalize Duke's result to k-walks or k-trees by using a suitable Dirac-type result. In fact, such a result exists for k-trees; Win [W12] proved that if $\delta \geq (v - 1)/k$ then a graph contains a k-tree (the special case $k = 2$ of this is a result on hamilton paths that follows from Dirac's result on hamilton cycles). We can modify Duke's argument in this way to obtain a result on k-trees. However, there is an alternative way to obtain a k-tree result for $k \geq 3$ which gives a better bound. The alternative method uses toughness-related conditions on the number of components that can be created when a set of vertices is deleted. Recall that $c(H)$ means the number of components of H.

Lemma 8.2 (Win, 1989 [W13]). *Let $k \geq 3$ be a positive integer and G a connected graph. Suppose that $c(G - S) \leq (k - 2)|S| + 2$ for every $S \subseteq V(G)$. Then G has a k-tree.*

Lemma 8.3 (Schmeichel and Bloom, 1979 [S10]). *Suppose G is an m-connected graph, $m \geq 3$, embedded on a surface of Euler characteristic χ. Then $c(G - S) \leq 2(|S| - \chi)/(m - 2)$ for all cutsets $S \subseteq V(G)$.*

Note that Schmeichel and Bloom considered only orientable surfaces, but again their argument is valid for nonorientable surfaces too.

By using Lemma 8.3 to satisfy the condition of Lemma 8.2, we arrive at the following result, which as far as we know is new.

Theorem 8.4. *Suppose that Σ is a surface of Euler characteristic $\chi < 0$. Let $m \geq 3$ and $k \geq 3$ be positive integers.*

(i) *If $m \geq 1 + \sqrt{\frac{k+2-2\chi}{k-2}}$, then every m-connected graph embeddable on Σ has a k-tree (and so, a k-walk).*

(ii) *If $m \leq 1 + \sqrt{\frac{k+3-2\chi}{k-1}}$ then $K_{m,(k-1)m+2}$ is an m-connected graph with no k-tree (and so, no $(k-1)$-walk) embeddable on Σ.*

Proof. (i) Consider a graph G satisfying (i), and let $S \subseteq V(G)$ be a cutset. The restriction on m implies that

$$m^2 - 2m - \frac{4-2\chi}{k-2} \geq 0.$$

Note that $|S| \geq m$, and if $k = 3$ then $m \geq 4$ by (ii), so that $k - 2 - \frac{2}{m-2} \geq 0$. Therefore, after some arithmetic we get

$$\frac{-2\chi}{m-2} \leq (k - 2 - \frac{2}{m-2})m + 2 \leq (k - 2 - \frac{2}{m-2})|S| + 2$$

from which it follows that

$$\frac{2(|S| - \chi)}{m-2} \leq (k-2)|S| + 2.$$

It then follows from Lemma 8.3 that the condition of Lemma 8.2 is satisfied for all cutsets S, and it is automatically satisfied when S is not a cutset, so by Lemma 8.2, G has a k-tree.

(ii) $K_{m,(k-1)m+2}$ has no k-tree by Lemma 2.2, is m-connected, and by Ringel's results [R3, R4] is embeddable in Σ if (ii) holds. ∎

Lemma 8.2 is not known to be tight, so it is possible that improvements in Theorem 8.4 (i) are possible if Win's result can be strengthened. It is also possible that (i) can be improved if we just want a k-walk, not a k-tree. A slightly stronger version of (ii) for walks can be found by using $K_{m,(k-1)m+1}$, which has no $(k-1)$-walk, instead of $K_{m,(k-1)m+2}$.

8c. Connectivity and local planarity

Now we return to the idea of working with a fixed connectivity. In this case, we know that extra conditions must be added to guarantee the existence of a k-walk or k-tree. One natural condition to add is that the graph be 'locally planar', i.e. that its edgewidth or representativity be large. Considering the result of Archdeacon, Hartsfield and Little [A5], and also Thomassen (personal communication), mentioned in subsection 8a, we see that it is not enough to impose a fixed representativity or edgewidth condition and a fixed connectivity condition. If the connectivity condition is fixed, the representativity or edgewidth condition must depend on the surface.

In fact, Thomassen [T9] has shown that even for 5-connected graphs, an edgewidth condition, even one depending on the surface, cannot guarantee the existence of a k-walk or k-tree for any k. However, his examples are not triangulations, and actually have bounded representativity for a given surface; this points out two directions in which to strengthen the conditions involved. Thomassen chose to restrict his attention to triangulations, and proved the following.

Theorem 8.5 (Thomassen, 1994 [T9])**.** *Suppose G is a triangulation of S_g with edgewidth (or representativity) at least 2^{3g+4}. Then G has a 4-tree.*

Thomassen's proof used the very important idea of a *collection of planarizing cycles* in a graph embedded on a surface, which was introduced in [T8]. The cycles in such a collection are disjoint and noncontractible, and such that if one cuts along each cycle, and glues a disc on each component of the boundary so created, one obtains a plane graph. Thomassen showed that triangulations of an orientable surface with sufficiently large edgewidth had a collection of planarizing cycles which were far apart. With this idea, the proof of Theorem 8.5 has the following outline.

Outline of proof. Given a triangulation of large edgewidth on S_g, we can find a collection of planarizing cycles, and cut along it to form a plane graph which is mostly triangulated except for some 'holes', whose boundaries represent the original cycles we cut along. We can show, by induction on the number of holes, that the plane graphs so constructed have 4-trees with special properties on the hole boundary cycles. (The induction works by cutting along a path between two holes to open them up into one larger hole.) The special properties on the hole boundary cycles allow us to undo the cutting operation and obtain a 4-tree in the original triangulation of S_g. ∎

Ellingham and Gao [E1] adapted Thomassen's argument to prove that all 4-connected triangulations of an orientable surface with sufficiently high edgewidth (or representativity) have 3-trees. Thomassen conjectured that in fact all 5-connected triangulations of a surface with sufficiently high edgewidth have a hamilton cycle. Yu [Y1] proved this conjecture, and also managed to strengthen both Theorem 8.5 and Ellingham and Gao's result in three ways: by replacing a tree result with a walk result, by proving them for nonorientable surfaces as well as for orientable, and by applying them to general graphs instead of triangulations by realizing that the local planarity condition should be stated in terms of representativity, not edgewidth. Specifically, Yu showed that on any surface, all 3-connected graphs of sufficiently high representativity have a 2-walk, and all 4-connected graphs of sufficiently high representativity have a 3-walk. The following still remains open.

Question 8.6 (Yu, 1993 [Y1])**.** *Do all 5-connected graphs of sufficiently high representativity on a given surface have a hamilton cycle? (Yu has verified this for triangulations.)*

9. Infinite graphs

It is natural to ask whether results on the existence of spanning subgraphs in finite graphs on surfaces, especially the plane, can be extended to infinite graphs. We would like to find connected spanning subgraphs where each vertex has finite degree. A graph in which every vertex has finite degree is called *locally finite*. If we want locally finite connected spanning subgraphs, we must consider *countable* graphs, those with a countable number of vertices. At a very general level, Halin [H3] showed that every 3-connected countable planar graph has a locally finite spanning tree. This does not guarantee a uniform finite bound on the degree of each vertex of the spanning subgraph, unlike the results below, which are more like the results for finite graphs.

The infinite analogue of a hamilton cycle is a *spanning* 2*-way (infinite) path*, and the analogue of a hamilton path is a *spanning* 1*-way (infinite) path*. It is also possible to investigate the existence of 1*-way* and 2*-way spanning infinite* k*-walks*, for any k. An important concept here is indivisibility: a graph is k*-indivisible* if the deletion of any finite set of vertices leaves less than k infinite components. It is not difficult to see that a graph with a 2-way spanning path or walk must be 3-indivisible, and a graph with a 1-way spanning path or walk must be 2-indivisible.

Nash-Williams [N1] conjectured that Tutte's Theorem could be generalized as follows: for $i = 1$ or 2, a countable 4-connected $(i + 1)$-indivisible planar graph has an i-way spanning path. Jung [J7, J8] verified the case $i = 1$ in the case of triangulations with a certain topological embedding property, and Dean, Thomas and Yu [D3] verified the complete conjecture for $i = 1$. Relaxing the degree and connectivity conditions, Jung [J7] has shown that a countable locally finite planar 3-connected 2-indivisible graph has a 3-tree. This has been improved by C. C. Timar (personal communication, 1995), who showed that such graphs have a spanning 1-way 2-walk. Timar also showed that a countable locally finite 3-connected 3-indivisible but not 2-indivisible planar graph has a spanning 2-way 2-walk.

Finally, Schmidt-Steup [S12] has constructed an infinite family of examples of infinite locally finite (in fact, all vertices have degree 3 or 4) hypohamiltonian planar graphs; here, *hypohamiltonian* means that the graph has no 2-way infinite spanning path, but each of its single-vertex-deleted subgraphs does have such a path.

10. Other spanning subgraphs

In this section we discuss some other spanning subgraphs which have been investigated for graphs embedded on surfaces.

10a. Figure-eight subgraphs

Using Theorem 3.4 and the existence of a nonhamiltonian 3-connected cubic planar graph, both established by Tutte, Rosenfeld investigated the existence of spanning figure-eight subgraphs in 4-connected planar graphs. Given a vertex v of a graph, a *figure-eight subgraph based at* v is a subgraph consisting of two edge-disjoint cycles which are vertex-disjoint except that both contain v. Rosenfeld [R6] showed that for every vertex in a 4-connected planar graph, there is a spanning figure-eight subgraph based at that vertex, which can be found in polynomial time. Figure-eight subgraphs do not always exist in 3-connected planar graphs or in 4-connected nonplanar graphs, so this result is best possible.

10b. 2-connected and minimum degree 2 subgraphs

As a variation on the idea of a k-tree, which may be regarded as a connected spanning subgraph of bounded degree, there has recently been some work on finding 2-connected spanning subgraphs of bounded degree in graphs on surfaces. For brevity, let us call a 2-connected spanning subgraph in which all vertices have degree at most k a *k-trestle*. Barnette [B9] showed that all 3-connected planar graphs have a 15-trestle, and gave an example of such a graph with no 5-trestle. Gao [G1] proved that all circuit and annulus graphs have a 6-trestle, from which it follows by results in [F4] and [B26] that all 3-connected graphs on the plane, projective plane, torus and Klein bottle have a 6-trestle. Sanders and Zhao [S7] showed that all 3-connected graphs on a surface of characteristic χ have a k-trestle, where

$$k = \begin{cases} 10 - 2\chi & \text{if } \chi \geq -4; \\ 8 - 2\chi & \text{if } -5 \geq \chi \geq -9; \\ 6 - 2\chi & \text{if } \chi \leq -10. \end{cases}$$

They give examples of graphs with no $(5 - 2\chi)$-trestle for $\chi \leq 0$, so these results are best possible for $\chi \leq -10$. They also show that each 4-connected graph on the plane, projective plane, torus or Klein bottle has a 3-trestle; for the plane and projective plane hamilton cycles exist, giving 2-trestles. Mohar [M6] gives an argument, based on Barnette's 15-trestle result and Yu's theorem on the existence of planarizing cycles [Y1], to show that 3-connected graphs embedded on a surface with large enough representativity have a 32-trestle; using Gao's 6-trestle result, this can be improved to a 14-trestle.

Enomoto, Iida and Ota [E2] show that every 3-connected planar graph with minimum degree at least 4 has a connected spanning subgraph in which all degrees are 2 or 3. They provide examples to show that 'connected' cannot be improved to '2-connected' and that the minimum degree condition is necessary.

10c. Eulerian subgraphs

Gao and Wormald [G4] have looked at the existence of k-walks restricted not to use any edge more than once; we may call such a walk a *k-trail*. They showed that every triangulation of a disc or an annulus has a 4-trail, or in other words, a spanning eulerian subgraph of maximum degree at most 8; it then follows from results in [F4], [D1] and [B26] that every triangulation of the plane, projective plane, torus or Klein bottle has a 4-trail. It is necessary to restrict attention to triangulations here, as nonhamiltonian 3-connected planar graphs do not have a k-trail for any k, and examples show that the result cannot be improved from 4-trails to 3-trails.

10d. Homeomorphically irreducible trees

Finally, we consider homeomorphically irreducible spanning trees, which were introduced by Hill [H6, H7]. A tree is *homeomorphically irreducible* if it has no vertex of degree 2; we abbreviate 'homeomorphically irreducible spanning tree' to *HIST*. HISTs do not fall within the stated goal of this survey, which is to examine spanning subgraphs of bounded degree, but they are related to Halin graphs, and 'restricted degree' is not too far from 'bounded degree'.

Hill [H6] used HISTs in Halin graphs to produce planar graphs with trivial automorphism groups. He conjectured that all plane triangulations have a HIST, and asked whether there were non-self-dual 3-connected planar graphs such that both the graph and its dual had no HIST. Malkevitch [M2] gave necessary conditions for a 3-connected cubic planar graph to have a HIST, used this to give examples of 3-connected cubic planar graphs with no HIST, mentioned some results by his student Joffe (see below), and made several conjectures, of which the following most appeals to us.

Conjecture 10.1 (Malkevitch, 1979 [M2]). *All 4-connected planar graphs have a HIST.*

This conjecture appears to still be open. Joffe [J4] found 3-connected 4-regular planar graphs without HISTs, but they were not 4-connected. He answered one of Hill's questions by finding an infinite family of 3-connected planar graphs for which neither they nor their duals had HISTs.

Joffe showed that determining whether a non-cubic 3-connected planar graph has a HIST is NP-complete; Lemke [L2] showed this for (not necessarily planar) graphs of maximum degree 3 and Douglas [D11] showed this for planar graphs of maximum degree 3.

Results on the existence of HISTs for graphs on surfaces were obtained by Albertson et al. [A2], who showed that plane near-triangulations have HISTs, verifying Hill's conjecture about plane triangulations [H6] and a similar conjecture by Malkevitch [M2]. An extended version of Conjecture 5.4 would, if true, strengthen this result. Based on Albertson et al.'s result, Fiedler et al. observed [F4] that triangulations of the projective plane also have HISTs. Davidow et al. [D1] showed that triangulated annuli have HISTs, and as a consequence triangulations of the torus have HISTs; it follows from [B26] that triangulations of the Klein bottle also have HISTs.

We may make the following series of successively weaker conjectures; (i) and (ii) are due to Albertson et al. [A2].

Conjecture 10.2. *The following have HISTs:*
(i) *all graphs in which every edge is in at least two triangles; or*
(ii) *all triangulations of any surface; or*
(iii) *all triangulations of a given surface with sufficiently large representativity.*

References

The sections or subsections where a reference is cited are given at the end of the reference.

A1. Takanori Akiyama, Takao Nishizeki and Nobuji Saito, NP-completeness of the Hamiltonian cycle problem for bipartite graphs, *J. Inform. Process.* **3** No. 2 (1980) 73-76. MR 82a:68119. §4.

A2. Michael O. Albertson, David M. Berman, Joan P. Hutchinson and Carsten Thomassen, Graphs with homeomorphically irreducible spanning trees, *J. Graph Theory* **14** No. 2 (1990) 247-258. MR 91b:05055. §10d.

A3. B. Alspach and Cun Quan Zhang, Hamilton cycles in cubic Cayley graphs on dihedral groups, *Ars Combin.* **28** (1989) 101-108. MR 91b:05124. §3e.

A4. Amos Altshuler, Hamiltonian circuits in some maps on the torus, *Discrete Math.* **1** No. 4 (1972) 299-314. MR 45#6651. §3e, 8b.

A5. Dan Archdeacon, Nora Hartsfield and C. H. C. Little, Nonhamiltonian triangulations with large connectivity and representativity, preprint (1994). §8a, 8c.

A6. Takao Asano, Shunji Kikuchi and Nobuji Saito, A linear algorithm for finding Hamiltonian cycles in 4-connected maximal planar graphs, *Discrete Appl. Math.* **7** No. 1 (1984) 1-15. MR 85e:05116. §4.

A7. Takao Asano, Nobuji Saito, Geoffrey Exoo and Frank Harary, The smallest 2-connected cubic bipartite planar non-Hamiltonian graph, *Discrete Math.* **38** No. 1 (1982) 1-6. MR 83k:05043. §5b.

B1. M. L. Balinski, On the graph structure of convex polyhedra in n-space, *Pacific J. Math.* **11** (1961) 431-434. MR 23A#4059. §5a.

B2. C. A. Barefoot, Hamiltonian connectivity of the Halin graphs, Eighteenth Southeastern International Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, Fla., 1987), *Congr. Numer.* **58** (1987) 93-102. MR 89d:05123. §5d.

B3. C. A. Barefoot, 4-connected Halin graphs are 2-Hamiltonian connected, *Ars Combin.* **35 A** (1993) 109-128. MR 95a:05067. §5d.

B4. David Barnette, Trees in polyhedral graphs, *Canad. J. Math.* **18** (1966) 731-736. MR 33#3951. §1a, 7.

B5. David Barnette, Path problems and extremal problems for convex polytopes, in "Relations between combinatorics and other parts of mathematics," (Proc. Sympos. Pure Math., Ohio State Univ., Columbus, Ohio, 1978), *Proc. Sympos. Pure Math.* **34**, Amer. Math. Soc., Providence, R.I. (1979) 25-34. MR 80b:52011. §5b.

B6. David W. Barnette, Every simple 3-polytope with 34 vertices is Hamiltonian, *Discrete Math.* **62** No. 1 (1986) 1-20. MR 87k:05113. §5a.

B7. D. W. Barnette, Decomposition theorems for the torus, projective plane and Klein bottle, *Discrete Math.* **70** No. 1 (1988) 1-16. MR 89f:05065. §3e, 7.

B8. D. W. Barnette, 3-trees in polyhedral maps, *Israel J. Math.* **79** (1992) 251-256. MR 94j:05045. §7.

B9. D. W. Barnette, 2-connected spanning subgraphs of planar 3-connected graphs, *J. Combin. Theory Ser. B* **61** No. 2 (1994) 210-216. §10b.

B10. David Barnette and Ernest Jucovič, Hamiltonian circuits on 3-polytopes, *J. Combin. Theory* **9** (1970) 54-59. MR 42#4437, MR 42#4437. §3b.

B11. David Barnette and Gerd Wegner, Hamiltonian circuits in simple 3-polytopes with up to 26 vertices, *Israel J. Math.* **19** (1974) 212-216. MR 51#6584. §5a.

B12. Ilker Baybars, On k-path Hamiltonian maximal planar graphs, *Discrete Math.* **40** No. 1 (1982) 119-121. MR 83k:05072. §3d.

B13. Claude Berge, "Graphs," second revised edition, *North Holland Mathematical Library* **6** Part 1, North-Holland, Amsterdam (1985). §1a.

B14. J.-C. Bermond, Hamiltonian graphs, in "Selected Topics in Graph Theory," edited by L. W. Beineke and R. J. Wilson, Academic Press, London (1978) Ch. 6, 127-167. §1a.

B15. Norman L. Biggs, E. Keith Lloyd and Robin J. Wilson, "Graph Theory 1739-1936," Clarendon Press, Oxford (1986). §5a.

B16. J. A. Bondy, Pancyclic graphs: recent results, in "Infinite and finite sets, Vol. I," (Colloq., Keszthely, 1973; dedicated to P. Erdos on his 60th birthday), *Colloq. Math. Soc. Janos Bolyai* **10**, North-Holland, Amsterdam (1975) 181-187. MR 51#10157. §5d.

B17. J. A. Bondy and R. Häggkvist, Edge-disjoint Hamilton cycles in 4-regular planar graphs, *Aequationes Math.* **22** No. 1 (1981) 42-45. MR 83a:05090. §6a.

B18. J. A. Bondy and L. Lovász, Lengths of cycles in Halin graphs, *J. Graph Theory* **9** No. 3 (1985) 397-410. MR 87a:05092. §5d.

B19. J. A. Bondy and U. S. R. Murty, "Graph Theory with Applications," Elsevier, New York (1976). §1b.

B20. J. Bosák, Hamiltonian lines in cubic graphs, in "Theory of graphs," Proc. Int. Seminar on Graph Theory and its Applications (Rome, 1966), Gordon and Breach, New York (1967) 35-46. MR 36#5022. §5a.

B21. I. Z. Bouwer and W. W. Chernoff, The toroidal graphs $\{6,3\}_{b,c}$ are Hamiltonian, Eleventh British Combinatorial Conference (London, 1987), *Ars Combin.* **25 A** (1988) 179-186. MR 89d:05126. §3e.

B22. H. R. Brahana, Systems of circuits of two-dimensional manifolds, *Ann. Math.* **30** (1923) 234-243. §1c.

B23. T. A. Brown, Simple paths on convex polyhedra, *Pacific J. Math.* **11** (1961) 1211-1214. MR 25#3864. §5b.

B24. T. A. Brown, Maximal simple paths on convex polyhedra, Abstract 61T-1, *Notices Amer. Math. Soc.* **8** (1961) 53. §5b.

B25. R. Brunet and R.B. Richter, Hamiltonicity of 5-connected toroidal triangulations, to appear, *J. Graph Theory*. §3d, 3e.

B26. Richard Brunet, Mark N. Ellingham, Zhicheng Gao, Alice Metzlar and R. Bruce Richter, Spanning planar subgraphs of graphs in the torus and Klein bottle, to appear, *J. Combin. Theory Ser. B*. §7, 10b, 10c, 10d.

B27. Jean W. Butler, Hamiltonian circuits on simple 3-polytopes, *J. Combinatorial Theory Ser. B* **15** (1973) 69-73. MR 48#3796. §5a.

B28. Jean W. Butler, Non-Hamiltonian simple 3-polytopes, in "Recent advances in graph theory," (Proc. Second Czechoslovak Sympos., Prague, 1974), Academia, Prague (1975) 135-151. MR 52#7964. §5a.

C1. Michael Capobianco and John C. Molluzzo, "Examples and Counterexamples in Graph Theory," North-Holland, New York (1978). MR 58#10536. §1a.

C2. Norishige Chiba and Takao Nishizeki, A theorem on paths in planar graphs, *J. Graph Theory* **10** No. 4 (1986) 449-450. MR 87k:05103. §3a, 3c.

C3. Norishige Chiba and Takao Nishizeki, The Hamiltonian cycle problem is linear-time solvable for 4-connected planar graphs, *J. Algorithms* **10** No. 2 (1989) 187-211. MR 90c:05124. §4.

C4. Kiran B. Chilakamarri, Peter Hamburger and Raymond E. Pippert, Hamilton cycles in planar graphs and Venn diagrams, preprint (1995). §3a.

C5. J. Chuard, Les reseaux cubiques et le problème des quarte couleurs, *Mem. Soc. Vaudoise Sci. Nat.* **25** No. 4 (1932) 41-101. §5a.

C6. V. Chvátal, Tough graphs and hamiltonian circuits, *Discrete Math.* **5** (1973) 215-228. MR 47#4849. §2a.

C7. V. Chvátal, Hamiltonian cycles, in "The Traveling Salesman Problem: A Guided Tour of Combinatorial Optimization," edited by E. L. Lawler, J. K. Lenstra, A. H. G. Rinnooy Kan, and D. B. Shmoys, Wiley, Chichester (1985). §4.

C8. Robert J. Cimikowski, Finding Hamiltonian cycles in certain planar graphs, *Inform. Process. Lett.* **35** No. 5 (1990) 249-254. MR 91h:68145. §5c.

C9. Robert J. Cimikowski, On certain Hamiltonian inner triangulations, *Discrete Appl. Math.* **43** No. 3 (1993) 185-195. MR 94m:05122. §5c.

C10. G. Cornuejols, D. Naddef and W. R. Pulleyblank, Halin graphs and the travelling salesman problem, *Math. Programming* **26** No. 3 (1983) 287-294. MR 84m:90045. §5d.

C11. H. S. M. Coxeter, Problem E 711: Sir William Hamilton's Icosian Game, *Amer. Math. Monthly* **53** (1946) 156 (solution on 593). §3b.

C12. H. S. M. Coxeter, "Regular polytopes," 2nd edition, London (1963). §3b.

D1. Amy L. Davidow, Joan P. Hutchinson and J. Philip Huneke, Planar and toroidal graphs with homeomorphically irreducible spanning treess, preprint (1992). §10c, 10d.

D2. Nathaniel Dean and Katsuhiro Ota, 2-factors, connectivity and graph minors, in "Graph Structure Theory," ed. Neil Robertson and Paul Seymour, Proc. Conf. on Graph Minors, Univ. of Washington, Seattle, 1991, *Contemporary Mathematics* **147**, American Math. Soc., Providence, Rhode Island (1993) 381-386. §3d, 3e.

D3. Nathaniel Dean, Robin Thomas and Xingxing Yu, Spanning paths in infinite planar graphs, preprint (1994). §9.

D4. Michael B. Dillencourt, A non-Hamiltonian, nondegenerate Delaunay triangulation, *Inform. Process. Lett.* **25** No. 3 (1987) 149-151. MR 88g:68101. §5c.

D5. Michael B. Dillencourt, An upper bound on the shortness exponent of inscribable polytopes, *J. Combin. Theory Ser. B* **46** No. 1 (1989) 66-83. MR 90k:05066. §5c.

D6. Michael B. Dillencourt, Hamiltonian cycles in planar triangulations with no separating triangles, *J. Graph Theory* **14** No. 1 (1990) 31-49. MR 91c:05117. §3b, 3d.

D7. M. B. Dillencourt, An upper bound on the shortness exponent of 1-tough, maximal planar graphs, *Discrete Math.* **90** (1991) 93-97. §3b.

D8. Michael B. Dillencourt, Polyhedra of small order and their hamiltonian properties, *J. Combin. Theory Ser. B* **66** (1996) 87-122. §3b, 5c.

D9. Michael B. Dillencourt, Finding hamiltonian cycles in Delaunay triangulations is NP-complete, to appear, *Discrete Appl. Math.* §5c.

D10. G. A. Dirac, Some theorems on abstract graphs, *Proc. London Math. Soc. (3)* **2** (1952) 69-81. MR 13,856e. §8b.

D11. Robert James Douglas, NP-completeness and degree restricted spanning trees, *Discrete Math.* **105** No. 1-3 (1992) 41-47. MR 93i:68085. §7, 10d.

D12. Richard A. Duke, On the genus and connectivity of hamiltonian graphs, *Discrete Math.* **2** No. 3 (1972) 199-206. MR 47#3221. §8b.

E1. M. N. Ellingham and Zhicheng Gao, Spanning trees in locally planar triangulations, *J. Combin. Theory Ser. B* **61** No. 2 (1994) 178-198. §8c.

E2. Hikoe Enomoto, Tadashi Iida and Katsuhiro Ota, Connected spanning subgraphs of 3-connected planar graphs, preprint (1995). §10b.

E3. E. Etourneau, Existence and connectivity of planar graphs having 12 vertices of degree 5 and $n-12$ vertices of degree 6, in "Infinite and finite sets, Vol. II," Proc. Colloq. (Keszthely, 1973), *Colloq. Math. Soc. János Bolyai* **10**, North-Holland, Amsterdam (1975) 645-655. MR 52#158. §5b.

E4. Günter Ewald, On shortness exponents of families of graphs, *Israel J. Math.* **16** (1973) 53-61. MR 49#8892. §5b.

E5. Günter Ewald, Hamiltonian circuits in simplicial complexes, *Geometriae Dedicata* **2** No. 1 (1973) 115-125. MR 47#9424. §3e, 5b.

E6. G. Ewald, Review 80b:52011, *Math. Reviews* **80b** (1980). §5b.

E7. Geoffrey Exoo and Frank Harary, The smallest cubic multigraphs with prescribed bipartite, block, Hamiltonian and planar properties, *Indian J. Pure Appl. Math.* **13** No. 3 (1982) 309-312. MR 84h:05075. §5a.

E8. Geoffrey Exoo and Frank Harary, Smallest cubic graphs with prescribed properties, *Southeast Asian Bull. Math.* **8** No. 1 (1984) 1-4. MR 86k:05094. §5a.

F1. S. Fanelli and A. Ghiraldini, On Hamiltonian maximal planar graphs, *Riv. Mat. Univ. Parma (4)* **13** (1987) 355-366. MR 90i:05063. §5b.

F2. G. B. Faulkner and D. H. Younger, Non-Hamiltonian cubic planar maps, *Discrete Math.* **7** (1974) 67-74. MR 48#10898. §5a.

F3. M. Fellows, F. Hickling and M. Sysło, A topological parameterization and hard graph problems (extended abstract), Eighteenth Southeastern International Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, Fla., 1987), *Congr. Numer.* **59** (1987) 69-78. MR 89i:05102. §4.

F4. J. R. Fiedler, J. P. Huneke, R. B. Richter and N. Robertson, Computing the orientable genus of projective planar graphs, to appear, *J. Graph Theory.* §7, 10b, 10c, 10d.

F5. Martin Fürer and Balaji Raghavachari, Approximating the minimum degree spanning tree to within one from the optimal degree, in "Proceedings of the Third Annual ACM-SIAM Symposium on Discrete Algorithms," (Orlando, FL, 1992), ACM, New York (1992) 317-324. §7.

G1. Zhicheng Gao, 2-connected coverings of bounded degree in 3-connected graphs, to appear, *J. Graph Theory*. §10b.

G2. Zhicheng Gao and R. Bruce Richter, 2-walks in circuit graphs, *J. Combin. Theory Ser. B* **62** No. 2 (1994) 259-267. §7.

G3. Zhicheng Gao, R. Bruce Richter and Xingxing Yu, 2-walks in 3-connected planar graphs, to appear, *Australas. J. Combin.* §7.

G4. Zhicheng Gao and Nicholas C. Wormald, Spanning Eulerian subgraphs of bounded degree in triangulations, *Graphs Combin.* **10** No. 2 (1994) 123-131. §10c.

G5. M. R. Garey, D. S. Johnson and R. Endre Tarjan, The planar Hamiltonian circuit problem is NP-complete, *SIAM J. Comput.* **5** No. 4 (1976) 704-714. MR 56#2867. §4.

G6. P. R. Goodey, Hamiltonian circuits on simple 3-polytopes, *J. London Math. Soc. (2)* **5** (1972) 504-510. MR 47#4850. §5a.

G7. P. R. Goodey, Hamiltonian paths on 3-polytopes, *J. Combinatorial Theory Ser. B* **12** (1972) 143-152. MR 45#5866. §3b.

G8. P. R. Goodey, Hamiltonian circuits in polytopes with even sided faces, *Israel J. Math.* **22** No. 1 (1975) 52-56. MR 53#14314. §5b.

G9. P. R. Goodey, A class of hamiltonian polytopes, *J. Graph Theory* **1** No. 2 (1977) 181-185. MR 57#1282. §5b.

G10. D. Gouyou-Beauchamps, The Hamiltonian circuit problem is polynomial for 4-connected planar graphs, *SIAM J. Comput.* **11** No. 3 (1982) 529-539. MR 84c:68029. §4.

G11. E. Ja. Grinberg, Plane homogeneous graphs of degree three without hamiltonian circuits, *Latvian Math. Yearbook* **4**, Izdat. 'Zinatne', Riga (1968) 51-58. MR 39#96. §2b, 5a.

G12. Branko Grünbaum, Polytopes, graphs and complexes, *Bull. Amer. Math. Soc.* **76** (1970) 1131-1201. MR 42#959. §1a, 3d, 3e, 5a, 5b.

G13. Branko Grünbaum, Polytopal graphs, in "Studies in Graph Theory, Part II," edited by D. R. Fulkerson, *Studies in Mathematics* **12**, Math. Assoc. of America (1975) 201-224. MR 53#10654. §1a.

G14. Branko Grünbaum and Joseph Malkevitch, Pairs of edge-disjoint Hamiltonian circuits, *Aequationes Math.* **14** No. 1/2 (1976) 191-196. MR 54#2544b. §6a.

G15. B. Grünbaum and T. S. Motzkin, Longest simple paths in polyhedral graphs, *J. London Math. Soc.* **37** (1962) 152-160. MR 25#2598. §5a.

G16. Branko Grünbaum and Hansjoachim Walther, Shortness exponents of families of graphs, *J. Combin. Theory Ser. A* **14** (1973) 364-385. MR 47#3242. §1a, 5, 5a, 5b.

G17. Branko Grünbaum and Joseph Zaks, The existence of certain planar maps, *Discrete Math.* **10** (1974) 93-115. MR 50#1949. §5b.

H1. S. L. Hakimi, E. F. Schmeichel and C. Thomassen, On the number of Hamiltonian cycles in a maximal planar graph, *J. Graph Theory* **3** No. 4 (1979) 365-370. MR 80k:05075. §6a.

H2. R. Halin, Studies on minimally n-connected graphs, in "Combinatorial Mathematics and its Applications," (Proc. Conf., Oxford, 1969), Academic Press, London (1971) 129-136. MR 43#4706. §5d.

H3. R. Halin, Some problems and results on infinite graphs, in "Graph Theory in Memory of G. A. Dirac," ed. L. Andersen et al., Proc. Conf. at Sandbjerg, Denmark, 1985, *Ann. Discrete Math.* **41**, North-Holland, Amsterdam (1989) 195-210. §9.

H4. Jochen Harant and Hansjoachim Walther, Some new results about the shortness exponent in polyhedral graphs, *Casopis Pest. Mat.* **112** No. 2 (1987) 114-122. MR 88f:05064. §5b.

H5. Wolfgang Hatzel, Ein planarer hypohamiltonscher Graph mit 57 Knoten, *Math. Ann.* **243** No. 3 (1979) 213-216. MR 80m:05072. §6b.

H6. Anthony Hill, Graphs with homeomorphically irreducible spanning trees, in "Combinatorics," (Proc. British Combinatorial Conf., Univ. Coll. Wales, Aberystwyth, 1973), *London Math. Soc. Lecture Note Ser.* **13**, Cambridge Univ. Press, London (1974) 61-68. MR 50#4359. §5d, 10d.

H7. Anthony Hill, Some topics in 3-polyhedral graphs, in "Recent advances in graph theory," (Proc. Second Czechoslovak Sympos., Prague, 1974), Academia, Prague (1975) 263-266. MR 52#2934. §5d, 10d.

H8. Anthony Hill and David Singmaster, Labelled and unlabelled Hamiltonian circuits in a class of 3-polytopes, in "Proceedings of the Fifth British Combinatorial Conference," (Univ. Aberdeen, Aberdeen, 1975), *Congressus Numer.* **15**, Utilitas Math., Winnipeg, Man. (1976) 293-305. MR 53#1410. §5d.

H9. D. A. Holton, B. Manvel and B. D. McKay, Hamiltonian cycles in cubic 3-connected bipartite planar graphs, *J. Combin. Theory Ser. B* **38** No. 3 (1985) 279-297. MR 86j:05097. §5b.

H10. D. A. Holton and B. D. McKay, The smallest non-Hamiltonian 3-connected cubic planar graphs have 38 vertices, *J. Combin. Theory Ser. B* **45** No. 3 (1988) 305-319. MR 90f:05090. §5a.

H11. D. A. Holton and B. D. McKay, Erratum: The smallest non-Hamiltonian 3-connected cubic planar graphs have 38 vertices (*J. Combin. Theory Ser. B* **45** No. 3 (1988) 305-319), *J. Combin. Theory Ser. B* **47** No. 2 (1989) 248. MR 91a:05067. §5a.

H12. Hong Yuan, On the spectral radius and genus of graphs, to appear, *J. Combin. Theory Ser. B.* §7.

H13. H. F. Hunter, "On non-hamiltonian maps and their duals," Ph. D. thesis, Rensselaer Polytechnic Inst. (1962). §5a.

J1. Bill Jackson and Nicholas C. Wormald, k-walks of graphs, *Australas. J. Combin.* **2** (1990) 135-146. MR 92g:05128. §1b, 2a, 7.

J2. Stanislav Jendrol' and Roman Kekeňák, Longest circuits in triangular and quadrangular 3-polytopes with two types of edges, *Math. Slovaca* **40** No. 4 (1990) 341-357. MR 92m:05113. §5b.

J3. Stanislav Jendrol' and Peter Mihók, On a class of Hamiltonian polytopes, *Discrete Math.* **71** No. 3 (1988) 233-241. MR 90c:05137. §5b.

J4. Peter Melvyn Joffe, "Some properties of 3-polytopal graphs," Ph.D. dissertation, CUNY (1982). *Diss. Abs. Internat.* 43-09, 2925-B (March 1983). §10d.

J5. Ernest Jucovič, A note on paths in quadrangular polyhedral graphs, *Časopis Pěst Mat.* **93** (1968) 69-72. MR 39#2658. §5b.

J6. Ernest Jucovič and Hansjoachim Walther, Über längste Kreise in flächenregulären Polyedergraphen, *Mat. Časopis Sloven. Akad. Vied* **23** (1973) 164-169. MR 49#138. §5b.

J7. Hwan-Ok Jung, "3-Gerüste und hamiltonsche Wege in unendlichen planaren Graphen," doctoral dissertation, Universität Hamburg (1988). §9.

J8. Jung Hwan Ok, Hamiltonian paths in infinite strong triangulations, *J. Korean Math. Soc.* **30** No. 2 (1993) 275-284. MR 94e:05176. §9.

K1. Vitit Kantabutra, Traveling salesman cycles are not always subgraphs of Voronoï duals, *Inform. Process. Lett.* **16** (1983) 11-12. §5c.

K2. A. K. Kelmans, Constructions of cubic bipartite 3-connected graphs without hamiltonian cycles, in "Selected topics in discrete mathematics (Moscow, 1972-1990)," *Amer. Math. Soc. Transl. Ser. 2* **158**, Amer. Math. Soc, Providence, Rhode Island (1994) 127-140. MR 95d:05083. §5b.

K3. T. P. Kirkman, On the enumeration of x-edra having triedral summits, and an $(x-1)$-gonal base, *Phil. Trans. Royal Soc. London* **146** (1856) 399-411. §5d.

K4. Victor Klee, Longest paths and circuits on polytopes, in "Convex polytopes," by Branko Grünbaum with the cooperation of Victor Klee, M. A. Perles and G. C. Shephard, Wiley Interscience, London (1967) Ch. 17, 356-389. §1a, 5a.

K5. Jan Kratochvíl and Dainis Zeps, On the number of Hamiltonian cycles in triangulations, *J. Graph Theory* **12** No. 2 (1988) 191-194. MR 89g:05074. §6a.

L1. Joshua Lederberg, Hamilton circuits of convex trivalent polyhedra (up to 18 vertices), *Amer. Math. Monthly* **74** No. 5 (1967) 522-527. MR 35#2770. §5a.

L2. P. Lemke, The maximum leaf spanning tree problem for cubic graphs is NP-complete, *IMA Preprint Series* **No. 428** (1988). §10d.

M1. Joseph Malkevitch, Cycle lengths in polytopal graphs, in "Theory and applications of graphs," (Proc. Internat. Conf., Western Mich. Univ., Kalamazoo, Mich., 1976), *Lecture Notes in Math.* **642**, Springer, Berlin (1978) 364-370. MR 58#10550. §5d.

M2. Joseph Malkevitch, Spanning trees in polytopal graphs, in "Second International Conference on Combinatorial Mathematics," (New York, 1978), *Ann. New York Acad. Sci.* **319**, New York Acad. Sci., New York (1979) 362-367. MR 81h:05062. §10d.

M3. Joseph Malkevitch, Polytopal graphs, in "Selected topics in graph theory, 3," Academic Press, San Diego, CA (1988) 169-188. §1a, 3b, 5b.

M4. Pierre Martin, Cycles hamiltoniens dans les graphes 4-réguliers 4-connexes (Hamiltonian cycles in 4-regular 4-connected graphs), *Aequationes Math.* **14** (1976) 37-40. MR 54#2544a. §6a.

M5. Bojan Mohar, Search for minimal non-Hamiltonian simple 3-polytopes, in "Graph theory," (Novi Sad, 1983), Univ. Novi Sad, Novi Sad (1984) 191-208. MR 85g:05098. §5a.

M6. Bojan Mohar, Face-width of embedded graphs, preprint (1995). §1a, 1c, 10b.

M7. John C. Molluzzo, Some Hamiltonian counterexamples, in "Topics in graph theory," (New York, 1977), *Ann. New York Acad. Sci.* **328**, New York Acad. Sci., New York (1979) 157-165. MR 81k:05072. §1a.

M8. J. W. Moon and L. Moser, Simple paths on polyhedra, *Pacific J. Math.* **13** (1963) 629-631. MR 27#4225. §5b.

N1. C. St. J. A. Nash-Williams, Hamiltonian lines in infinite graphs with few vertices of small valency, *Aequationes Math.* **7** (1971) 59-81. MR 46#94. §9.

N2. C. St. J. A. Nash-Williams, Unexplored and semi-explored territories in graph theory, in "New Directions in the Theory of Graphs," edited by Frank Harary, (Proc. Third Ann Arbor Conf., Univ. Michigan, Ann Arbor, Mich., 1971), Academic Press, New York (1973) 149-186. MR 52#7944. §3e.

N3. C. St. J. A. Nash-Williams, Hamiltonian circuits, in "Studies in Graph Theory, Part II," edited by D. R. Fulkerson, *Studies in Mathematics* **12**, Math. Assoc. of America (1975) 301-360. MR 53#10649. §1a.

N4. Ja. Ninčak, Estimation of the number of Hamiltonian cycles in cubic multigraphs, *Vesci Akad. Navuk BSSR Ser. Fiz.-Mat. Navuk* **1975** No. 2 (1975) 20-22, 138. MR 53#2740. §6a.

N5. Takao Nishizeki, A 1-tough non-Hamiltonian maximal planar graph, *Discrete Math.* **30** No. 3 (1980) 305-307. MR 81i:05099. §3b.

N6. T. Nishizeki and N. Chiba, "Planar Graphs: Theory and Algorithms," *Ann. Discrete Math.* **32**, North-Holland, Amsterdam (1988). §4.

O1. J. O'Rourke, The computational geometry column, *SIGACT News* **18** No. 1 (1986) 17-19. §5c.

O2. Haruko Okamura, Hamiltonian circuits on simple 3-polytopes with up to 30 vertices, *J. Math. Soc. Japan* **34** No. 2 (1982) 365-369. MR 83e:05080a. §5a.

O3. Haruko Okamura, Every simple 3-polytope of order 32 or less is Hamiltonian, *J. Graph Theory* **6** No. 2 (1982) 185-196. MR 83e:05080b. §5a.

O4. Oystein Ore, "The Four-Color Problem," *Pure and Applied Mathematics* **27**, Academic Press, New York (1967). MR 36#74. §3b.

O5. P. J. Owens, On regular graphs and Hamiltonian circuits, including answers to some questions of Joseph Zaks, *J. Combin. Theory Ser. B* **28** No. 3 (1980) 262-277. MR 81j:05075. §5a, 6a.

O6. P. J. Owens, Non-Hamiltonian simple 3-polytopes whose faces are all 5-gons or 7-gons, *Discrete Math.* **36** No. 2 (1981) 227-230. MR 83m:05056. §5b.

O7. P. J. Owens, Cyclically 5-edge-connected cubic planar graphs and shortness coefficients, *J. Graph Theory* **6** No. 4 (1982) 473-479. MR 84d:05077. §5b.

O8. P. J. Owens, Shortness parameters of families of regular planar graphs with two or three types of face, *Discrete Math.* **39 no. 2** (1982) 199-209. MR 84c:05063. §5b.

O9. P. J. Owens, Non-Hamiltonian simple 3-polytopes with only one type of face besides triangles, in "Convexity and graph theory," (Jerusalem, 1981), *North-Holland Math. Stud.* **87**, North-Holland, Amsterdam-New York (1984) 241-251. MR 86j:05092. §5b.

O10. P. J. Owens, Regular planar graphs with faces of only two types and shortness parameters, *J. Graph Theory* **8** No. 2 (1984) 253-275. MR 85j:05017. §5b.

O11. P. J. Owens, Non-Hamiltonian 3-polytopes whose faces are all pentagons, in "Cycles in graphs," (Burnaby, B.C., 1982), *North-Holland Math. Stud.* **115**, North-Holland, Amsterdam-New York (1985) 151-158. MR 87g:05141b. §5b.

O12. P. J. Owens, Shortness parameters for planar graphs with faces of only one type, *J. Graph Theory* **9** No. 3 (1985) 381-395. MR 87g:05141a. §5b.

O13. P. J. Owens, Simple 3-polytopal graphs with edges of only two types and shortness coefficients, *Discrete Math.* **59** No. 1-2 (1986) 107-114. MR 87i:05088. §5b.
P1. E. Pannwitz, Review, *Jahrbuch über die Fortschritte der Math.* **58** (1932) 1204. §5a.
P2. Douglas Lee Peterson, "Hamiltonian cycles in bipartite plane cubic maps," Ph.D. thesis, Texas A&M University (1977). *Diss. Abs. Internat.* 38-04, 1753-B. §5b.
P3. D. L. Peterson, A note on Hamiltonian cycles in bipartite plane cubic maps having connectivity 2, *Discrete Math.* **36** No. 3 (1981) 327-332. MR 83m:05085. §5b.
P4. Ján Plesník, The NP-completeness of the Hamiltonian cycle problem in bipartite cubic planar graphs, *Acta Math. Univ. Comenian.* **42/43** (1983) 271-273. MR 85d:68036. §4.
P5. M. D. Plummer, Problems, in "Infinite and Finite Sets, Vol. III," *Colloq. Math. Soc. János Bolyai* **10**, North-Holland, Amsterdam (1975) 1549-1550. §3b, 3c, 3d, 5d.
P6. W. L. Price, An algorithm which generates the Hamiltonian circuits of a cubic planar map, *J. London Math. Soc. (2)* **18** No. 2 (1978) 193-201. MR 80f:05060. §6a.
R1. C. N. Reynolds, Jr., Circuits upon polyhedra, *Ann. Math.* **33** (1932) 367-372. §3a.
R2. L. Bruce Richmond, R. W. Robinson and N. C. Wormald, On Hamilton cycles in 3-connected cubic maps, in "Cycles in graphs," (Burnaby, B.C., 1982), *North-Holland Math. Stud.* **115**, North-Holland, Amsterdam-New York (1985) 141-149. MR 87d:05115. §5a.
R3. Gerhard Ringel, Das Geschlecht des vollständigen paaren Graphen (The genus of the complete bipartite graph), *Abh. Math. Sem. Univ. Hamburg* **28** (1965) 139-150. MR 32#6439. §8b.
R4. Gerhard Ringel, Der vollständige paare Graph auf nichtorientbaren Flächen (The complete bipartite graph on nonorientable surfaces), *J. Reine Angew. Math.* **220** (1965) 88-93. MR 32#445. §8b.
R5. Neil Robertson and Richard Vitray, Representativity of surface embeddings, in "Paths, Flows and VLSI-Layout," (Bonn, 1988), edited by B. Korte, L. Lovász and A. Schrijver, *Algorithms and Combinatorics* **9**, Springer, Berlin (1990). MR 91m:05072. §1c.
R6. Moshe Rosenfeld, On spanning subgraphs of 4-connected planar graphs, *Discrete Appl. Math.* **25** No. 3 (1989) 279-287. MR 90m:05051. §6a, 10a.
R7. Moshe Rosenfeld, Pairs of edge disjoint Hamiltonian circuits in 5-connected planar graphs, *Aequationes Math.* **38** No. 1 (1989) 50-55. MR 90g:05129. §6a.
R8. M. Rosenfeld, Addendum: Pairs of edge disjoint Hamiltonian circuits in 5-connected planar graphs (*Aequationes Math.* **38** No. 1 (1989) 50-55), *Aequationes Math.* **40** No. 1 (1990) 109. MR 91b:05129. §6a.
S1. Thomas L. Saaty and Paul C. Kainen, "The Four-Color Problem: Assaults and Conquest," Dover, New York (1986). §1a.
S2. H. Sachs, Construction of non-hamiltonian planar regular graphs of degrees 3, 4, 5 with highest possible connectivity, in "Theory of graphs," (Internat. Sympos., Rome, 1966), Gordon and Breach, New York (1967) 373-382. MR 36#3671. §5a.
S3. H. Sachs, Ein von Kozyrev und Grinberg angegebener nicht-Hamiltonischer kubischer planarer Graph, in "Beitrage zur Graphentheorie," Internat. Kolloq. Manebach, 1967, Teubner, Leipzig (1968) 127-130. MR 39#5403. §2b, 5a.
S4. H. Sachs, "Einführung in die Theorie der endlichen Graphen II," Teubner, Leipzig (1972). §3b.
S5. Daniel P. Sanders, On hamiltonian cycles in certain planar graphs, to appear, *J. Graph Theory.* §3d.
S6. Daniel P. Sanders, On paths in planar graphs, preprint, Georgia Institute of Technology (1992). §3d, 3e.
S7. Daniel P. Sanders and Yue Zhao, On 2-connected spanning subgraphs with low maximum degree, preprint (1995). §10b.
S8. V. I. Sarvanov, A class of Hamiltonian planar graphs, *Dokl. Akad. Nauk BSSR* **24** No. 9 (1980) 792-794, 860. MR 82j:05082. §5b.
S9. V. I. Sarvanov, Sufficient conditions for traceability of planar 3-connected graphs, *Vestsi Akad. Navuk BSSR Ser. Fiz. Mat. Navuk* **1984** No. 3 (1984) 11-14. MR 85h:05067. §5d.
S10. Edward F. Schmeichel and Gary S. Bloom, Connectivity, genus, and the number of components in vertex-deleted subgraphs, *J. Combin. Theory Ser. B* **27** No. 2 (1979) 198-201. MR 81a:05084. §8b.

S11. Monika Schmidt and Tudor Zamfirescu, Shortness exponents for polytopes which are k-gonal modulo n, *J. Combin. Theory Ser. B* **33** No. 2 (1982) 101-120. MR 84b:52011. §5b.
S12. Monika Schmidt-Steup, Infinite locally finite hypo-Hamiltonian graphs, *Math. Scand.* **58** No. 1 (1986) 139-148. MR 87i:05138. §9.
S13. F. Schoblik, Zum Problem des Kartenfärbens, *Jahresbericht Deutsch. Math. Verein.* **39** (1930) 51-52. §5a.
S14. M. I. Shamos, "Computational Geometry," Ph.D. dissertation, Yale University, New Haven, Conn. (1978). §5c.
S15. David Singmaster and Anthony Hill, Some classes of Hamiltonian 3-polytopes, in "Proceedings of the Fifth British Combinatorial Conference," (Univ. Aberdeen, Aberdeen, 1975), *Congressus Numer.* **15**, Utilitas Math., Winnipeg, Man. (1976) 551-558. MR 53#1412. §5d.
S16. M. Skowrońska and M. M. Sysło, Hamiltonian cycles in skirted trees, Proceedings of the International Conference on Combinatorial Analysis and its Applications (Pokrzywna, 1985), *Zastos. Mat.* **19** No. 3-4 (1987) 599-610. MR 90c:05139. §5d.
S17. Miroslawa Skowrońska, Hamiltonian properties of Halin-like graphs, *Ars Combin.* **16 B** (1983) 97-109. MR 85i:05163. §5d.
S18. Miroslawa Skowrońska, The pancyclicity of Halin graphs and their exterior contractions, in "Cycles in graphs," (Burnaby, B.C., 1982), *North-Holland Math. Stud.* **115**, North-Holland, Amsterdam-New York (1985) 179-194. MR 87c:05078. §5d.
S19. Zdzislaw Skupień, Some examples in Hamiltonian graph theory, *Bull. Polish Acad. Sci. Math.* **36** No. 5-6 (1988) 363-374. MR 91m:05129. §5d.
S20. E. Steinitz, Polyeder und Raumeinteilungen, *Enzykl. Math. Wiss.* **3** (1922) Geometrie part 3AB12, 1-139. §1c.
S21. Maciej M. Sysło and Andrzej Proskurowski, On Halin graphs, in "Graph theory," (Lagow, 1981), *Lecture Notes in Math.* **1018**, Springer, Berlin-New York (1983) 248-256. MR 86c:05060. §5d.
T1. P. G. Tait, Listing's *Topologie*, *Phil. Mag. (5)* **17** (1884) 30-46. §5a.
T2. Robin Thomas and Xingxing Yu, 5-connected toroidal graphs are hamiltonian, preprint (1994). §3e.
T3. Robin Thomas and Xingxing Yu, 4-connected projective-planar graphs are hamiltonian, *J. Combin. Theory Ser. B* **62** No. 1 (1994) 114-132. §3d.
T4. Carsten Thomassen, Planar and infinite hypohamiltonian and hypotraceable graphs, *Discrete Math.* **14** (1976) 377-389. MR 54#10078. §6b.
T5. Carsten Thomassen, Hypohamiltonian graphs and digraphs, in "Theory and applications of graphs," (Proc. Internat. Conf., Western Mich. Univ., Kalamazoo, Mich., 1976), *Lecture Notes in Math.* **642**, Springer, Berlin (1978) 557-571. MR 80e:05079. §6b.
T6. Carsten Thomassen, Planar cubic hypo-Hamiltonian and hypotraceable graphs, *J. Combin. Theory Ser. B* **30** No. 1 (1981) 36-44. MR 83f:05042. §6b.
T7. Carsten Thomassen, A theorem on paths in planar graphs, *J. Graph Theory* **7** No. 2 (1983) 169-176. MR 84i:05075. §3a, 3c, 3e.
T8. Carsten Thomassen, Five-coloring maps on surfaces, *J. Combin. Theory Ser. B* **59** No. 1 (1993) 89-105. MR 94h:05031. §8c.
T9. Carsten Thomassen, Trees in triangulations, *J. Combin. Theory Ser. B* **60** No. 1 (1994) 56-62. §2a, 8a, 8c.
T10. Michal Tkáč, Simple 3-polytopal graphs with edges of only two types and shortness coefficients, *Math. Slovaca* **42** No. 2 (1992) 147-152. MR 93h:52013. §5b.
T11. W. T. Tutte, On hamiltonian circuits, *J. London Math. Soc.* **21** (1946) 98-101. MR 8,397d. §4, 5a, 5d, 6a.
T12. W. T. Tutte, A theorem on planar graphs, *Trans. Amer. Math. Soc.* **82** (1956) 99-116. MR 18,408e. §1a, 3a, 3b.
T13. W. T. Tutte, A non-Hamiltonian planar graph, *Acta Math. Acad. Sci. Hungar.* **11 1960** (1960) 371-375. MR 25#4517. §5a.
T14. W. T. Tutte, Non-Hamiltonian planar maps, in "Graph theory and computing," Academic Press, New York (1972) 295-301. MR 49#4840. §2b, 5a.
T15. W. T. Tutte, Bridges and hamiltonian circuits in planar graphs, *Aequationes Math.* **15** No. 1 (1977) 1-33. MR 57#5826. §1a, 3b.

W1. K. Wagner, Über eine Eigenschaft der ebenen Komplexe, *Math. Ann.* **114** (1937) 570-590. §3b.
W2. Klaus Wagner, "Graphentheorie," Bibliographisches Institut, Mannheim (1970). §3b.
W3. H. Walther, Über Extremalkreise in regulären Landkarten, *Wiss. Z. Techn. Hochsch. Ilmenau* **15** No. 4-5 (1965) 139-142. MR 42#4438. §5a.
W4. H. Walther, Ein kubischer, planarer, zyklisch funffach zusammenhangender Graph, der keinen Hamiltonkreis besitzt, *Wiss. Z. Techn. Hochsch. Ilmenau* **11** (1965) 163-166. MR 33#70. §5a.
W5. H. Walther, "Über das Problem der Existenz von Hamiltonkreisen in planaren regulären Graphen der Grade 3, 4 und 5," thesis, Techn. Hochschule Ilmenau (1966). §5a.
W6. H. Walther, Über di Anzahl der Knotenpunkte eines längsten Kreises in planaren, kubischen, dreifach zusammenhängenden Graphen, *Studia Sci. Math. Hungar.* **2** (1967) 391-398. MR 36#1364. §5a.
W7. H. Walther, On the problem of the existence of Hamilton-lines in planar regular graphs, Proc. Colloq. Theory of Graphs (Tihany, 1966), Academic Press, New York (1968) 341-343. §5a.
W8. Hansjoachim Walther, Über das Problem der Existenz von Hamiltonkreisen in planaren, regulären Graphen, *Math. Nachr.* **39** (1969) 277-296. MR 41#3315. §5a.
W9. Hansjoachim Walther, Note on two problems of J. Zaks concerning Hamiltonian 3-polytopes, *Discrete Math.* **33** No. 1 (1981) 107-109. MR 82k:52015. §5b.
W10. Hansjoachim Walther, Longest cycles in polyhedral graphs, *Israel J. Math.* **83** (1993) 203-212. MR 95a:05059. §5b.
W11. H. Whitney, A theorem on graphs, *Ann. Math.* **32** (1931) 378-390. §1a, 3a.
W12. Sein Win, Existenz von Gerüsten mit vorgeschriebenem Maximalgrad in Graphen, *Abh. Math. Sem. Univ. Hamburg* **43** (1975) 263-267. MR 54#2529. §8b.
W13. Sein Win, On a connection between the existence of k-trees and the toughness of a graph, *Graphs Combin.* **5** No. 2 (1989) 201-205. MR 90d:05081. §8b.
Y1. Xingxing Yu, Disjoint paths, planarizing cycles, and k-walks, preprint (1993). §8c, 10b.
Z1. Joseph Zaks, Pairs of Hamiltonian circuits in 5-connected planar graphs, *J. Combinatorial Theory Ser. B* **21** No. 2 (1976) 116-131. MR 55#180. §5a, 6a.
Z2. Joseph Zaks, Recent results in graph theory, Proc. 7th Southeastern Conf. on Combinatorics, Graph Theory and Computing (Baton Rouge, 1976), *Congressus Numer.* **17**, Utilitas Math., Winnipeg (1976) 527-532. MR 56#2858. §5b.
Z3. J. Zaks, Non-hamiltonian non-Grinbergian graphs, *Discrete Math.* **17** (1977) 317-321. MR 57#184. §5b.
Z4. J. Zaks, Some hamiltonian results in polytopal graphs, in "Problèmes Combinatoires et Théorie des Graphes," (Orsay, 1976), *Colloq. Internat. CNRS* **260**, CNRS, Paris (1978) 435-436. §5b.
Z5. Joseph Zaks, Non-Hamiltonian simple 3-polytopes having just two types of faces, *Discrete Math.* **29** No. 1 (1980) 87-101. MR 80m:05079. §5b.
Z6. Joseph Zaks, Non-Hamiltonian cubic planar graphs having just two types of faces, Combinatorics 79 (Proc. Colloq., Univ. Montreal, Montreal, Que., 1979), Part II, *Ann. Discrete Math.* **9** (1980) 225-227. MR 81m:05094. §5b.
Z7. Joseph Zaks, Non-Hamiltonian simple planar graphs, in "Theory and practice of combinatorics," *North-Holland Math. Stud.* **60**, North-Holland, Amsterdam-New York (1982) 255-263. MR 86j:05095. §5b.
Z8. Joseph Zaks, Shortness coefficient of cyclically 5-connected cubic planar graphs, *Aequationes Math.* **25** No. 1 (1982) 97-102. MR 85d:05167. §5a.
Z9. Tudor Zamfirescu, Three small cubic graphs with interesting Hamiltonian properties, *J. Graph Theory* **4** No. 3 (1980) 287-292. MR 81h:05097. §5a.