

Orientable embeddings and orientable cycle double covers of projective-planar graphs*

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Abstract

In a *closed 2-cell embedding* of a graph each face is homeomorphic to an open disk and is bounded by a cycle in the graph. The Orientable Strong Embedding Conjecture says that every 2-connected graph has a closed 2-cell embedding in some orientable surface. This implies both the Cycle Double Cover Conjecture and the Strong Embedding Conjecture. In this paper we prove that every 2-connected projective-planar cubic graph has a closed 2-cell embedding in some orientable surface. The three main ingredients of the proof are (1) a surgical method to convert nonorientable embeddings into orientable embeddings; (2) a reduction for 4-cycles for orientable closed 2-cell embeddings, or orientable cycle double covers, of cubic graphs; and (3) a structural result for projective-planar embeddings of cubic graphs. We deduce that every 2-edge-connected projective-planar graph (not necessarily cubic) has an orientable cycle double cover.

1 Introduction

In this paper all graphs are finite and may have multiple edges but no loops. A graph is *simple* if it has no multiple edges. A *pseudograph* may have multiple edges and loops. By a *surface* we mean a connected compact 2-manifold without boundary. The nonorientable surface of genus k is denoted N_k . By an *open* or *closed disk* in a surface we mean a subset of the surface homeomorphic to such a subset of \mathbb{R}^2 .

A *closed 2-cell embedding* of a graph is an embedding such that every face is an open disk bounded by a cycle (no repeated vertices) in the graph. A graph must be 2-connected to have a closed 2-cell embedding. The *Strong Embedding Conjecture* due to Haggard [4] says that every 2-connected graph has a closed 2-cell embedding in some surface. The even stronger *Orientable Strong Embedding Conjecture* [7] says that every 2-connected graph has a closed 2-cell embedding in some

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orientable surface. The facial walks of every closed 2-cell embedding form a *cycle double cover*, a set of cycles in the graph such that each edge is contained in exactly two of these cycles. Therefore, both embedding conjectures imply the well-known *Cycle Double Cover Conjecture*, which says that every 2-edge-connected graph has a cycle double cover,

Every spherical embedding of a 2-connected planar graph is an orientable closed 2-cell embedding. For cubic graphs (but not in general) we can go from a cycle double cover back to a closed 2-cell embedding; thus, cubic graphs with cycle double covers (see for example [1, 3, 5]) have closed 2-cell embeddings. Some special classes of graphs are known to have minimum genus embeddings with all faces bounded by cycles; these are closed 2-cell embeddings. For example, the complete graph K_n has embeddings with all faces bounded by 3-cycles in a nonorientable surface if $n \equiv 0, 1, 3$ or $4 \pmod{6}$, and in an orientable surface if $n \equiv 0, 3, 4$ or $7 \pmod{12}$ (see [12]). The complete bipartite graph $K_{m,n}$ has embeddings with all faces bounded by 4-cycles in a nonorientable surface if mn is even, and in an orientable surface if $(m-2)(n-2)$ is divisible by 4 [10, 11]. However, in general not many graphs are known to have closed 2-cell embeddings.

Even though the Orientable Strong Embedding Conjecture is very strong, the study of orientable closed 2-cell embeddings seems to be promising. One approach is to try to prove the following:

Conjecture 1.1 (Robertson and Zha [personal communication]). *If a 2-connected graph has a nonorientable closed 2-cell embedding then it has an orientable closed 2-cell embedding.*

While this conjecture appears weaker than the Orientable Strong Embedding Conjecture, it is actually equivalent to it, via a result in [15] based on techniques of Little and Ringelsen [8]. This result says that if a graph G has an orientable closed 2-cell embedding, and e is a new edge, then $G + e$ has a closed 2-cell embedding in *some* surface, which may or may not be orientable.

This paper is a first step towards verifying Conjecture 1.1. Our goal is to develop techniques to turn nonorientable closed 2-cell embeddings into orientable closed 2-cell embeddings. Using techniques of this kind, we show that the Orientable Strong Embedding Conjecture is true for projective-planar cubic graphs. The following is the main result of this paper.

Theorem 1.2. *Every 2-connected projective-planar cubic graph has a closed 2-cell embedding in some orientable surface.*

A standard construction then provides a result for general 2-edge-connected projective-planar graphs. Given such an embedded graph G , we may construct a 2-edge-connected (hence 2-connected) cubic projective-planar graph H by expanding each vertex v of degree $d_v \geq 4$ to a contractible cycle of length d_v . By Theorem 1.2, H has an orientable closed 2-cell embedding, whose oriented faces give a compatibly oriented (each edge occurs once in each direction) cycle double cover \mathcal{C} of H . Contracting the new cycles to recover G , \mathcal{C} becomes a compatibly oriented cycle double cover \mathcal{C}' of G . (\mathcal{C}' may not correspond to a surface embedding of G , but that does not matter.) Some cycles of \mathcal{C} may become oriented eulerian subgraphs, rather than cycles, of \mathcal{C}' , but those can always be decomposed into oriented cycles, preserving the compatible orientation. This proves the following.

Corollary 1.3. *Every 2-edge-connected projective-planar graph has an orientable cycle double cover.*

In Section 2 we introduce some notation. In Section 3 we develop surgeries that convert nonorientable surfaces into orientable surfaces, and use them to prove a special case of our main result. In Section 4 we describe some reductions for our problem. In Section 5 we arrive at our main result by proving a structural result which shows that either the special case from Section 3 occurs, or some kind of reduction applies.

2 Definitions and notation

Let Ψ denote an embedding of a graph G in a surface Σ . We usually identify the graph and the point-set of its image under the embedding. If $S \subseteq \Sigma$, then \overline{S} denotes the closure of S in Σ . A *face*

is a component of $\Sigma - G$. The boundary of the face f is denoted by ∂f . Each component of ∂f is traced out by a closed walk in G , which we call a *facial boundary component walk* of f . A *k-cycle face* is a face with exactly one boundary component, which is a *k-cycle*.

An embedding in which every face is an open disk (2-cell) is an *open 2-cell embedding*; then each face has a single boundary component walk, called the *facial walk*. If every facial walk of an open 2-cell embedding is in fact a cycle, we have a *closed 2-cell embedding*. If Σ is not the sphere, then the *representativity* of any embedding Ψ is defined to be $\rho(\Psi) = \min\{|\Gamma \cap G| : \Gamma \text{ is a noncontractible simple closed curve in } \Sigma\}$. We say Ψ is *k-representative* if $\rho(\Psi) \geq k$. Robertson and Vitray [14] showed that Ψ is open 2-cell exactly when G is connected and Ψ is 1-representative, and closed 2-cell exactly when G is 2-connected and Ψ is 2-representative.

Suppose C is a cycle with a given orientation, and u and v are two vertices on C . Denote by uCv the path on C from u to v in the given direction. If we have a graph embedded on an orientable surface, then all cycles may be oriented in a consistent clockwise direction, and we assume this orientation unless otherwise specified. If P is a path, uPv is defined to be the subpath from u to v along P .

We say two sets (usually faces) f and g *touch* if $\partial f \cap \partial g \neq \emptyset$; we say they *touch k times* if $\partial f \cap \partial g$ has k components. A sequence of sets $f_0, f_1, f_2, \dots, f_n$ is an *$f_0 f_n$ -face chain of length n* if for $1 \leq i \leq n-1$ each f_i is a distinct face of Ψ and for $0 \leq i < j \leq n$, f_i and f_j touch when $j = i+1$ and do not touch otherwise. The sets f_0 and f_n may be faces of Ψ , but need not be; in this paper they are usually paths. If $R \subseteq \Sigma$ and $f_i \subseteq R$ for $1 \leq i \leq n-1$, we say the face chain *goes through* R .

A cyclic sequence of distinct faces (f_1, f_2, \dots, f_n) is called a *face ring of length n* if (i) $n = 2$, and f_1 and f_2 touch at least twice, or (ii) $n \geq 3$ and the sets $\partial f_i \cap \partial f_j$, $i \neq j$, are pairwise disjoint, nonempty when $j = i-1$ or $i+1$, and empty otherwise (subscripts interpreted modulo n). We will use face rings only in closed 2-cell embeddings, so we do not need to consider situations in which a face ‘touches itself’, i.e., face rings of length 1. A face ring is *elementary* if (i) $n = 2$ and the two faces touch exactly twice, or (ii) $n \geq 3$ and any two faces touch at most once. A face ring is *noncontractible* if $R = \bigcup_{i=0}^{n-1} \overline{f_i}$ contains a noncontractible simple closed curve.

The following observations will be useful for showing that face rings are elementary.

Observation 2.1. *If Ψ is an embedding of a 3-connected graph, and two faces are contained in some open disk, then the faces touch at most once.*

Observation 2.2. *If Ψ is a 3-representative embedding of a 3-connected graph then any two faces touch at most once.*

3 Converting nonorientable surfaces to orientable surfaces

It is not hard to turn an orientable embedding into a nonorientable embedding, by adding a crosscap in an arbitrary location. On the other hand, it is not easy in general to construct an orientable embedding from an existing nonorientable embedding of a graph. The authors of [2, 13] developed some surgeries which turn embeddings on the projective plane and on the Klein bottle into orientable embeddings. However, the resulting orientable embeddings are not necessarily closed 2-cell embeddings. In this section we develop some surgeries to convert nonorientable embeddings into orientable embeddings. We show that under certain conditions this can be applied to convert a closed 2-cell projective-planar embedding into a closed 2-cell orientable embedding.

Our overall strategy will be as follows. Suppose G is embedded in a nonorientable surface Σ_1 , obtainable by inserting a set \mathcal{X} of crosscaps in an orientable surface Σ_0 . We insert an additional set \mathcal{X}' of crosscaps to get an embedding of G on a nonorientable surface Σ_2 , in which all facial boundary component walks are cycles. Then we embed in Σ_2 a pseudograph H disjoint from G , such that cutting along H destroys all the crosscaps in $\mathcal{X} \cup \mathcal{X}'$. Capping any holes due to cutting we obtain

an embedding of G on an orientable surface Σ_3 , in which all facial boundary component walks are still cycles. Removing any faces that are not open disks, and capping again, we finish with a closed 2-cell embedding of G in an orientable surface Σ_4 .

We first define some concepts related to crosscaps, next discuss insertion of crosscaps, then examine cutting to remove nonorientability, and finally apply these ideas to certain projective-planar embeddings.

We regard a *crosscap* in a 2-manifold (with or without boundary) as just a one-sided simple closed curve in the interior of the manifold (rather than the usual definition, where a crosscap is a neighborhood of such a curve, homeomorphic to a Möbius strip). To *add a crosscap* to a 2-manifold we remove a point or a set homeomorphic to a closed disk not intersecting the boundary, locally close the result by adding a boundary component homeomorphic to a circle, then identify antipodal points of this circle to obtain a one-sided simple closed curve. If we removed a point p or closed disk Δ , we call this *inserting a crosscap at p or Δ* . If p is an interior point of an edge e of an embedded graph, we call this *inserting a crosscap on e* . In figures we represent a crosscap by a circle (representing the added boundary component) with an X inside it. To recover the original 2-manifold (up to homeomorphism) we may *collapse* the crosscap by identifying all its points into a single point (the result is homeomorphic to what we obtain by cutting out a Möbius-strip neighborhood of the crosscap and capping the resulting hole with a disk).

We define the following surgeries for inserting crosscaps. These, or closely related operations, have been used in earlier papers such as [15], and generalize ideas used by Haggard [4].

Operation 3.1 (Inserting crosscaps between faces or along a face ring).

- (a) Let f_1 and f_2 be two distinct face occurrences at a vertex v . (They may be different occurrences of the same face.) Then f_1 and f_2 partition the edges incident with v into two intervals, I_1 and I_2 . Choose a closed disk Δ close to v such that all edges of I_1 cross it once, and no edges of I_2 intersect it. Add a crosscap at Δ , re-embedding the parts of the edges of I_1 between Δ and v so that their order around v is reversed. We call this *inserting a crosscap between f_1 and f_2 near v* . It does not matter whether we insert the crosscap across I_1 or I_2 : using the other one just amounts to pulling v through the crosscap, which does not change the embedding. If one of I_1 or I_2 is a single edge, then this is equivalent to inserting a crosscap on that edge.
- (b) If f_1 and f_2 are distinct faces and $\partial f_1 \cap \partial f_2$ is a path P (possibly a single vertex) then the effect of inserting a crosscap between f_1 and f_2 at any vertex of P , or on any edge of P , is the same, so we just talk about *inserting a crosscap between f_1 and f_2* .
- (c) Suppose $\mathcal{F} = (f_1, f_2, \dots, f_m)$ is an elementary face ring of length $m \geq 3$. Interpreting subscripts modulo m , insert a crosscap between f_i and f_{i+1} for $1 \leq i \leq m$. We call this *inserting crosscaps along \mathcal{F}* . (This can also be defined for elementary face rings of length 2, or for suitable face chains, although we do not need this here.)

While the above definition allows us some freedom in exactly how we place the crosscaps, in practice it may be convenient to make a definite choice about the location of the crosscaps, to help in tracing boundary component walks in the new embedding.

Now we introduce our cutting operation. Our approach is fairly general, although we will need only a special case of it for our projective-planar results.

We first need to examine what happens if we cut through an individual crosscap on a surface. Suppose Δ is a neighborhood of a crosscap X , which we think of as a circle \tilde{X} with antipodal points identified. There is a consistent local orientation ω on $\Delta - X$. Cut along a curve Γ which crosses X at a single point p . If antipodal points of \tilde{X} were not identified, we would cut Δ into two pieces Δ_1 and Δ_2 . Because we do identify antipodal points, Δ_1 and Δ_2 remain joined along the cut crosscap X^* , a line segment joining two copies of p . Δ_1 and Δ_2 are joined with a twist so we now have a

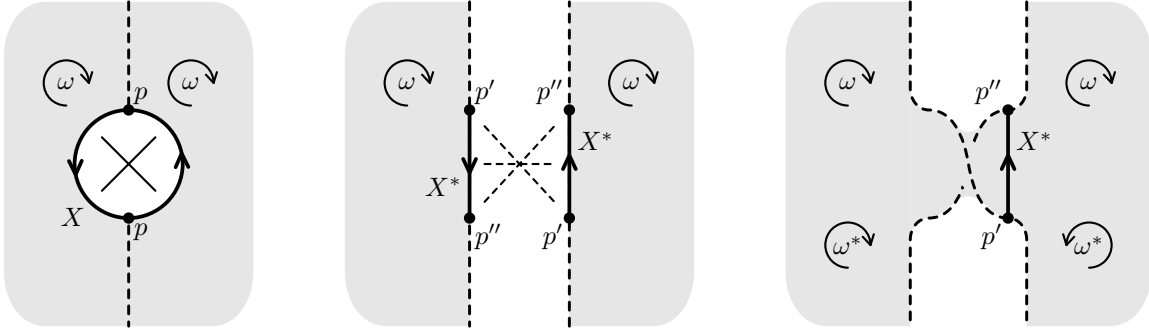


Figure 1: Cutting a crosscap

local orientation ω^* everywhere in the neighborhood of the cut crosscap X^* , including on X^* itself, which reverses relative to ω when we cross X^* . See Figure 1: at left is a planned cut; in the middle the cut crosscap is shown as two copies of the line segment X^* that are to be identified; and at right we see the result of the identification.

As mentioned earlier, we consider a nonorientable surface built up by adding crosscaps to an orientable surface; we then cut through a pseudograph H intersecting those crosscaps to remove the nonorientability.

Lemma 3.2. *Suppose Φ is a 2-face-colorable embedding of a pseudograph H on an orientable surface Σ . Suppose we choose a finite set of points P so that each point of P is an interior point of an edge of H . If we insert a crosscap X_p at every $p \in P$, cut the resulting surface along all edges of H , and use disks to cap the boundary components of the resulting 2-manifold with boundary, the result is a finite union of pairwise disjoint orientable surfaces.*

Proof. It suffices to show that the 2-manifold with boundary Σ^* obtained by cutting along H is orientable; then by capping we obtain a union of orientable surfaces.

Consider a fixed global orientation ω (local choice of clockwise direction) for Σ . When we add the crosscaps, ω gives a consistent local orientation on a neighborhood of each crosscap X_p , excluding X_p itself. Let C_0 and C_1 be the two color classes of faces of Φ . Define an orientation ω^* on Σ^* that is equal to ω on each face in C_0 and opposite to ω on each face in C_1 . When we pass between two faces of Φ in Σ^* , we pass between a face in C_0 and a face in C_1 via a cut crosscap. As previously discussed, crossing the cut crosscap X_p^* reverses the orientation, in agreement with ω^* . Therefore ω^* provides a consistent global orientation for Σ^* . ■

In Lemma 3.2, H need not be connected. Also, we may allow H to contain free loops, edges incident with no vertex, which map to simple closed curves in an embedding; we can always insert vertices to turn these into ordinary loops.

Instead of starting with the pseudograph H and inserting the crosscaps, we may start with the crosscaps and add H . Therefore, the following is equivalent to Lemma 3.2.

Operation 3.3. Suppose Σ' is a nonorientable surface containing disjoint crosscaps X_1, X_2, \dots, X_k . Suppose Φ' is an embedding on Σ' of a pseudograph H , such that each X_i contains exactly one point of H , an interior point of an edge that crosses X_i . Suppose that when we collapse every X_i , $1 \leq i \leq k$, we get a 2-face-colorable embedding of H in an orientable surface Σ . Then if we cut Σ' along all edges of H and use disks to cap the boundary components of the resulting 2-manifold with boundary, the result, Σ'' , is a finite union of pairwise disjoint orientable surfaces.

In Operation 3.3, the condition that each crosscap X_i be crossed by exactly one edge of H , implying that the pseudograph after collapsing every X_i is still H , is important. If this is not the

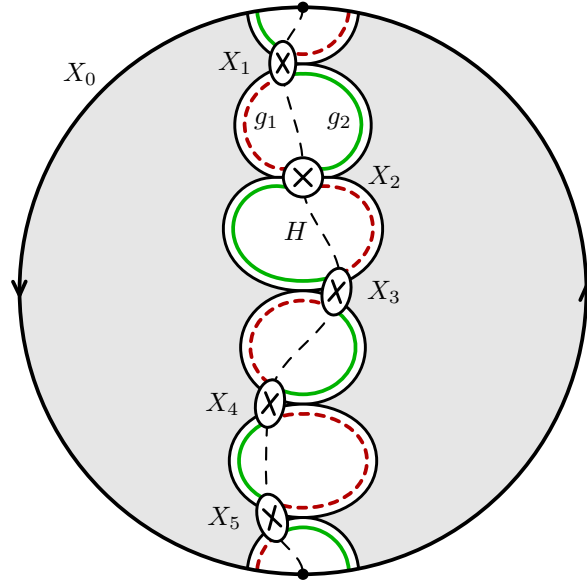


Figure 2: Orienting using an elementary face ring of odd length

case, then the embedding obtained by cutting along H may not be orientable, even if collapsing the crosscaps yields a 2-face-colorable orientable embedding of some graph. For example, suppose we add four crosscaps X_1, X_2, X_3, X_4 to a sphere to get $\Sigma' = N_4$, and take H to consist of two disjoint loops Γ_1 and Γ_2 , where Γ_1 and Γ_2 both cross X_1 and X_2 , X_3 is crossed only by Γ_1 and X_4 is crossed only by Γ_2 . When we collapse the crosscaps we get a 2-face-colorable embedding of a graph on a sphere, but if we cut Σ' along Γ_1 and Γ_2 the result is not orientable.

With some extra conditions it is possible to allow more than one edge of H to cross a crosscap X_i , but we will not pursue the details here.

The pseudograph H in Operation 3.3 indicates where to cut. The graph we wish to embed is a different graph, G . We assume that G also has an embedding Ψ' in Σ' , disjoint from the embedding Φ' of H . Note that G may cross some or all of the crosscaps X_i and still be disjoint from H ; there may be several edges of G crossing each crosscap, and an edge of G may cross several crosscaps. In this context vertex-splitting in H preserving 2-face-colorability allows us to assume, if we wish, that H is a union of vertex-disjoint cycles or free loops.

If G is connected, then when we cut along H and cap, we get an embedding Ψ'' of G in an orientable surface, one of the connected components of Σ'' . We do not disturb the order of the edges around any of the vertices of G , so Ψ'' has the same set of facial boundary component walks as Ψ' , although the faces may not be open disks.

Now we combine Operations 3.1 and 3.3 to prove an easy case of Theorem 1.2, which applies to graphs with arbitrary vertex degrees, not just cubic graphs.

Theorem 3.4. *Let Ψ be a closed 2-cell embedding of a 2-connected graph G in the projective plane. Suppose Ψ contains a noncontractible elementary face ring $\mathcal{F} = (f_1, f_2, \dots, f_l)$ which has odd length $l \geq 3$. Then G has a closed 2-cell embedding in some orientable surface.*

Proof. Represent the projective plane in the standard way as a disk with antipodal boundary points identified; the boundary represents a crosscap added to the sphere, which we call the *outer crosscap* X_0 . By homotopic shifting we may ensure that the closure of exactly one face of \mathcal{F} intersects X_0 , and X_0 cuts that face into exactly two nonempty pieces. See Figure 2.

Insert crosscaps X_1, X_2, \dots, X_l along \mathcal{F} as in Operation 3.1(c), to obtain a new embedding Ψ' of G in N_{l+1} . All faces of \mathcal{F} turn into a new face g in Ψ' whose boundary has two components g_1 and g_2 . All other facial walks in Ψ remain unchanged in Ψ' . Because faces in \mathcal{F} are disjoint except that consecutive faces intersect in a single component, each of g_1 and g_2 is a cycle. Thus, all facial boundary component walks of Ψ' are cycles.

By following \mathcal{F} , we may construct a simple closed curve H that passes through each face of \mathcal{F} and each inserted crosscap exactly once, and is disjoint from G . H crosses each of the $l+1$ crosscaps X_0, X_1, \dots, X_l . If we consider H as an embedded pseudograph with one vertex (any point of $H - \bigcup_{i=0}^l X_i$) and one loop, then collapsing all crosscaps leaves H as a simple closed curve in the sphere, which is a 2-face-colorable embedding of H in an orientable surface. Therefore, by Operation 3.3, cutting along H and capping yields an orientable surface, in which G is embedded with all facial boundary components cycles. Removing any faces that are not open disks and capping any resulting holes with disks, we obtain a closed 2-cell embedding of G in an orientable surface, as required. ■

Theorem 3.4 does not work with elementary face rings of even length because then we get a face with a single boundary component that self-intersects, instead of two components g_1 and g_2 .

In terms of our original strategy, here Σ_0 is the sphere, $\mathcal{X} = \{X_0\}$, Σ_1 is the projective plane, $\mathcal{X}' = \{X_1, X_2, \dots, X_l\}$, Σ_2 is N_{l+1} , and Σ_3 and Σ_4 are the orientable surfaces in the last two sentences of the proof.

The proof of Theorem 3.4 uses only a very simple version of our strategy, and there is an alternative way to obtain the final embedding in this proof, namely the surgery of Fiedler et al. [2, Lemma A]. However, we have examples of noncubic 2-connected projective-planar graphs where we can find orientable closed 2-cell embeddings using more complicated applications of Operations 3.1 and 3.3. Also, our approach continues the development of a coherent set of tools for constructing closed 2-cell embeddings, begun in papers such as [15].

4 Reductions

In this section we describe reductions that allow us to restrict our attention to a smaller class of projective-planar cubic graphs. Most of these reductions are standard, but the reduction we give for 4-cycles is new. We state our results in a general setting and then apply them to projective-planar cubic graphs. We also prove a technical result which will be used to deal with certain ‘planar 4-edge-cuts’.

Let \mathcal{C}_2 be the class of 2-connected cubic graphs, and let \mathcal{G} be a subclass of \mathcal{C}_2 closed under taking minors in \mathcal{C}_2 (i.e., if $G_1 \in \mathcal{G}$ and $G_2 \in \mathcal{C}_2$ is a minor of G_1 , then $G_2 \in \mathcal{G}$ also). In constructing orientable closed 2-cell embeddings for graphs in \mathcal{G} , standard reductions used for cycle double covers (see [7, pp. 4–5]) can be applied to deal with any 2-edge-cut, or nontrivial 3-edge-cut (a 3-edge-cut that is not just the edges incident with a single vertex). Since we can exclude 2-edge-cuts, we can also exclude multiple edges. A 2-connected cubic graph with no 2-edge-cuts or nontrivial 3-edge-cuts is said to be *cyclically-4-edge-connected*, so we may state the following.

Lemma 4.1. *Given \mathcal{G} as above, an element G of \mathcal{G} having no orientable closed 2-cell embedding (i.e., no orientable cycle double cover) and, subject to that, fewest edges is simple and cyclically 4-edge-connected.*

Since G has no nontrivial 3-edge-cuts, and since K_4 has an orientable embedding, it follows that G as in Lemma 4.1 has no triangles (3-cycles). For the Cycle Double Cover Conjecture there are also reductions to exclude 4-cycles and in fact cycles of length up to 11, due to Goddyn [3] (see also [16, pp. 155–158]) and Huck [6]. However, these reductions do not work if we add the condition of orientability. Here we show that there is at least a 4-cycle reduction for the Orientable Cycle Double Cover Conjecture.

Lemma 4.2. *Given \mathcal{G} as above, an element G of \mathcal{G} having no orientable closed 2-cell embedding (i.e., no orientable cycle double cover) and, subject to that, fewest edges has no 4-cycle.*

Proof. By Lemma 4.1 G is simple, cyclically-4-edge-connected, and has no triangles. Since $K_{3,3}$ has a toroidal closed 2-cell embedding (in which there are three faces, all hamilton cycles), we know that $|V(G)| \geq 8$.

Assume that G contains a 4-cycle $C = (v_1 v_2 v_3 v_4)$. Since G has no triangles, each v_i has a neighbour $u_i \notin V(C)$. Since $|V(G)| \geq 8$, cyclic-4-edge-connectivity implies that u_1, u_2, u_3 and u_4 are distinct, and moreover that $G' = (G - V(C)) \cup \{u_1 u_2, u_3 u_4\}$ is 2-connected. Hence $G' \in \mathcal{G}$, so since G' has fewer edges than G it has an orientable closed 2-cell embedding Ψ' . Suppose the faces of Ψ' containing $u_1 u_2$ are f_1 and f_2 , where ∂f_1 traversed clockwise uses directed edge $u_2 u_1$, and ∂f_2 uses $u_1 u_2$. Suppose the faces containing $u_3 u_4$ are g_1 and g_2 , where ∂g_1 uses directed edge $u_3 u_4$ when traversed clockwise, and ∂g_2 uses $u_4 u_3$. In Ψ' subdivide $u_1 u_2$ to obtain the path $u_1 v_1 v_2 u_2$, and $u_3 u_4$ to obtain the path $u_3 v_3 v_4 u_4$.

If $f_1 = g_2$ we may add the edges $v_1 v_4$ and $v_2 v_3$ inside f_1 to obtain an orientable closed 2-cell embedding of G . A similar argument applies if $f_2 = g_1$. Thus, $f_1 \neq g_2$ and $f_2 \neq g_1$.

If in addition $f_1 \neq g_1$ then we add a handle from f_1 to g_1 and add edges $v_1 v_4, v_2 v_3$ along the handle, to obtain an orientable embedding of G . In the new embedding the faces f_1 and g_1 are replaced by f'_1 and g'_1 , where $\partial f'_1 = (\partial f_1 - v_2 v_1) \cup v_2 v_3 v_4 v_1$ and $\partial g'_1 = (\partial g_1 - v_3 v_4) \cup v_3 v_2 v_1 v_4$. Now $\partial f'_1$ is a cycle because $f_1 \neq g_2$, and $\partial g'_1$ is a cycle because $g_1 \neq f_2$, so this is an orientable closed 2-cell embedding of G . Therefore, $f_1 = g_1$, and similarly $f_2 = g_2$.

Since $f_1 = g_1$ and $f_2 = g_2$ we may add the edge $v_1 v_4$ inside f_1 and add the edge $v_2 v_3$ inside f_2 , to obtain an orientable closed 2-cell embedding of G .

Since G has an orientable closed 2-cell embedding in all cases, our assumption was false, and G has no 4-cycle, as required. ■

We may apply our results when \mathcal{G} is the class of projective-planar cubic graphs.

Corollary 4.3. *Let G be a 2-connected projective-planar cubic graph that has fewest edges subject to having no orientable closed 2-cell embedding. Then G is simple, cyclically-4-edge-connected, and has no 3- or 4-cycles.*

Now we introduce the technical result for dealing with certain ‘planar 4-edge-cuts.’ It is easier to prove it first in a dual form, for near-triangulations. A *near-triangulation* is a graph embedded in the plane so that every face is a triangle, except possibly for the outer face, which is a cycle. A *separating cycle* in an embedded graph is a cycle whose removal disconnects the surface into two components, each of which contains at least one vertex of the graph.

If G is a connected graph embedded in the plane, ∂G represents its outer walk. An *interior* vertex of G is a vertex not on ∂G . An *interior path* in G is a path none of whose internal vertices lie on ∂G (although one or both ends may lie on ∂G). If $C = (v_1 v_2 v_3 \dots v_k)$ is a cycle of G , then $I_G(C)$ or $I_G(v_1 v_2 v_3 \dots v_k)$ represents the embedded subgraph of G on and inside C . If G is understood we just write $I(C)$ or $I(v_1 v_2 v_3 \dots v_k)$.

$N_G(v)$ represents the set of neighbors of vertex v in G , and $N_G[v]$ represents $N_G(v) \cup \{v\}$. If H is a subgraph of G , then a *chord* of H is an edge that is not in H but whose two ends are in H . If x is a cutvertex of H , then a chord of H is *x-jumping* if its ends lie in different components of $H - x$.

Observation 4.4. *In a near-triangulation, a minimal cutset separating two given nonadjacent vertices induces either a chordless separating cycle, or a chordless interior path with ends on the outer cycle.*

Observation 4.5. *Suppose G is a near-triangulation with outer cycle ∂G and x, y are vertices of ∂G with $xy \notin E(\partial G)$. Then either G has an interior xy -path, or $\partial G - y$ has an x -jumping chord.*

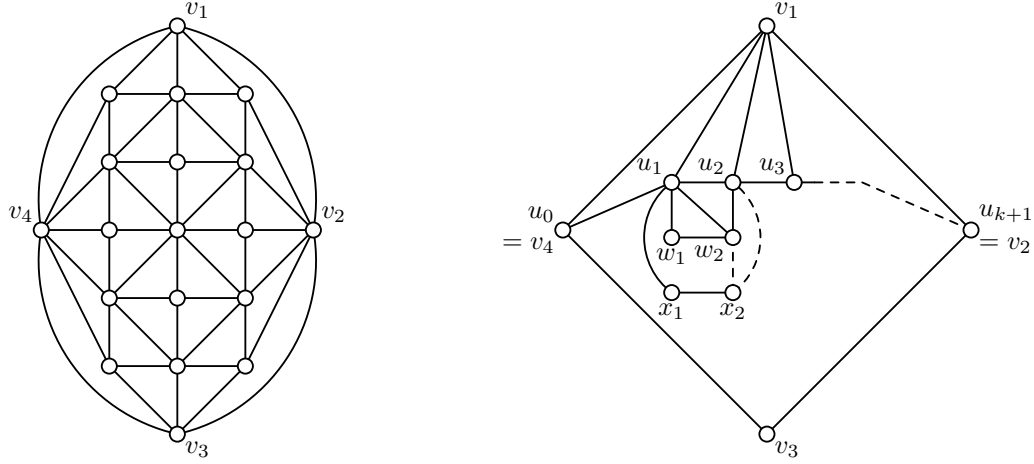


Figure 3: Example with only even paths, and structure for the proof of Proposition 4.6

Proposition 4.6. *Let G be a simple graph (no loops or multiple edges) embedded in the plane so that all faces are triangles except the outer face, which is a 4-cycle $(v_1v_2v_3v_4)$ in that clockwise order; there is at least one interior vertex; and there are no separating triangles.*

- (i) *Then there is a chordless interior v_1v_3 -path in G .*
- (ii) *Moreover, if all chordless interior v_1v_3 -paths in G have length of the same parity (all even, or all odd), then G has an interior vertex of degree 4.*

Nontrivial examples as in (ii) above do occur. At left in Figure 3 is an example where all chordless interior v_1v_3 -paths have even length; it has several interior vertices of degree 4.

Proof. For (i), $v_2v_4 \notin E(G)$ because there are no separating triangles, and hence by Observation 4.5 there is an interior v_1v_3 -path. A shortest such path is chordless.

We prove (ii) by contradiction. Assume it does not hold, and G is a counterexample with fewest vertices. So, all interior v_1v_3 -paths have length of the same parity, but there is no interior vertex of degree 4. If $|V(G)| = 5$, G is a wheel with central vertex of degree 4, so we must have $|V(G)| \geq 6$. Since G is a counterexample and there are no separating triangles, every interior vertex has degree at least 5. Since there are no separating triangles, $v_1v_3, v_2v_4 \notin E(G)$.

Claim 1. If $C = (t_1t_2t_3t_4)$ is a separating 4-cycle then there are chordless interior t_1t_3 -paths in $I(C)$ whose lengths have different parities.

Proof of Claim. If not, then by minimality of G there is an interior vertex of $I(C)$ of degree 4 in $I(C)$. But this is also an interior vertex of G of degree 4 in G , a contradiction. \square

Claim 2. There is no interior vertex adjacent to both v_1 and v_3 , or both v_2 and v_4 .

Proof of Claim. Suppose there is an interior vertex v with $vv_1, vv_3 \in E(G)$. Since $|V(G)| \geq 6$, at least one of the subgraphs $I(v_1v_2v_3v)$ or $I(v_1vv_3v_4)$ has an interior vertex. But this subgraph contradicts Claim 1.

If there is an interior vertex v adjacent to both v_2 and v_4 then at least one of $G' = I(v_1v_2vv_4)$ and $G'' = I(vv_2v_3v_4)$ has an interior vertex; without loss of generality assume it is G' . There is a chordless interior vv_3 -path Q'' in G'' , by (i) if G'' has an interior vertex, and because $v_2v_4 \notin E(G)$ so that $vv_3 \in E(G)$ if G'' has no interior vertex. For every chordless interior v_1v -path Q' in G' , $Q' \cup Q''$ is a chordless interior v_1v_3 -path in G , so all such paths Q' have length of the same parity, contradicting Claim 1. \square

Let the neighbors of v_1 in anticlockwise order be $v_4 = u_0, u_1, u_2, \dots, u_k, u_{k+1} = v_2$. By Claim 2, $k \geq 2$. As all interior faces are triangles and there are no separating triangles, $U = u_0 u_1 u_2 \dots u_k u_{k+1}$ is a chordless path in G . By Claim 2, no vertex of $U - \{u_0, u_{k+1}\}$ is adjacent to v_3 .

Claim 3. Every 4-cycle containing v_1 is ∂G , or has the form $(v_1 u_{i+2} u_{i+1} u_i)$, or has the form $(v_1 u_{i+1} x u_i)$ where x is an interior vertex of $I(U \cup v_2 v_3 v_4)$.

Proof of Claim. Every 4-cycle containing v_1 has the form $C = (v_1 u_j x u_i)$ with $i < j$. If $i = 0$ and $j = k + 1$ then by Claim 2, C is ∂G . So, without loss of generality, we may assume that $j \leq k$. Then, also by Claim 2, $x \neq v_3$. If x is on U then since U is chordless, C must be $(v_1 u_{i+2} u_{i+1} u_i)$, as specified. So we may assume that x is an interior vertex of $I(U \cup v_2 v_3 v_4)$. If $j = i + 1$ then C is as specified, so suppose $j \geq i + 2$. Let $G' = I(C)$ and $G'' = I(u_0 u_1 \dots u_i x u_j u_{j+1} \dots u_{k+1} v_3)$. Since U is chordless, Observation 4.5 implies that G'' has an interior $x v_3$ -path. Let Q'' be a shortest, hence chordless, such path. For any chordless interior $v_1 x$ -path Q' in G' , $Q' \cup Q''$ is a chordless interior $v_1 v_3$ -path in G , so all such paths Q' have length of the same parity. Since G' has an interior vertex, u_{i+1} , this contradicts Claim 1. \square

The following are immediate consequences of Claim 3.

Claim 4. There is no interior vertex x not adjacent to v_1 but adjacent to u_i and u_j with $|j - i| \geq 2$. \square

Claim 5. There is no separating 4-cycle in G of the form $(v_1 v_2 x y)$ or $(v_1 x y v_4)$, or (by symmetry) $(v_3 v_2 x y)$ or $(v_3 x y v_4)$. \square

Since u_1 has degree at least 5, there exist w_1, w_2 so that the neighbors of u_1 in clockwise order are $v_4 = u_0, v_1, u_2, w_2, w_1, \dots$. Since U is chordless, neither w_1 nor w_2 is on U , and by Claim 2, neither w_1 nor w_2 is v_3 . There are triangular faces $(u_1 u_2 w_2)$ and $(u_1 w_2 w_1)$. Since there are no separating triangles, $u_0 w_2, w_1 u_2 \notin E(G)$. By Claim 5, $w_2 v_3 \notin E(G)$. See the right side of Figure 3.

Let $U' = u_0 u_1 w_1 w_2 u_2 u_3 \dots u_{k+1}$, and let $H = I(U' \cup v_2 v_3 v_4)$. Since U is chordless, since $u_0 w_2 \notin E(G)$, and since $u_1 u_2$ and $u_1 w_2$ are edges of G but not H , U' has no w_1 -jumping chord in H .

Claim 6. There is an interior $w_1 v_3$ -path in H avoiding all neighbors (in G) of v_1, u_1, u_2 and w_2 , except w_1 itself.

Proof of Claim. Let $S = (V(U) \cup N_H[u_1] \cup N_H[u_2] \cup N_H[w_2]) - \{w_1\}$. Since $u_1 v_3, w_2 v_3, u_2 v_3 \notin E(G)$, we know that $v_3 \notin S$. It suffices to show that there is a $w_1 v_3$ -path in $H - S$.

Let $S_1 = (N_H[u_1] - \{w_1\})$, and $S_2 = (N_H[w_2] - \{w_1\}) \cup N_H[u_2] \cup \{u_4, u_5, \dots, u_{k+1}\}$. Then $S = S_1 \cup S_2$. Suppose $x \in S_1 \cap S_2$. Since U' has no w_1 -jumping chord in H , $x \notin V(U')$ and hence $x \in N_H(u_1) \cap (N_H(w_2) \cup N_H(u_2))$. But then either $(u_1 w_2 x)$ or $(u_1 u_2 x)$ is a separating triangle in G , a contradiction. Hence $S_1 \cap S_2 = \emptyset$.

Assume there is no $w_1 v_3$ -path in $H - S$. Then there is a minimal cutset contained in S separating w_1 and v_3 in H , which by Observation 4.4 induces a chordless path R starting on $u_0 u_1$ and ending on $w_2 u_2 u_3 \dots u_{k+1}$. Since U' has no w_1 -jumping chord in H , there are no vertices of S_2 on $u_0 u_1$, and no vertices of S_1 on $w_2 u_2 u_3 \dots u_{k+1}$. Hence, the first vertex of R belongs to S_1 and the last to S_2 . Let x_1 denote the last vertex of R that belongs to S_1 , and x_2 its immediate successor, which must belong to S_2 . Since U' has no w_1 -jumping chord in H , we cannot have both $x_1, x_2 \in V(U')$.

Suppose $x_1 \in V(U')$, so $x_1 = u_0$ or u_1 . Then $x_2 \in S_2 - V(U') = (N_H(w_2) \cup N_H(u_2)) - V(U')$. If $x_1 = u_1$ then $x_2 \in S_1 \cap S_2$, a contradiction. If $x_1 = u_0$ and $x_2 \in N(u_2)$ then $(v_1 u_2 x_2 u_0)$ contradicts Claim 4. Thus, $x_1 = u_0$ and $x_2 \in N(w_2)$. Since U' has no w_1 -jumping chord in H , by Observation 4.5 there is an interior $x_2 v_3$ -path in $I(u_0 x_2 w_2 u_2 u_3 \dots u_{k+1} v_3)$. Let Q'' be a shortest, hence chordless, such path. Since $I(u_0 u_1 w_2 x_2)$ has an interior vertex w_1 , by Claim 1 it has interior $u_1 x_2$ -paths Q'_1 and Q'_2 whose lengths have different parities. But then $v_1 u_1 \cup Q'_1 \cup Q''$ and $v_1 u_1 \cup Q'_2 \cup Q''$ are chordless interior $v_1 v_3$ -paths whose lengths have different parities, a contradiction.

Therefore, $x_1 \in S_1 - V(U') = N(u_1) - V(U')$. We consider three possibilities for x_2 . Cases (2) and (3) are shown at right in Figure 3.

(1) Suppose that $x_2 \in V(U')$, so that x_2 is a vertex of $w_2u_2u_3 \dots u_{k+1}$. Since $x_1u_1 \in E(G)$, we either get a separating triangle in G if $x_2 = w_2$ or u_2 , or violate Claim 4 if $x_2 = u_i$, $3 \leq i \leq k+1$.

(2) Suppose that $x_2 \in N_H(w_2) - V(U')$. Let $U'' = u_0u_1x_1x_2w_2u_2u_3 \dots u_{k+1}$ and let $J = I(U'' \cup v_2v_3v_4)$. If x_1 is adjacent to a vertex z of $w_2u_2u_3 \dots u_{k+1}$ we get a separating triangle if $z = w_2$ or u_2 , or violate Claim 4 otherwise. Also, $u_1w_2, u_1u_2 \notin E(J)$, and U' has no w_1 -jumping chord in H . Therefore, U'' has no x_2 -jumping chord in J .

If there is an interior x_2v_3 -path in J that avoids $N_J[u_1]$, then we may take Q'' to be a shortest, hence chordless, such path. Then for any chordless interior u_1x_2 -path Q' in $G' = I(u_1w_2x_2x_1)$, $v_1u_1 \cup Q' \cup Q''$ is a chordless interior v_1v_3 -path in G , so all such paths Q' have length of the same parity. Since G' has an interior vertex, w_1 , this contradicts Claim 1.

Therefore, $S' = (V(U'') \cup N_J[u_1]) - \{x_2\}$ separates x_2 and v_3 in J . Let $S'_1 = N_J[u_1]$ and $S'_2 = \{w_2, u_2, u_3, \dots, u_{k+1}\}$, so that $S' = S'_1 \cup S'_2$. Because U'' has no x_2 -jumping chord in J , $S'_1 \cap S'_2 = \emptyset$. Applying Observation 4.4 to a minimal cutset contained in S' separating x_2 and v_3 , which induces a chordless path R' which starts on a vertex of $S'_1 \cap V(U'')$ and ends on a vertex of S'_2 , we see that R' has an edge y_1y_2 with $y_1 \in S'_1$, $y_2 \in S'_2$. Since U'' has no x_2 -jumping chord in J , we cannot have both $y_1, y_2 \in V(U'')$, so $y_1 \in N_J(u_1) - V(U'')$. But then if $y_2 = w_2$ or u_2 we have a separating triangle $(u_1y_2y_1)$, and otherwise y_1 violates Claim 4.

(3) Suppose that $x_2 \in N_H(u_2) - V(U')$. After modifying U'' to be $u_0u_1x_1x_2u_2u_3 \dots u_{k+1}$, G' to be $I(u_1u_2x_2x_1)$, and S'_2 to be $\{u_2, u_3, \dots, u_{k+1}\}$, the proof is almost identical to (2) above.

So our assumption was incorrect, and there is a w_1v_3 -path in $H - S$. \square

Let Q be a w_1v_3 -path as in Claim 6 that is as short as possible; then Q is chordless. As Q contains no neighbor of v_1 , u_1 , u_2 or w_2 except w_1 , the paths $v_1u_1w_1 \cup Q$ and $v_1u_2w_2w_1 \cup Q$ are chordless interior v_1v_3 -paths whose lengths have different parities, giving the final contradiction that proves (ii). \blacksquare

Corollary 4.7. *Let G be a cyclically-4-edge-connected cubic graph embedded in a surface, and let C be a cycle of G with a fixed orientation bounding a closed disk D in the surface. Suppose that precisely four edges touch C from outside of D , at vertices u_1, u_2, u_3, u_4 in that order around C , and let $B_i = u_iCu_{i+1}$ (taking $u_5 = u_1$). If all B_1B_3 -face chains through D have length of the same parity, then there is a 4-cycle face contained in D .*

Proof. Let H be the intersection graph of the sets B_1, B_2, B_3, B_4 and the closures of all faces inside D , and for each i let g_i denote the face outside D intersecting D along B_i . Essentially H is a subgraph of the dual of G , but we replace each face g_i by $B_i = \overline{g_i} \cap D$ to make sure that there are four distinct vertices representing g_1, g_2, g_3 and g_4 , even if some of these faces are actually the same. Since G is cubic, H is a near-triangulation, with outer 4-cycle $(B_1B_2B_3B_4)$. Since G is cyclically-4-edge-connected, H is simple and has no separating triangles. Apply Proposition 4.6 (ii) to H . \blacksquare

5 Main theorem

We are now ready to prove our main theorem. The main step is a structural result, Theorem 5.2.

Lemma 5.1. *Suppose Ψ is a projective-planar embedding of a 3-connected cubic graph, with $\rho(\Psi) = m \geq 2$. Then for any noncontractible simple closed curve Γ with $|\Gamma \cap G| = m$, the faces traversed by Γ , in their order along Γ , form a noncontractible elementary face ring.*

Proof. Label the faces along Γ as f_1, f_2, \dots, f_m . Interpreting subscripts modulo m , clearly f_i and f_{i+1} touch for each i . If f_i touches f_j for some $j \neq i-1, i$ or $i+1$, then we can find a noncontractible

closed curve intersecting G in fewer points than Γ , a contradiction. Thus, we have a noncontractible face ring.

If $m \geq 3$ then the face ring is elementary by Observation 2.2, so assume that $m = 2$. If $\partial f_1 \cap \partial f_2$ has three or more components, then we can find a contractible simple closed curve lying in $f_1 \cup f_2$ cutting G at exactly two vertices that form a cutset in G , contradicting 3-connectivity. Thus, the face ring is elementary. ■

Theorem 5.2. *If G is a cyclically-4-edge-connected cubic graph with a 2-representative embedding Ψ in the projective plane, then the embedding has a 4-cycle face or a noncontractible elementary face ring of odd length or both.*

Proof. Assume for a contradiction that G has neither a 4-cycle face nor a noncontractible elementary face ring of odd length. By Lemma 5.1, Ψ has a noncontractible elementary face ring of length $\rho(\Psi)$, so $\rho(\Psi) = 2n$ for some $n \geq 1$. Let this face ring be $\mathcal{F} = (f_1, f_2, \dots, f_{2n})$. Subscripts i for f_i are to be interpreted modulo $2n$.

Since G is cubic and 3-connected, if $n = 1$ then each component of $\partial f_1 \cap \partial f_2$ is a single edge. Similarly, if $n \geq 2$ then each $\partial f_i \cap \partial f_{i+1}$ has one component which is a single edge. Therefore, $F = \bigcup_{i=1}^{2n} f_i$ is a closed Möbius strip bounded by a cycle L . L contains distinct vertices v_1, v_2, \dots, v_{4n} in that order such that $\partial f_{i-1} \cap \partial f_i$ is the edge $v_i v_{2n+i}$. Let $L_i = v_i v_{i+1}$ (subscripts modulo $4n$), so that the boundary of f_i is $L_i \cup L_{2n+i} \cup \{v_i v_{2n+i}, v_{i+1} v_{2n+i+1}\}$. Subscripts i for v_i , L_i and related objects are to be interpreted modulo $4n$. Removing the interior of F from the projective plane leaves a closed disk D , which is the union of the closures of the faces not in \mathcal{F} , and which contains all vertices of G . We assume that L goes around D clockwise, and all cycles contained in D will also be oriented clockwise.

Suppose $\rho(\Psi) = 2$, so that $n = 1$. Every $L_1 L_3$ -face chain through D extends to a noncontractible elementary face ring using Observation 2.1, and is therefore of even length. But then by Corollary 4.7 there is a 4-cycle face contained in D , contradicting the fact that G has no 4-cycles.

Thus, $\rho(\Psi) \geq 4$, so that $n \geq 2$. By Observation 2.2, every face ring is elementary. Thus, every noncontractible face ring has even length. For each i , $1 \leq i \leq 4n$, let \mathcal{D}_i be the set of faces in D whose closures intersect L_i , and $\mathcal{D} = \bigcup_{i=1}^{4n} \mathcal{D}_i$. If $i, j \in \{1, 2, \dots, 4n\}$, let $d(i, j)$ denote the distance between i and j in the cyclic sequence $(1, 2, \dots, 4n)$.

We proceed by proving a sequence of claims. Note that we implicitly use the fact that $\rho \geq 4$. In particular, all face rings we construct are valid if $\rho \geq 4$.

Claim 1. Since G is cubic and 3-connected and since $\rho \geq 3$, for any two faces f and g , $\partial f \cap \partial g$ is either empty or a single edge. ■

Claim 2. If $d(i, j) \geq 3$ then $\mathcal{D}_i \cap \mathcal{D}_j = \emptyset$.

Proof of Claim. Suppose not. Without loss of generality we may assume that $i = 1$ and $4 \leq j \leq 2n+1$. Let $g \in \mathcal{D}_i \cap \mathcal{D}_j$. We may construct a noncontractible simple closed curve Γ passing through $f_1, g, f_j, f_{j+1}, \dots, f_{2n+1} = f_1$ and intersecting G at $2n+3-j < 2n = \rho$ points, a contradiction. ■

Claim 3. For every $d \in \mathcal{D}$, $\partial d \cap L$ has exactly one component, which is a subpath of L with at least one edge.

Proof of Claim. Any component of $\partial d \cap L$ cannot be a single vertex because G has no vertices of degree 4 or more, so it is a subpath of L with at least one edge.

Suppose $\partial d \cap L$ has two distinct components, C_1 and C_2 . Suppose C_1 intersects L_i and C_2 intersects L_j where we choose i and j so that $d(i, j)$ is as small as possible. By Claim 2, $d(i, j) \leq 2$. If $i = j$ then G has a 2-edge-cut, and if $d(i, j) = 1$ then G has a nontrivial 3-edge-cut, contradicting the fact that G is cyclically-4-edge-connected.

Suppose that $d(i, j) = 2$. Without loss of generality assume that $i = 1$ and $j = 3$. Using Claim 1, suppose that $C_1 \cap L_1 = \partial d \cap \partial f_1$ is the edge $x_1 y_1$ where x_1, y_1 occur on that order along L_1 , and

$C_2 \cap L_3 = \partial d \cap \partial f_3$ is the edge $y_2 x_2$ where y_2, x_2 occur in that order along L_3 . Since G is cubic, $x_1 \neq v_1$, $y_1 \neq v_2$, $y_2 \neq v_3$ and $x_2 \neq v_4$. Let D' be the closed disk bounded by the cycle $y_1(\partial d)y_2 \cup y_1 L y_2$. Let $B_1 = y_1 L v_2$ and $B_3 = v_3 L y_2$. Then for every $B_1 B_3$ -face chain $(B_1, g_1, g_2, \dots, g_{k-1}, B_3)$ of length k through D' , we have a noncontractible face ring $(f_1, g_1, g_2, \dots, g_{k-1}, f_3, f_4, \dots, f_{2n})$, which must have even length, so that k is always even. Therefore, by Corollary 4.7 there is a 4-cycle face contained in D' , contradicting the fact that G has no 4-cycles. \square

By Claim 3, the elements of \mathcal{D} can be cyclically ordered along L according to their intersection with L . Within each \mathcal{D}_i we may linear order the elements of \mathcal{D}_i along L_i as $d_{i,1}, d_{i,2}, \dots, d_{i,n_i}$, where $d_{i,1}$ contains v_i and d_{i,n_i} contains v_{i+1} . By Claim 1, n_i is the length of L_i . Possibly $n_i = 1$. Note that $d_{i,n_i} = d_{i+1,1}$. The following is immediate.

Claim 4. If $1 < j < n_i$ and $i \neq k$ then $d_{i,j}$ and f_k do not touch. \square

Claim 5. We do not have $n_i = n_{i+1} = 1$ for any i .

Proof of Claim. If $n_i = n_{i+1} = 1$ then $d_{i-1,n_{i-1}} = d_{i,1} = d_{i+1,1} = d_{i+2,1}$, violating Claim 2. \square

Claim 6. Since G has no nontrivial 3-edge-cut, if $j \leq k - 2$ then $d_{i,j}$ and $d_{i,k}$ do not touch. \square

Claim 7. If $j < n_i$ and $k > 1$ then $d_{i,j}$ and $d_{i+1,k}$ do not touch.

Proof of Claim. Without loss of generality assume that $i = 1$. Suppose that $j < n_1$, $k > 1$, and $d_{1,j}$ and $d_{2,k}$ touch. Let m be the largest m so that $d_{2,n_2} = d_{m,1}$; by Claim 5, $m = 3$ or 4 . Let $\partial d_{1,j} \cap L = x_1 L y_1$ and $\partial d_{2,k} \cap L = y_2 L x_2$. Then y_1 and y_2 are internal vertices of L_1 and L_2 , respectively. Let y_3 be the first vertex of $\partial d_{2,k}$ encountered when travelling clockwise along $\partial d_{1,j}$ from y_1 . Let D' be the closed disk bounded by the cycle $y_1(\partial d_{1,j})y_3 \cup y_3(\partial d_{2,k})y_2 \cup y_1 L y_2$. Let $B_1 = y_1 L v_2$ and $B_3 = y_3(\partial d_{2,k})y_2$. Then for every $B_1 B_3$ -face chain $(B_1, g_1, g_2, \dots, g_{l-1}, B_3)$ of length l through D' , by Claims 4 and 6 we have a noncontractible face ring $(f_1, g_1, g_2, \dots, g_{l-1}, d_{2,k}, d_{2,k+1}, \dots, d_{2,n_2} = d_{m,1}, f_m, f_{m+1}, \dots, f_{2n})$, which must have even length, so that l always has the same parity. Therefore, by Corollary 4.7 there is a 4-cycle face contained in D' , contradicting the fact that G has no 4-cycles. \square

Claim 8. If $d(i, j) \geq 4$ then no face in \mathcal{D}_i touches a face in \mathcal{D}_j .

Proof of Claim. This is similar to the proof of Claim 2. \square

Claim 9. If $n_{i-1} > 1$ and $n_{i+1} > 1$ then n_i is odd. Equivalently, if n_i is even then $n_{i-1} = 1$ or $n_{i+1} = 1$.

Proof of Claim. Without loss of generality assume that $i = 2$. If $n_1 > 1$ and $n_3 > 1$ then, by Claims 4 and 6, $(f_1, d_{1,n_1} = d_{2,1}, d_{2,2}, d_{2,3}, \dots, d_{2,n_2} = d_{3,1}, f_3, f_4, \dots, f_{2n})$ is a noncontractible face ring, which must have even length, so that n_2 must be odd. \square

Claim 10. If $n_{i-1} > 1$ and $n_{i+1} = 1$, or $n_{i-1} = 1$ and $n_{i+1} > 1$, then n_i is even.

Proof of Claim. Without loss of generality assume that $i = 2$, where $n_1 > 1$ and $n_3 = 1$. By Claim 5, $n_4 > 1$. Thus, and by Claims 4 and 6, $(f_1, d_{1,n_1} = d_{2,1}, d_{2,2}, d_{2,3}, \dots, d_{2,n_2} = d_{3,1} = d_{4,1}, f_4, f_5, \dots, f_{2n})$ is a noncontractible face ring, which must have even length, so that n_2 must be even. \square

Claim 11. If $n_{i-1} = n_{i+1} = 1$ then n_i is odd and $n_i \geq 3$.

Proof of Claim. Without loss of generality assume that $i = 3$, and $n_2 = n_4 = 1$. By Claim 5, $n_1 > 1$ and $n_5 > 1$. Thus, and by Claims 4 and 6, $(f_1, d_{1,n_1} = d_{2,1} = d_{3,1}, d_{3,2}, d_{3,3}, \dots, d_{3,n_3} = d_{4,1} = d_{5,1}, f_5, f_6, \dots, f_{2n})$ is a noncontractible face ring, which must have even length, so that n_3 must be odd. By Claim 5, $n_3 \geq 3$. \square

Claim 12. If n_i is even then either $n_{i-1} = n_{i+2} = 1$ and n_{i+1} is even, or $n_{i+1} = n_{i-2} = 1$ and n_{i-1} is even.

Proof of Claim. Without loss of generality assume that $i = 3$. By Claim 9, $n_2 = 1$ or $n_4 = 1$: again without loss of generality assume that $n_2 = 1$. By Claim 5, $n_1 > 1$, and by Claim 11, $n_4 > 1$. If $n_5 > 1$, then n_4 is odd by Claim 9, but then $(f_1, d_{1,n_1} = d_{2,1} = d_{3,1}, d_{3,2}, d_{3,3}, \dots, d_{3,n_3} = d_{4,1}, d_{4,2}, d_{4,3}, \dots, d_{4,n_4} = d_{5,1}, f_5, f_6, \dots, f_{2n})$ is a noncontractible face ring (by Claims 4, 6 and 7) of odd length, a contradiction. Therefore $n_5 = 1$, and from Claim 10, n_4 is even. \square

Claim 13. If $n_i = 1$ then either n_{i+1} and n_{i+2} are even and $n_{i+3} = 1$, or n_{i+1} is an odd number at least 3 and $n_{i+2} = 1$.

Proof of Claim. Without loss of generality assume that $i = 1$. If n_2 is even then by Claim 12 we know that n_3 is even and $n_4 = 1$. If n_2 is odd then by Claim 10 $n_3 = 1$. \square

Claim 14. We cannot have $n_i, n_{i+1}, n_{i+2}, n_{i+3}$ all greater than 1.

Proof of Claim. Without loss of generality assume that $i = 1$. Suppose that $n_1, n_2, n_3, n_4 > 1$. By Claim 9, n_2 and n_3 are odd. By Claims 4, 6 and 7, $(f_1, d_{1,n_1} = d_{2,1}, d_{2,2}, d_{2,3}, \dots, d_{2,n_2} = d_{3,1}, d_{3,2}, d_{3,3}, \dots, d_{3,n_3} = d_{4,1}, f_4, f_5, \dots, f_{2n})$ is a noncontractible face ring of odd length, a contradiction. \square

Suppose now that no n_i is even. By Claim 14, some n_i is equal to 1, say $n_1 = 1$. Since no n_i is even, Claim 13 implies that $n_2, n_4, n_6, \dots, n_{4n}$ are all odd numbers at least 3, while $n_3 = n_5 = \dots = n_{4n-1} = 1$. In particular, $n_1 = n_{2n+1} = 1$, which means that f_1 is a 4-cycle face, a contradiction.

Therefore, some n_i is even, and by Claim 12 either $n_{i-1} = 1$ or $n_{i+1} = 1$. Without loss of generality we may assume that n_3 is even and $n_2 = 1$. By Claim 5, $n_1 > 1$, and by Claim 13, n_4 is even and $n_5 = 1$, so that, by Claim 5 again, $n_6 > 1$. By Claims 4, 6 and 7, $(f_1, d_{1,n_1} = d_{2,1} = d_{3,1}, d_{3,2}, d_{3,3}, \dots, d_{3,n_3} = d_{4,1}, d_{4,2}, d_{4,3}, \dots, d_{4,n_4} = d_{5,1} = d_{6,1}, f_6, f_7, \dots, f_{2n})$ is a noncontractible face ring of odd length, a contradiction, if $\rho \geq 6$.

Therefore, $\rho = 4$. To satisfy Claims 13 and 14, and since some n_i is even, we must have the following situation, or one rotationally equivalent to it: $n_2 = 1$, n_3 even, n_4 even, $n_5 = 1$, n_6 even, n_7 even, $n_8 = 1$, and n_1 odd and at least 3.

By Lemma 5.1, $(f_1, d_{1,n_1} = d_{2,1} = d_{3,1}, f_3, f_4)$ is a noncontractible face ring. The union of the closures of these faces is a closed Möbius strip with boundary cycle L' , which we may divide into 8 subpaths L'_i in the same way that L is divided into subpaths L_i . We see that $L_4, L_8 \subseteq L'$ and we may orient L' and label its subpaths so that $L'_4 = L_4$, $L'_8 = L_8$, and the orientation of L' agrees with that of L on these two subpaths. If n'_i is the length of L'_i we see that $n'_1 = n_1 - 1$ is even, n'_2 is unknown, $n'_3 = n_3 - 1$ is odd, $n'_4 = n_4$ is even, $n'_5 = n_5 + 1 = 2$, $n'_6 = 1$, $n'_7 = n_7 + 1$ is odd, and $n'_8 = n_8 = 1$. By Claim 12, $n'_3 = 1$ so that $n_3 = 2$. By symmetry, $n_7 = 2$ also.

In the same way, $(f_4, d_{4,n_4} = d_{5,1} = d_{6,1}, f_6, f_7)$ is a noncontractible face ring, and we get a closed Möbius strip whose boundary cycle is divided into 8 subpaths of lengths n''_i , $1 \leq i \leq 8$, where we see that $n''_1 = 1$, $n''_2 = n_2 + 1 = 2$, $n''_3 = n_3 = 2$, $n''_4 = n_4 - 1$ is odd, n''_5 is unknown, $n''_6 = n_6 - 1$ is odd, $n''_7 = n_7 = 2$, and $n''_8 = n_8 + 1 = 2$. By Claim 12, $n''_4 = n''_6 = 1$ so that $n_4 = n_6 = 2$.

Now all even numbers n_i (n_3, n_4, n_6, n_7) have been shown to equal 2. By the same reasoning, all even numbers n'_i must be 2. In particular, $n'_1 = n_1 - 1 = 2$, so that $n_1 = 3$.

Since $n_1 = 3$ and $n_3 = n_4 = n_6 = n_7 = 2$ we may write $L = (v_1 w_{1,1} w_{1,2} v_2 v_3 w_3 v_4 w_4 v_5 v_6 w_6 v_7 w_7 v_8)$. For each $u \in \{w_{1,1}, w_{1,2}, w_3, w_4, w_6, w_7\}$, let u' denote the neighbor of u to which u is joined by an edge that is not an edge of L . By Claim 3, no such u' is a vertex of L .

If $d_{3,2}$ does not touch $d_{1,2}$ then $(f_4 = f_8, d_{8,1} = d_{1,1}, d_{1,2}, d_{1,3} = d_{2,1} = d_{3,1}, d_{3,2} = d_{4,1})$ is a noncontractible face ring of length 5, which is odd, a contradiction. So $d_{3,2}$ touches $d_{1,2}$.

Write $\partial d_{1,2} \cap d_{3,2} = x_2 y_2$ where x_2 precedes y_2 in the clockwise order around $\partial d_{3,2}$. By Claim 3, x_2 and y_2 are not vertices of L . If $w'_{1,2}, w'_3$ and y_2 are not all equal, then $\{w_{1,2} w'_{1,2}, w_3 w'_3, x_2 y_2\}$ is a nontrivial 3-edge-cut, a contradiction. Therefore, $y_2 = w'_{1,2} = w'_3$ is a vertex shared by $d_{1,2}$, $d_{1,3} = d_{2,1} = d_{3,1}$, and $d_{3,2}$. Similarly, there is a vertex $y_1 = w'_{1,1} = w'_7$ shared by $d_{1,2}$, $d_{1,1} = d_{8,1} =$

$d_{7,2}$, and $d_{7,1}$, with neighbor $x_1 \neq w_{1,1}, w_7$, where neither x_1 nor y_1 is on L . Since $d_{1,2}$ is not a 3- or 4-cycle face, $y_1 \neq y_2$ and $y_1 y_2 \notin E(G)$, so that $x_1 \neq y_2$ and $x_2 \neq y_1$.

Since $d_{5,1}$ is not a 4-cycle face, $w_4 w_6 \notin E(G)$. Since the neighbors of y_1 are x_1, w_7 and $w_{1,1}$, y_1 is adjacent to neither w_4 nor w_6 ; similarly, y_2 is adjacent to neither w_4 nor w_6 . Thus, $\{y_1, y_2, w_4, w_6\}$ is an independent set and x_1, x_2, w'_4, w'_6 do not belong to this set.

Now there is a contractible simple closed curve intersecting the embedding at precisely four points, one interior point of each of the four edges $y_1 x_1, y_2 x_2, w_4 w'_4, w_6 w'_6$, in that order. This curve bounds an open disk Δ containing x_1, x_2, w'_4 and w'_6 . Let H be the subgraph of G induced by $V(G) \cap \Delta$. Since G is cyclically-4-edge-connected, H is either a single edge, or is a 2-connected graph embedded in a closed disk D' bounded by a cycle C' through distinct vertices x_1, x_2, w'_4, w'_6 in that order (and possibly containing other vertices). If H is a single edge, then either $x_1 = x_2, w'_4 = w'_6$, $x_1 w'_4 \in E(G)$, and there is a noncontractible simple closed curve through $f_3, d_{3,2}$ and $d_{7,1}$ that intersects G at only 3 points; or $x_1 = w'_6, x_2 = w'_4, x_1 w'_4 \in E(G)$, and there is a noncontractible simple closed curve through $f_1, d_{1,2}$ and $d_{5,1}$ that intersects G at only 3 points. In either case we have a contradiction, so H is 2-connected and we have D' and C' as described above. Let $B_1 = x_1 C' x_2 = x_2 (\partial d_{1,2}) x_1$ and $B_3 = w'_4 C' w'_6 = w'_6 (\partial d_{5,1}) w'_4$. Then for every $B_1 B_3$ -face chain $(B_1, g_1, g_2, \dots, g_{l-1}, B_3)$ of length l through D' , $(f_1, d_{1,2}, g_1, g_2, \dots, g_{l-1}, f_{5,1})$ is a noncontractible face ring, which must have even length, so that l is always even. Therefore, by Corollary 4.7 there is a 4-cycle face contained in D' , a contradiction.

Since every possibility leads to a contradiction, our original assumption must be wrong, and Ψ does have either a 4-cycle face or a noncontractible elementary face ring of odd length. ■

Now we prove the main result, which we restate.

Theorem 1.2. *Every 2-connected projective-planar cubic graph has a closed 2-cell embedding in some orientable surface.*

Proof. Suppose G is a 2-connected cubic graph with a projective-planar embedding Ψ but with no orientable closed 2-cell embedding and that, subject to these conditions, G has a minimum number of vertices. By Corollary 4.3, G is simple, cyclically-4-edge-connected (hence 3-connected), and has no 3- or 4-cycles.

If $\rho(\Psi) \leq 1$ then G can be embedded in the plane [9, 14], and hence has a spherical closed 2-cell embedding. Otherwise, by Theorem 5.2, since G has no 4-cycle, Ψ must have a noncontractible elementary face ring of odd length. Then by Theorem 3.4, G has an orientable closed 2-cell embedding, as required. ■

We would like to strengthen Theorem 1.2 to say that we have a k -face-colorable orientable closed 2-cell embedding for some fixed small k . Then we could improve Corollary 1.3 to say that we have an orientable k -cycle double cover (the cycles can be colored using at most k colors so that each edge is contained in two cycles of different colors). However, this seems difficult. The obstacle is our 4-cycle reduction: we see no obvious way to avoid increasing the number of face colors needed when $f_1 \neq g_2$ and $f_2 \neq g_1$ in the proof of Lemma 4.2.

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