

# Minimal quadrangulations of surfaces

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In memory of Nora Hartsfield

## Abstract

A *quadrangular* embedding of a graph in a surface  $\Sigma$ , also known as a *quadrangulation of*  $\Sigma$ , is a cellular embedding in which every face is bounded by a 4-cycle. A quadrangulation of  $\Sigma$  is *minimal* if there is no quadrangular embedding of a (simple) graph of smaller order in  $\Sigma$ . In this paper we determine  $n(\Sigma)$ , the order of a minimal quadrangulation of a surface  $\Sigma$ , for all surfaces, both orientable and nonorientable. Letting  $S_0$  denote the sphere and  $N_2$  the Klein bottle, we prove that  $n(S_0) = 4$ ,  $n(N_2) = 6$ , and  $n(\Sigma) = \lceil (5 + \sqrt{25 - 16\chi(\Sigma)})/2 \rceil$  for all other surfaces  $\Sigma$ , where  $\chi(\Sigma)$  is the Euler characteristic. Our proofs use a ‘diagonal technique’, introduced by Hartsfield in 1994. We explain the general features of this method.

*Keywords:* surface, quadrangular embedding, minimal quadrangulation

## 1 Introduction

All graphs considered in this paper are simple. Let  $G$  be a graph with vertex set  $V(G)$  and edge set  $E(G)$ . For convenience, we use  $E_k$  to denote a subset of  $E(G)$  with exactly  $k$  edges. A *surface* is a connected compact 2-manifold without boundary. The orientable surface of genus  $g$  is denoted  $S_g$ , and the nonorientable surface of genus  $q$  is denoted  $N_q$ . The Euler characteristic of a surface  $\Sigma$  is denoted  $\chi(\Sigma)$ , which is  $2 - 2g$  for  $S_g$ , and  $2 - q$  for  $N_q$ . The *Euler genus* of  $\Sigma$  is defined as  $\gamma(\Sigma) = 2 - \chi(\Sigma)$ .

An embedding of a graph in a surface  $\Sigma$  is *cellular* if every face of the embedding is homeomorphic to an open disc. All embeddings considered in the paper are cellular. An embedding is *quadrangular*, or a *quadrangulation of*  $\Sigma$ , if every face is bounded by a 4-cycle. A face bounded by a 4-cycle is a *quadrangle*, or to use a shorter word, a *square*. A quadrangulation of  $\Sigma$  is *minimal* if there is no quadrangular embedding of a graph of smaller order in  $\Sigma$ . Similarly, a *triangulation of*  $\Sigma$  is an embedding of a graph in  $\Sigma$  such that every face is bounded by a 3-cycle. A triangulation of  $\Sigma$  is *minimal* if there is no triangular embedding of a graph with a smaller order in  $\Sigma$ .

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Thomassen [26, 27] showed that given a graph  $G$  and an integer  $k$ , it is NP-complete to determine whether  $G$  has an embedding in a surface of orientable (or nonorientable) genus at most  $k$ . In other words, determining the minimum genus of an embedding of  $G$  is difficult. A minimum genus embedding of a graph maximizes the number of faces over all its embeddings, and hence often has many small faces. Triangular embeddings of a given  $G$  are always minimum genus embeddings. However, we can also consider triangular embeddings from the perspective of surfaces. Peschl (see [9]) asked how many vertices a triangulation of a given surface  $\Sigma$  must have.

A triangular embedding of a complete graph  $K_n$  in a given surface  $\Sigma$  is both a minimum genus embedding of  $K_n$ , and a minimum order triangulation of  $\Sigma$ . Such embeddings played a key role in the proof of the Map Color Theorem (see [25]). These embeddings were generalized in two ways. For some values of  $n$ , there is no triangular embedding of  $K_n$ , so to determine the minimum genus of  $K_n$ , embeddings were used where most, but not all, of the faces are triangular (again see [25]). For most surfaces  $\Sigma$ , there is no complete graph with a triangular embedding in  $\Sigma$ , so to find minimal triangulations of  $\Sigma$  we must use graphs close to complete graphs. Ringel [22] did this for nonorientable surfaces, and Jungerman and Ringel [9] for orientable surfaces.

Quadrangular embeddings are also of interest. For bipartite graphs, quadrangular embeddings have minimum genus. For non-bipartite graphs, quadrangular embeddings have minimum genus over all embeddings with face degrees 4 or more, or with even face degrees. Ringel [23, 24] determined the minimum genus of complete bipartite graphs, which used quadrangular embeddings in many cases; Bouchet [1] provided a simpler proof. Quadrangular embeddings of nearly complete bipartite graphs and graphs obtained from some graph operations were studied in [2, 19, 20, 21, 28, 29]. Hartsfield and Ringel [6, 7] found quadrangular embeddings of the complete graph  $K_n$  in orientable surfaces for  $n \equiv 5 \pmod{8}$ , and in nonorientable surfaces for  $n \equiv 1 \pmod{4}$  and  $n \neq 1, 5$ . They also found both orientable and nonorientable quadrangular embeddings of the general octahedral graph  $O_{2n}$ , obtained by removing a perfect matching from  $K_{2n}$ . Using the ‘diagonal technique’, which we discuss in more detail below, Hartsfield [4] outlined a proof that a complete multipartite graph  $K_{n_1, n_2, \dots, n_t}$  with an even number of edges, other than  $K_5$  and  $K_{1, m, n}$ , has a quadrangular embedding in a nonorientable surface. This includes nonorientable quadrangular embeddings of  $K_n$  when  $n \equiv 0 \pmod{4}$ . Korzhik and Voss [10, 11] constructed exponentially many nonisomorphic quadrangular embeddings of the complete graph  $K_{8s+5}$ .

Recently, the authors and others [15] determined the minimum genus of an embedding of  $K_n$  with even face degrees. (Lawrencenko, Chen and Yang, a subset of the authors of [15], also have alternative current graph proofs [13] of some of these results, although some modification of the index 2 current graphs is required.) This completed the proof of the Even Map Color Theorem, a strengthening of the Map Color Theorem for embeddings with even face degrees, and included a complete characterization of when  $K_n$  has a quadrangular embedding.

**Theorem 1.1** ([6, 7, 15]). *The complete graph  $K_n$  has a quadrangular embedding in an orientable surface if and only if  $n \equiv 0$  or  $5 \pmod{8}$ , and in a nonorientable surface if and only if  $n \equiv 0$  or  $1 \pmod{4}$  and  $n \neq 1, 5$ .*

The quadrangular embeddings of the complete graphs  $K_n$  and the general octahedral graphs  $O_{2n}$  given in [6, 7, 15] are all minimal quadrangulations of surfaces. Other prior results on minimal

quadrangulations, for which we provide details later in this section, appear in [6, 12, 15]. The purpose of this paper is to construct, and hence determine the order of, minimal quadrangulations for all surfaces. Our main results are as follows; Theorem 1.2 provides the embeddings needed to prove Theorems 1.3 and 1.4.

**Theorem 1.2.** *Let  $(n, t)$  be a pair of integers with  $n \geq 4$  and  $0 \leq t \leq n - 4$ .*

*If  $t \equiv \frac{1}{2}n(n - 5) \pmod{4}$  then there is an orientable quadrangular embedding of an  $n$ -vertex graph with  $\binom{n}{2} - t$  edges. There is also a quadrangulation of  $S_0$  for  $(n, t) = (4, 2)$ .*

*If  $t \equiv \frac{1}{2}n(n - 5) \pmod{2}$  then there is a nonorientable quadrangular embedding of an  $n$ -vertex graph with  $\binom{n}{2} - t$  edges, unless  $(n, t) = (5, 0)$ , in which case no such embedding exists. There is also a quadrangulation of  $N_2$  for  $(n, t) = (6, 3)$ .*

**Theorem 1.3.** *Let  $\Sigma$  be a surface with Euler characteristic  $\chi(\Sigma)$  and Euler genus  $\gamma(\Sigma)$ . Let  $n(\Sigma)$  be the number of vertices of a minimal quadrangulation of  $\Sigma$ . If  $\Sigma \neq S_0$  and  $N_2$ , then*

$$n(\Sigma) = \left\lceil \frac{5 + \sqrt{25 - 16\chi(\Sigma)}}{2} \right\rceil = \left\lceil \frac{5 + \sqrt{16\gamma(\Sigma) - 7}}{2} \right\rceil.$$

Moreover,  $n(S_0) = 4$  and  $n(N_2) = 6$ .

An embedding is *face-simple* if its dual is simple, i.e., two face boundaries share at most one edge. We can strengthen Theorem 1.3 slightly in the orientable case.

**Theorem 1.4.** *Let  $n'(\Sigma)$  be the minimum number of vertices of a face-simple quadrangular embedding of a simple graph in  $\Sigma$ . Then  $n'(S_0) = 8$  and  $n'(S_g) = n(S_g)$  for all  $g \geq 1$ .*

We show in Section 2 that all quadrangulations given in Theorem 1.2 are minimal, and that this proves Theorems 1.3 and 1.4. The main tool used to prove Theorem 1.2 is an approach due to Hartsfield, which we call the ‘diagonal technique’ and describe in Section 3. As we explain there, Hartsfield wrote two papers (one published, one not) using this idea, but her papers did not contain complete proofs. One of the contributions of this paper is to provide an explicit overview of how the diagonal technique works, and to demonstrate the rigorous use of this method. The actual proof of Theorem 1.2 is in Section 4, divided into orientable and nonorientable cases. Section 5 gives some final remarks regarding Theorem 1.4.

Prior results on minimal quadrangulations proved some special cases of Theorem 1.2, constructing quadrangulations with  $n$  vertices and  $\binom{n}{2} - t$  edges for suitable  $t$ . Theorem 1.1 deals with the case  $t = 0$ , and Hartsfield and Ringel’s results on octahedral graphs [6, 7] deal with the case where  $n$  is even and  $t = n/2$ . They also proved the orientable case when  $n$  is even and  $t = n/2 + 4$  [6, Section 6]. Lawrencenko [12] used a result originally due to White [29], which can also be proved using Craft’s graphical surface technique [2], to prove the orientable cases where  $n$  is even and  $n/2 \leq t \leq n - 4$ . Liu et al. [15, Corollary 7.2] extended this idea to prove the nonorientable cases where  $n$  is even,  $n/2 \leq t \leq n - 4$ , and  $t \equiv \frac{1}{2}n(n - 5) \pmod{4}$  (but not  $t \equiv 2 + \frac{1}{2}n(n - 5) \pmod{4}$ ). Moreover, [15, Corollary 7.3] handles all cases (orientable and nonorientable) where  $8 \mid n$  and  $16 \mid t$ , and [15, Corollary 7.4] handles all nonorientable cases where  $t = n - 4$  and all orientable cases where  $n - 6 \leq t \leq n - 4$ .

Note that Magajna, Mohar and Pisanski [18] solved a problem related to minimal quadrangulations, by showing that for every surface  $\Sigma$  the minimum number of vertices of a bipartite graph with a quadrangular embedding in  $\Sigma$  is  $\lceil 4 + \sqrt{16 - 8\chi(\Sigma)} \rceil$ .

## 2 Relationship between the main theorems

In this section we show that the quadrangulations described in Theorem 1.2 are minimal, and that Theorem 1.2 implies Theorems 1.3 and 1.4. Suppose we have a quadrangular embedding in a surface  $\Sigma$  with  $n$  vertices,  $m = \binom{n}{2} - t$  edges, and  $r$  faces. Counting sides of edges gives  $2m = 4r$  or  $r = \frac{1}{2}m$ , and so from Euler's formula  $\chi(\Sigma) = n - m + r = n - \frac{1}{2}m$ . Hence

$$-2\chi(\Sigma) = m - 2n = \binom{n}{2} - t - 2n = \frac{1}{2}n(n - 5) - t. \quad (1)$$

We have a sufficient condition for such an embedding to be minimal.

**Lemma 2.1** ([15, Lemma 7.1]). *Suppose that  $n \geq 5$ ,  $0 \leq t \leq n - 4$ , and  $L$  is a graph with  $n$  vertices and  $m = \binom{n}{2} - t$  edges. Then any quadrangular embedding of  $L$  is minimal.*

We now consider properties of the right-hand side of (1).

**Lemma 2.2.** *Let  $f(x) = \frac{1}{2}x(x - 5)$ , defined on  $[3, \infty)$ .*

(a) *If  $n \geq 3$  is an integer, then  $f(n)$  is an integer.*

(b) *For every real number  $y > f(3) = -3$  there exists a unique pair  $(n, t)$  where  $n \geq 4$  is an integer,  $0 \leq t < n - 3$ , and  $y = f(n) - t$ . Moreover,  $n = \lceil f^{-1}(y) \rceil = \left\lceil \frac{5 + \sqrt{25 + 8y}}{2} \right\rceil$ .*

(c) *Therefore, if  $k \geq -2$  is an integer there exists a unique pair of integers  $(n, t)$  where  $n \geq 4$ ,  $0 \leq t \leq n - 4$ , and  $k = f(n) - t$  (or  $\frac{1}{2}n(n - 5) = t + k$ ). Moreover,  $n = \left\lceil \frac{5 + \sqrt{25 + 8k}}{2} \right\rceil$ .*

*Proof.* For (a), if  $n$  is an integer then  $2f(n) = n(n - 5)$  is even, so  $f(n)$  is an integer. For (b), since  $f'(x) = x - \frac{5}{2} > 0$  on  $[3, \infty)$ ,  $f$  is strictly increasing, and clearly  $f(x) \rightarrow \infty$  as  $x \rightarrow \infty$ . Therefore, every  $y > f(3)$  lies in a unique interval  $(f(n - 1), f(n)] = (f(n) - (n - 3), f(n)]$  for some integer  $n \geq 4$ , so that  $y = f(n) - t$  where  $0 \leq t < n - 3$ . Moreover,  $n - 1 < f^{-1}(y) \leq n$  so that  $n = \lceil f^{-1}(y) \rceil$ , and  $f^{-1}(y)$  is found by the quadratic formula. Now (c) follows from (a) and (b).  $\square$

*Proof that Theorem 1.2 implies Theorem 1.3.* A quadrangulation has  $n \geq 4$  vertices, so the special case  $(n, t) = (4, 2)$  and regular case  $(n, t) = (4, 0)$  of Theorem 1.2 verify Theorem 1.3 for  $\Sigma = S_0$  and  $N_1$ , respectively. By equation (1) a quadrangulation of  $N_2$  must have  $\frac{1}{2}n(n - 5) = -2\chi(N_2) + t = t \geq 0$ , so  $n \geq 5$ , and if  $n = 5$  then  $t = 0$ . By Theorem 1.2 there is no quadrangulation of  $N_2$  for  $(n, t) = (5, 0)$ , so the one for  $(n, t) = (6, 3)$  is minimal, verifying Theorem 1.3 for  $\Sigma = N_2$ .

So assume  $\Sigma \neq N_2$  is a surface with  $\chi = \chi(\Sigma) \leq 0$ . Applying Lemma 2.2(c) with  $k = -2\chi \geq 0$ , there are integers  $n$  and  $t$  that satisfy equation (1) (or  $\frac{1}{2}n(n - 5) = t - 2\chi$ ) and  $0 \leq t \leq n - 4$ . Moreover,  $n = \left\lceil \frac{5 + \sqrt{25 - 16\chi}}{2} \right\rceil \geq 5$ . If  $\Sigma$  is orientable then  $\chi$  is even, so  $\frac{1}{2}n(n - 5) = t - 2\chi \equiv t \pmod{4}$ . Thus, the first part of Theorem 1.2 gives an orientable quadrangulation with  $n$  vertices and  $\binom{n}{2} - t$  edges. This is embedded in  $\Sigma$  by (1), minimal by Lemma 2.1, and has order  $n$  satisfying Theorem 1.3. We use the second part of Theorem 1.2 in a similar way when  $\Sigma$  is nonorientable.  $\square$

We also verify Theorem 1.4, using the following.

**Observation 2.3** ([15, Observation 3.4]). *Suppose  $\Phi$  is a quadrangular embedding of a simple connected graph with minimum degree at least 3. If  $\Phi$  is not face-simple then it contains two squares of the form  $uvwx$  and  $wxvw$ . Thus, if  $\Phi$  is orientable or bipartite then it is face-simple.*

*Proof of Theorem 1.4.* Suppose  $\Phi$  is a face-simple quadrangulation of  $S_0$  with  $n$  vertices,  $m$  edges and  $r$  faces. By Euler's formula  $m = 2n - 4$  and  $r = \frac{1}{2}m = n - 2$ . Let  $H$  be the underlying graph of the dual  $\Phi^*$ . Then  $r = |V(H)| \geq 6$ , because  $H$  is a 4-regular simple graph that is planar and hence not  $K_5$ . Hence  $n = r + 2 \geq 8$ , and the usual quadrangular embedding of the cube (whose dual is the octahedron, which is simple) has  $n = 8$ . Thus,  $n'(S_0) = 8$ .

For  $g \geq 1$ , we know from above that there is a minimal quadrangulation  $\Phi$  of  $S_g$  as in Lemma 2.1. Since the underlying graph is obtained from a complete graph by deleting at most  $n - 4$  edges, the edge-connectivity, and hence also the minimum degree, is at least  $(n - 1) - (n - 4) = 3$ , and so  $\Phi$  is face-simple by Observation 2.3. Thus,  $n'(S_g) = n(S_g)$ .  $\square$

### 3 Hartsfield's diagonal technique

In this section, we describe the operations that form part of a method introduced by Hartsfield [4, 5], which we will call the *diagonal technique*. This technique applies specifically to constructing embeddings that are quadrangular, or where most faces are squares (quadrangles).

This technique was used by Hartsfield in two papers, one published [4] and one not [5]. In [4] Hartsfield claimed to construct nonorientable quadrangular embeddings of almost all complete multipartite graphs with an even number of edges, except for  $K_5$  and complete tripartite graphs  $K_{1,m,n}$ . This included complete graphs  $K_n$  with  $n \equiv 0 \pmod{4}$  (for  $n \equiv 1 \pmod{4}$  Hartsfield used embeddings from [7]). In [5] Hartsfield claimed to construct both orientable and nonorientable minimum genus embeddings with all faces of degree 4 or more for  $K_n$ ,  $n \geq 4$ . Unfortunately Hartsfield did not provide rigorous proofs in either of these papers. She illustrated the proof ideas with small examples, and seemed to assume that it was clear how to generalize these. But she did not provide an explicit overview of how the constructions are supposed to work and so from her papers it is hard to see how to extend the small examples. Sadly, Hartsfield died in 2011, so she cannot provide further elucidation. But we have distilled the main ideas from what she wrote, and we provide an outline at the end of this section, after defining necessary concepts and operations.

We hope that rigorous versions of Hartsfield's proofs will appear eventually. Lawrencenko et al. [14] are preparing a paper that gives alternative proofs for the main result of [15], which determined the minimum genus for orientable and nonorientable embeddings with all faces of even degree, and with all faces of degree at least 4, for complete graphs. This includes Theorem 1.1 as a special case. These alternative proofs are based both on current graphs as in [13], and on Hartsfield's proof outlines from [5] which use the diagonal technique.

As a byproduct, our results in this paper also provide a proof of Theorem 1.1 using Hartsfield's diagonal technique. However, since our goal is the construction of minimal quadrangulations, rather than embeddings of complete graphs with minimum genus subject to all faces having degree at least

4, our proof differs significantly from those in [4, 5, 14]. We add two vertices at a time, rather than eight, and we use additional operations (handle additions of Type III and crosscap additions; see below).

We now introduce some definitions that we will need to implement the diagonal technique. Let  $\Phi$  be a quadrangulation of a surface  $\Sigma$ . Every face of  $\Phi$  is a square, bounded by a 4-cycle. We describe squares by listing their vertices in order around the boundary. For an orientable embedding, we always list the vertices in clockwise order.

Two nonadjacent vertices  $v_i$  and  $v_j$  of a square form a *diagonal*, denoted by  $di(v_iv_j)$ . The square is called the *underlying square* of  $di(v_iv_j)$ . Clearly, each diagonal depends on an underlying square and this underlying square may be not unique. For example, in Figure 1,  $di(v_1v_3)$  has three different underlying squares. When it will not cause confusion, we pinpoint a diagonal but do not explicitly mention its underlying square. If we do wish to indicate the underlying square, we write  $di(v_iv_j, v_kv_\ell v_jv_\ell)$ . A *diagonal set* is a set of diagonals that have different underlying squares. A diagonal set is *full* if it contains at least one diagonal incident with every vertex, *perfect* if it contains exactly one diagonal incident with every vertex, and  *$v_i$ -nearly-perfect* if it contains no diagonal incident with  $v_i$  and exactly one diagonal incident with every vertex not equal to  $v_i$ .

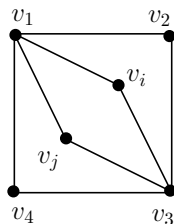


Figure 1: Disc addition

### Operation 1: Disc addition

Let  $f = v_1v_2v_3v_4$  be a square of the quadrangulation  $\Phi$  of a surface  $\Sigma$ . Add two new vertices  $v_i$  and  $v_j$  into the interior of the square  $f$ , and then join  $v_i$  and  $v_j$  to the diagonal  $di(v_1v_3)$  of  $f$  by four new edges  $v_1v_i$ ,  $v_1v_j$ ,  $v_3v_i$  and  $v_3v_j$ . The square  $f$  is divided into three new squares  $f_1 = v_1v_2v_3v_i$ ,  $f_2 = v_1v_iv_3v_j$  and  $f_3 = v_1v_jv_3v_4$ . All the new squares  $f_i$  with  $i \in \{1, 2, 3\}$  have  $di(v_1v_3)$  as one of their diagonals. This operation is called a *disc addition* (Hartsfield called this a *planar addition*). Applying a disc addition to a square of  $\Phi$  generates a new quadrangulation of the same surface  $\Sigma$ , with the same genus. See Figure 1 for an illustration. Disc additions preserve  $di(v_1v_3)$  as a diagonal, although the underlying square may change. Usually we add  $di(v_iv_j)$  to the current diagonal set.

### Operation 2: Handle addition

Let  $f_1 = v_1v_2v_3v_4$  and  $f_2 = u_1u_2u_3u_4$  be two squares of  $\Phi$ . First, cut the open discs bounded by  $f_1$  and  $f_2$  along their boundaries and remove them from the surface  $\Sigma$ . Second, add a handle (cylinder) by identifying its two boundaries with the boundaries of  $f_1$  and  $f_2$  respectively. Finally, add four new edges on the handle, each joining a vertex of  $f_1$  to a vertex of  $f_2$ , so that all resulting faces are squares. This operation is called a *handle addition*. The resulting embedding is also a

quadrangular embedding. After applying a handle addition, the genus of the new quadrangular embedding increases by one if  $\Sigma$  is orientable, and by two if  $\Sigma$  is nonorientable.

We represent handle additions by the planar diagrams shown in Figure 2. One of the two original squares is called the *outer square* ( $f_1 = v_1v_2v_3v_4$  in Figure 2) and the other is called the *inner square* ( $f_2 = u_1u_2u_3u_4$  in Figure 2). The handle is the annular region between the outer and inner squares. If the initial embedding is nonorientable, we may use the vertices of the inner and outer squares in either order, as convenient, and the resulting embedding is always nonorientable.

If the initial embedding is orientable we must take more care. Usually we want the resulting embedding to also be orientable. When we add a handle to an orientable surface  $\Sigma$  to create a new orientable surface, a given direction around the handle corresponds to the clockwise direction in  $\Sigma$  at one end of the handle and the counterclockwise direction in  $\Sigma$  at the other end. In particular, consider the clockwise direction around a handle as represented in the figure. We assume this corresponds to the clockwise direction in  $\Sigma$  for the outer square; then it must correspond to the counterclockwise direction in  $\Sigma$  for the inner square. So if  $v_1v_2v_3v_4$  and  $u_1u_2u_3u_4$  are in clockwise order in  $\Sigma$ , the figure has  $v_1v_2v_3v_4$  and  $u_1u_4u_3u_2$  in clockwise order. For the new squares created by a handle addition, the clockwise order of their vertices in the new surface is the clockwise order in the figure. If we did use clockwise order  $u_1u_2u_3u_4$  for the inner square in the figure, we would add a twisted handle, and the new embedding would be nonorientable.

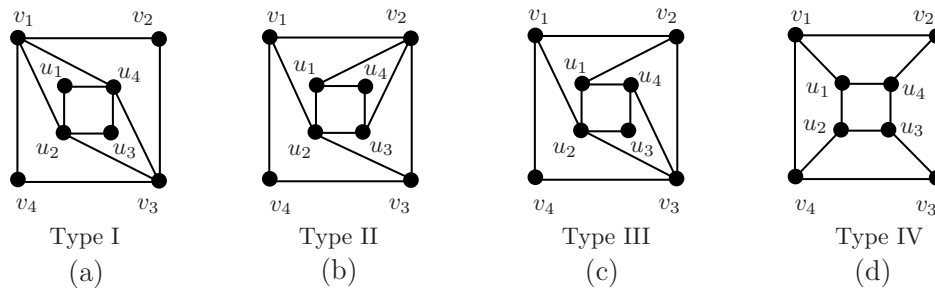


Figure 2: The four types of handle additions

For two given squares, there are four different types of handle additions between them based on the new edge connections, which are listed in Figure 2. If we wish to be specific, we use the labels in the figure and refer to a *handle addition of Type I, II, III or IV*, as appropriate. (Our Types I, II and IV correspond to Hartsfield's Types 1, 2 and 3, respectively. Hartsfield did not use handle additions of Type III.) Handle additions of Types I and III preserve  $di(v_1v_3)$  and  $di(u_2u_4)$  as diagonals, although the underlying squares may change. Handle additions of Type II preserve  $di(v_1v_3)$  and  $di(u_1u_3)$ . Handle additions of Type IV are not guaranteed to preserve diagonals of  $f_1$  or  $f_2$ .

### Operation 3: Crosscap addition

Let  $f = v_1v_2v_3v_4$  be a square of  $\Phi$ . Cut a disc from the interior of the square  $f_1$ , which leaves the surface  $\Sigma$  with a hole. Then identify antipodal points of the boundary of the hole, which generates a crosscap inside of  $f$ . Finally, add two new edges  $v_1v_3$  and  $v_2v_4$  passing through the

new crosscap. This operation is called a *crosscap addition*. See Figure 3. The resulting embedding is a nonorientable quadrangular embedding, and the Euler characteristic decreases by one (so the genus increases by one if the original embedding was also nonorientable). Neither diagonal of  $f$  is a diagonal of either of the new squares. (Hartsfield did not use crosscap additions.)

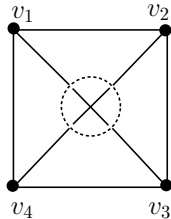


Figure 3: Crosscap addition

### Outline of the diagonal technique

We now provide a brief overview of the diagonal technique, using the concepts and operations defined above. The idea is to replace squares in a known embedding by new squares while adding edges and sometimes also vertices. In this paper the known embedding will be a quadrangulation, but in general it may have some faces that are not squares. All or most of the vertices of the known embedding are partitioned into a diagonal set of vertex pairs, so that the vertices in each pair occur as diagonally opposite vertices in an existing square. New vertices are first added in pairs using disc additions, adding pairs to the diagonal set. Then most edges incident with the new vertices are added using handle additions of Type I. Each such addition usually takes two pairs of vertices in the current diagonal set, uses an underlying square of one pair as the outer square and an underlying square of the other pair as the inner square, and adds four edges between the two pairs.

A small number of edges may be added using handle additions of Types II, III and IV and (in the nonorientable case) crosscap additions. Often the exact details (in particular, which underlying square is used for each diagonal pair) of the handle additions of Type I do not matter, so they can be done in a fairly arbitrary way, except that the necessary faces must be constructed for any additions of handles of Types II, III and IV and crosscaps.

Hartsfield's diagonal technique is particularly suited for constructing minimal quadrangulations, because it adds four (or sometimes two) edges at a time, which is precisely what we need to get the embeddings described in Theorem 1.2. The 'graphical surface' construction due to Craft [2], which was used in [12, 15] to construct some minimal quadrangulations, can be regarded as a special case of the diagonal technique. It is just the case where we start with an embedding of  $C_4$  in the sphere and use only disc additions and handle additions of Type I that preserve the current diagonal set.

Hartsfield's diagonal technique belongs to a more general class of methods that construct embeddings, particularly orientable embeddings, by adding handles, sometimes called *tubes*, that carry specific edges. We mention a few examples of such methods. White [28] and Pisanski [20] (see also [3, Subsection 3.5.4]) used tubes to construct orientable quadrangular embeddings of cartesian products of bipartite graphs; their operations are similar to our handle addition of Type IV. Lv and Chen [16] used handle insertions where each handle carries four or five edges to construct minimum

orientable genus embeddings of  $K_{n,n,1}$  when  $n$  is odd. Ma and Ren [17] used tubes, typically added between triangular faces and carrying five edges, to construct orientable minimum genus embeddings of  $C_m + K_n$  for  $n \geq 5$  and  $m \geq 6n - 13$ .

## 4 Proof of Theorem 1.2

In this section, we focus on the proof of Theorem 1.2, which is divided into two major cases – orientable surfaces and nonorientable surfaces. Apart from a few small cases, our proof is self-contained and does not rely on earlier constructions of minimal quadrangulations.

Denote the vertex set of a graph (or an embedding) of order  $n$  by  $\{v_1, v_2, \dots, v_n\}$ . We know that in a disc addition or a handle addition of Type I, if a diagonal of a diagonal set  $\mathcal{D}$  is used, then at least two new squares with the same diagonal are obtained. So this diagonal still occurs in the resulting embedding. We then replace the underlying square of the diagonal in  $\mathcal{D}$  by an arbitrary choice of one of the two new squares unless otherwise stated. For convenience, the resulting diagonal set is still denoted by  $\mathcal{D}$ . Such situations occur frequently in the proof of Theorem 1.2.

Except in some small cases, Theorem 1.2 does not mention the surface in which an embedding occurs. The surface can always be determined from equation (1), taking orientability into account.

### Orientable surfaces

Let  $Q(n, t)$  denote an orientable quadrangular embedding of a simple graph with  $n$  vertices and  $\binom{n}{2} - t$  edges, for any integers  $n \geq 1$  and  $t$  with  $0 \leq t \leq \binom{n}{2}$ . Let  $T_n = \{t \mid 0 \leq t \leq n - 4, t \equiv \frac{1}{2}n(n - 5) \pmod{4}\}$  for each integer  $n \geq 4$ . The elements of  $T_n$  form an arithmetic progression with difference 4. The condition  $t \equiv \frac{1}{2}n(n - 5) \pmod{4}$  is equivalent to  $8 \mid n(n - 5) - 2t$  and so we must consider the value of  $n \pmod{8}$  to determine  $T_n$ . We must show that there exist a  $Q(4, 2)$  in  $S_0$  and  $Q(n, t)$  for each  $n \geq 4$  and  $t \in T_n$ . We divide the proof into the cases where  $n$  is even and odd.

Recall that when working with orientable surfaces we must pay close attention to the clockwise order of vertices around each square.

**Lemma 4.1.** *There exist a  $Q(4, 2)$  and  $Q(n, t)$  for each even  $n \geq 4$  and  $t \in T_n$ .*

*Proof.* Clearly, a 4-cycle is a quadrangulation  $Q(4, 2)$  of the sphere  $S_0$ . We have  $T_4 = \{t \mid 0 \leq t \leq 0, t \equiv 2 \pmod{4}\} = \emptyset$  and  $T_6 = \{t \mid 0 \leq t \leq 2, t \equiv 3 \pmod{4}\} = \emptyset$ , so there is nothing else to prove for  $n \leq 6$ . For  $n \geq 8$  we proceed inductively.

**Basis.** There exist  $Q(8, t)$  for all  $t \in T_8$ . In particular, there exists a  $Q(8, 0)$  with a perfect diagonal set.

We have  $T_8 = \{0, 4\}$ . Hartsfield and Ringel [6] gave a quadrangular embedding  $\Phi_8$  of  $K_8$  in  $S_4$ , shown in Figure 4. This is the required  $Q(8, 0)$ , with a perfect diagonal set using the squares shaded in the figure, namely

$$\mathcal{D}_8 = \{di(v_1v_2, v_1v_6v_2v_5), di(v_3v_4, v_3v_7v_4v_8), di(v_5v_6, v_5v_8v_6v_4), di(v_7v_8, v_7v_1v_8v_2)\}.$$

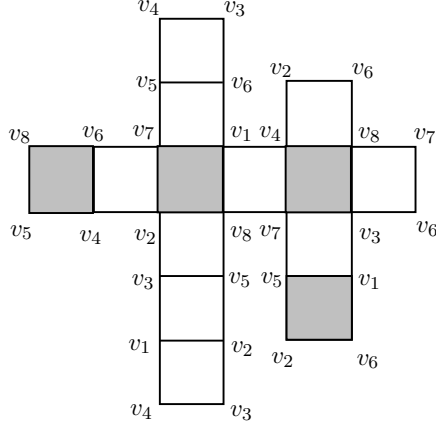


Figure 4

Also in [6], Hartsfield and Ringel constructed a quadrangular embedding of the octahedral graph  $O_8$ , which is the required  $Q(8, 4)$ .

**Induction step.** Given a  $Q(n, 0)$  with a perfect diagonal set, where  $n = 8k$ ,  $k \geq 1$ , there exist  $Q(p, t)$  for all  $p \in \{n + 2, n + 4, n + 6, n + 8\}$  and  $t \in T_p$ . In particular, there exists a  $Q(n + 8, 0)$  with a perfect diagonal set.

Suppose that  $n = 8k$ ,  $k \geq 1$ , and a  $Q(n, 0)$ , denoted by  $\Phi_n$ , with a perfect diagonal set  $\mathcal{D}_n$  exists. Without loss of generality, we assume that

$$\mathcal{D}_n = \{di(v_1v_2), di(v_3v_4), \dots, di(v_{n-1}v_n)\}.$$

We construct the necessary embeddings in four stages. The first two handle additions of Type I in Stages 1 and 2, and the last two handle additions of Type I in Stages 3 and 4, are setting up squares needed for a Type IV handle addition in Stage 4.

**(4.1) Stage 1.** Suppose  $p = n + 2$ .

Since  $p = n + 2 \equiv 2 \pmod{8}$  we have  $p(p - 5) \equiv 2 \pmod{8}$  and hence  $\frac{1}{2}p(p - 5) \equiv 1 \pmod{4}$ . Thus,  $T_{n+2} = T_p = \{t \mid 0 \leq t \leq p - 4, t \equiv 1 \pmod{4}\} = \{t \mid 0 \leq t \leq (n + 2) - 4, t \equiv 1 \pmod{4}\} = \{1, 5, \dots, n - 3\}$ .

We start with  $\Phi_n$ . First employing a disc addition, we add the two vertices  $v_{n+1}$  and  $v_{n+2}$  into the square with  $di(v_1v_2)$  from  $\mathcal{D}_n$  and obtain the square  $v_{n+1}v_2v_{n+2}v_1$  with a new diagonal  $di(v_{n+1}v_{n+2})$ . Then, we apply  $\frac{n}{2} - 1$  successive handle additions of Type I. During the process, the squares with  $di(v_{n+1}v_{n+2})$  resulting from previous additions are used as the outer squares and the squares with the diagonals from  $\mathcal{D}_n$  as the inner squares. Moreover, the diagonals of  $\mathcal{D}_n$  are used in the order  $di(v_3v_4), di(v_5v_6), \dots, di(v_{n-1}v_n)$ . See Figure 5.

In our figures we often do not label vertices whose identity does not matter, except that we use  $x_1, x_2, \dots$  to label vertices which help to identify a square in later parts of the construction. Handle additions of Type I connect two diagonals using an underlying square for each diagonal, unless otherwise specified. We shade squares that are *reserved* for later use; these should not be

used as the inner or outer square in a handle addition until that is explicitly specified. For example, the first two handle additions create reserved squares  $v_{n+1}v_4x_1v_3$  and  $v_{n+2}v_5x_2v_6$ , for use in Stage 2 below.

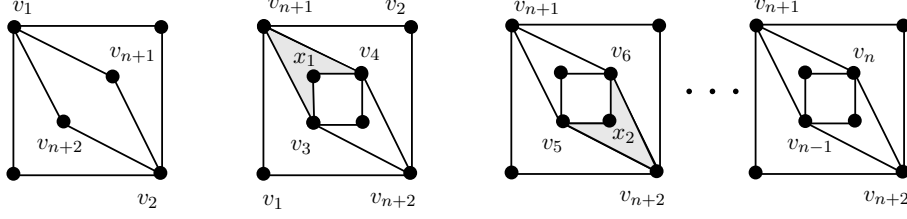


Figure 5

After the initial disc addition we have  $\binom{n}{2} + 4 = \binom{n+2}{2} - (2n-3)$  edges. So this process constructs embeddings  $Q(n+2, t)$  for  $t = 2n-3, 2n-7, 2n-11, \dots, n-3, \dots, 5, 1$ , which includes all values  $t \in T_{n+2}$ . Since the disc and handle additions join  $v_{n+1}$  and  $v_{n+2}$  to all of  $v_1, v_2, \dots, v_n$ , but do not provide an edge  $v_{n+1}v_{n+2}$ , the final result  $\Phi_{n+2}$  is an embedding of  $K_{n+2} - E_1$  where  $E_1 = \{v_{n+1}v_{n+2}\}$ . It has a perfect diagonal set

$$\mathcal{D}_{n+2} = \{di(v_1v_2), di(v_3v_4, v_{n+1}v_4x_1v_3), di(v_5v_6, v_{n+2}v_5x_2v_6), di(v_7v_8), di(v_9v_{10}), \dots, di(v_{n+1}v_{n+2})\}.$$

**(4.1) Stage 2.** Suppose  $p = n+4$ . Since  $n+4 \equiv 4 \pmod{8}$  we have  $T_{n+4} = \{t \mid 0 \leq t \leq (n+4) - 4, t \equiv 2 \pmod{4}\} = \{2, 6, \dots, n-2\}$ .

Similarly to Stage 1, starting from  $\Phi_{n+2}$  with  $\mathcal{D}_{n+2}$  we can employ a disc addition to add vertices  $v_{n+3}$  and  $v_{n+4}$  and obtain a diagonal  $di(v_{n+3}v_{n+4})$ , then employ  $\frac{n}{2}$  handle additions of Type I. The first two handle additions use the reserved squares from Stage 1 as inner squares. They create new reserved squares with new diagonals, for use in Stage 3 below. See Figure 6. After the initial disc addition we have  $\binom{n+2}{2} + 3 = \binom{n+4}{2} - (2n+2)$  edges. So this process creates embeddings  $Q(n+4, t)$  for  $t = 2n+2, 2n-2, 2n-6, \dots, n-2, \dots, 6, 2$ , which includes all  $t \in T_{n+4}$ .

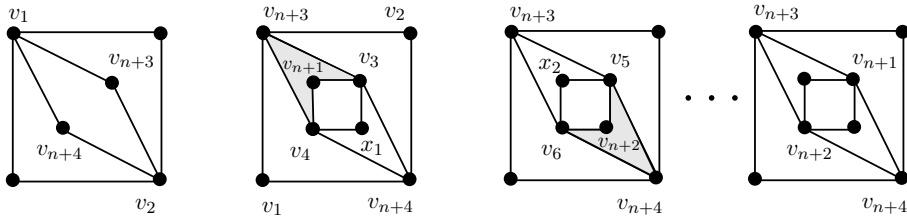


Figure 6

The final result  $\Phi_{n+4}$  is a quadrangular embedding of  $K_{n+4} - E_2$  where  $E_2 = \{v_{n+1}v_{n+2}, v_{n+3}v_{n+4}\}$ . It has a perfect diagonal set

$$\mathcal{D}_{n+4} = \{di(v_1v_2), di(v_3v_4), di(v_5v_6), \dots, di(v_{n-1}v_n), di(v_{n+1}v_{n+3}, v_{n+1}v_4v_{n+3}v_3), di(v_{n+2}v_{n+4}, v_{n+2}v_5v_{n+4}v_6)\}$$

where  $v_{n+1}v_4v_{n+3}v_3$  and  $v_{n+2}v_5v_{n+4}v_6$  are the two reserved squares.

(4.1) **Stage 3.** Suppose  $p = n + 6$ . Since  $n + 6 \equiv 6 \pmod{8}$  we have  $T_{n+6} = \{t \mid 0 \leq t \leq (n + 6) - 4, t \equiv 3 \pmod{4}\} = \{3, 7, \dots, n - 1\}$ .

Similarly to Stages 1 and 2, from  $\Phi_{n+4}$  with  $\mathcal{D}_{n+4}$  we can employ a disc addition to add vertices  $v_{n+5}$  and  $v_{n+6}$ , and then  $\frac{n}{2} + 1$  handle additions of Type I. The last two handle additions create reserved squares for use in Stage 4 below. See Figure 7. After the initial disc addition we have  $\binom{n+4}{2} + 2 = \binom{n+6}{2} - (2n + 7)$  edges. So this process creates embeddings  $Q(n + 6, t)$  for  $t = 2n + 7, 2n + 3, 2n - 1, \dots, n - 1, \dots, 7, 3$ , which includes all  $t \in T_{n+6}$ .

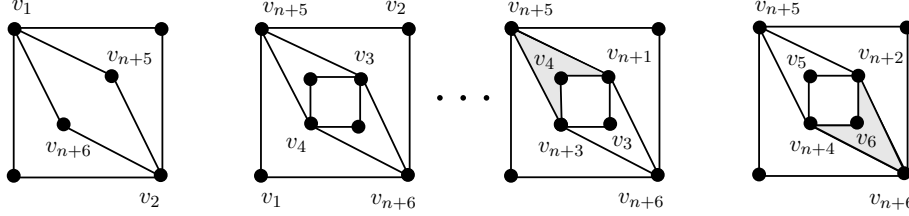


Figure 7

The final result  $\Phi_{n+6}$  is a quadrangular embedding of  $K_{n+6} - E_3$  where  $E_3 = \{v_{n+1}v_{n+2}, v_{n+3}v_{n+4}, v_{n+5}v_{n+6}\}$ . It has a perfect diagonal set

$$\mathcal{D}_{n+6} = \{di(v_1v_2), di(v_3v_4), \dots, di(v_{n-1}v_n), \\ di(v_{n+1}v_{n+3}, v_{n+1}v_4v_{n+3}v_{n+5}), di(v_{n+2}v_{n+4}, v_{n+2}v_{n+6}v_{n+4}v_6), di(v_{n+5}v_{n+6})\}$$

where  $v_{n+1}v_4v_{n+3}v_{n+5}$  and  $v_{n+2}v_{n+6}v_{n+4}v_6$  are the two reserved squares.

(4.1) **Stage 4.** Suppose  $p = n + 8$ . Since  $n + 8 \equiv 0 \pmod{8}$  we have  $T_{n+8} = \{t \mid 0 \leq t \leq (n + 8) - 4, t \equiv 0 \pmod{4}\} = \{0, 4, \dots, n + 4\}$ .

Similarly to the previous stages, from  $\Phi_{n+6}$  with  $\mathcal{D}_{n+6}$  we can employ a disc addition to add vertices  $v_{n+7}$  and  $v_{n+8}$ , and then  $\frac{n}{2} + 2$  handle additions of Type I. The last two handle additions of Type I create reserved squares  $v_{n+1}v_{n+7}v_{n+3}v_{n+5}$  and  $v_{n+2}v_{n+6}v_{n+4}v_{n+8}$ , which we then use for a Type IV handle addition. See Figure 8. After the initial disc addition we have  $\binom{n+6}{2} + 1 = \binom{n+8}{2} - (2n + 12)$  edges. So this process creates embeddings  $Q(n + 6, t)$  for  $t = 2n + 12, 2n + 8, 2n + 4, \dots, n + 4, \dots, 4, 0$ , which includes all  $t \in T_{n+8}$ .

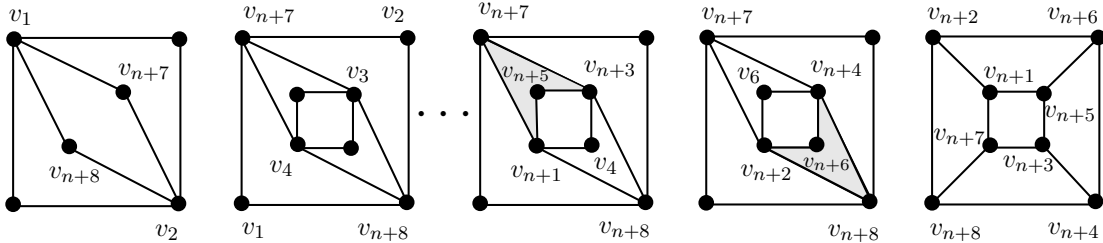


Figure 8

The final result  $\Phi_{n+8}$  is a quadrangular embedding of  $K_{n+8}$ . It is a  $Q(n+8, 0)$  with perfect diagonal set

$$\mathcal{D}_{n+8} = \{di(v_1v_2), di(v_3v_4), \dots, di(v_{n-1}v_n), \\ di(v_{n+1}v_{n+3}), di(v_{n+2}v_{n+4}), di(v_{n+5}v_{n+6}), di(v_{n+7}v_{n+8})\}.$$

This completes the proof of the induction step. Now the small cases ( $n = 4$  and  $6$ ), the basis, and the induction step together imply Lemma 4.1.  $\square$

**Lemma 4.2.** *There exists a  $Q(n, t)$  for each odd  $n \geq 5$  and  $t \in T_n$ .*

*Proof.* We proceed inductively.

**Basis.** There exist  $Q(5, t)$  for all  $t \in T_5$ . In particular, there exists a  $Q(5, 0)$  with a full diagonal set  $\mathcal{D}_5$ , which contains  $v_1$ -nearly-perfect and  $v_2$ -nearly-perfect subsets.

We have  $T_5 = \{0\}$  so we only need to find the specified embedding  $Q(5, 0)$ . We use the embedding  $\Phi_5$  of  $K_5$  in  $S_1$  with five squares, shown in Figure 9. It has a full diagonal set

$$\mathcal{D}_5 = \{di(v_1v_5, v_1v_4v_5v_2), di(v_3v_4, v_3v_5v_4v_2), di(v_4v_5, v_4v_1v_5v_3), di(v_2v_3, v_2v_4v_3v_1)\}$$

where the first two elements form a  $v_2$ -nearly-perfect subset and the last two elements form a  $v_1$ -nearly-perfect subset.

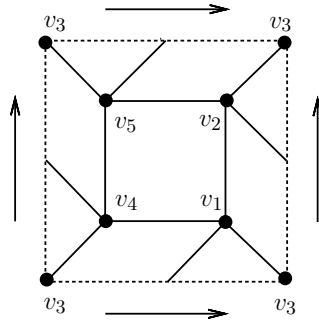


Figure 9

**Induction step.** Suppose  $n = 8k + 5$ ,  $k \geq 0$ , and we are given a  $Q(n, 0)$  with a full diagonal set having  $v_1$ -nearly-perfect and  $v_2$ -nearly-perfect subsets. Then there exist  $Q(p, t)$  for all  $p \in \{n+2, n+4, n+6, n+8\}$  and  $t \in T_p$ . In particular, there exists a  $Q(n+8, 0)$  with a full diagonal set having  $v_1$ -nearly-perfect and  $v_2$ -nearly-perfect subsets.

Suppose that  $n = 8k + 5$ ,  $k \geq 0$ , and there is a  $Q(n, 0)$ , denoted by  $\Phi_n$ , with a full diagonal set  $\mathcal{D}_n$ , as described. Write the  $v_2$ -nearly-perfect and  $v_1$ -nearly-perfect subsets as

$$\mathcal{D}'_n = \{di(y_1y_2), di(y_3y_4), \dots, di(y_{n-4}y_{n-3}), di(y_{n-2}v_1)\} \text{ and} \\ \mathcal{D}''_n = \{di(z_1z_2), di(z_3z_4), \dots, di(z_{n-4}z_{n-3}), di(z_{n-2}v_2)\},$$

respectively. Thus,  $\{y_1, y_2, \dots, y_{n-2}\} = \{z_1, z_2, \dots, z_{n-2}\} = \{v_3, v_4, \dots, v_n\}$ . We construct the necessary embeddings in four stages. Note that the last few handle additions in each stage help to set up squares needed for the handle additions of Type II and III in later stages.

**(4.2) Stage 1.** Suppose  $p = n + 2$ . Since  $n + 2 \equiv 7 \pmod{8}$  we have  $T_{n+2} = \{t \mid 0 \leq t \leq (n + 2) - 4, t \equiv 3 \pmod{4}\} = \{3, 7, \dots, n - 2\}$ .

We start with  $\Phi_n$ . First, by employing a disc addition, we add two vertices  $v_{n+1}$  and  $v_{n+2}$  into the square with  $di(y_1y_2)$  from  $\mathcal{D}_n$  and obtain the square  $v_{n+1}y_2v_{n+2}y_1$  with  $di(v_{n+1}v_{n+2})$ . Then, we apply  $\frac{n-3}{2}$  successive handle additions of Type I. During the process, the squares with  $di(v_{n+1}v_{n+2})$  resulting from previous additions are used as the outer squares and the squares with the diagonals from  $\mathcal{D}'_n$ , in the order  $di(y_3y_4), di(y_5y_6), \dots, di(y_{n-2}v_1)$ , as the inner squares. The final handle addition creates a square  $v_{n+1}x_1v_{n+2}v_1$  that is reserved for Stage 2 below. See Figure 10.

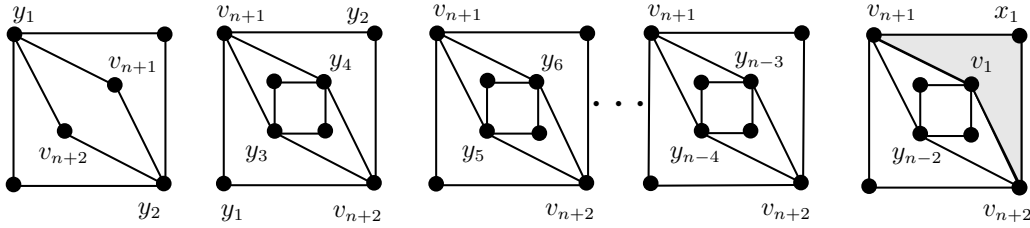


Figure 10

After the initial disc addition we have  $\binom{n}{2} + 4 = \binom{n+2}{2} - (2n - 3)$  edges. So this process constructs embeddings  $Q(n + 2, t)$  for  $t = 2n - 3, 2n - 7, 2n - 11, \dots, n - 2, \dots, 7, 3$ , which includes all values  $t \in T_{n+2}$ . Since the disc and handle additions join  $v_{n+1}$  and  $v_{n+2}$  to all of  $v_3, v_4, v_5, \dots, v_n$  and to  $v_1$ , but not to  $v_2$ , and do not provide an edge  $v_{n+1}v_{n+2}$ , the final result  $\Phi_{n+2}$  is an embedding of  $K_{n+2} - E_3$  where  $E_3 = \{v_2v_{n+1}, v_2v_{n+2}, v_{n+1}v_{n+2}\}$ . It has a full diagonal set  $\mathcal{D}_{n+2} = \mathcal{D}_n \cup \{di(v_{n+1}v_{n+2})\}$  with  $v_2$ -nearly-perfect and  $v_1$ -nearly perfect subsets

$$\mathcal{D}'_{n+2} = \mathcal{D}'_n \cup \{di(v_{n+1}v_{n+2})\} \quad \text{and} \quad \mathcal{D}''_{n+2} = \mathcal{D}''_n \cup \{di(v_{n+1}v_{n+2})\},$$

respectively.

**(4.2) Stage 2.** Suppose  $p = n + 4$ . Since  $n + 4 \equiv 1 \pmod{8}$  we have  $T_{n+4} = \{t \mid 0 \leq t \leq (n + 4) - 4, t \equiv 2 \pmod{4}\} = \{2, 6, \dots, n - 3\}$ .

Starting from  $\Phi_{n+2}$  with  $\mathcal{D}''_{n+2}$  we can employ a disc addition to add vertices  $v_{n+3}$  and  $v_{n+4}$ , then  $\frac{n-1}{2}$  handle additions of Type I. The second last of these creates a reserved square, which is used in a final handle addition of Type II, along with the reserved square from Stage 1. This creates two reserved squares which provide new diagonals with specific underlying squares, for use in Stage 3 below. See Figure 11. After the initial disc addition we have  $\binom{n+2}{2} + 1 = \binom{n+4}{2} - (2n + 4)$  edges. So this process creates embeddings  $Q(n + 4, t)$  for  $t = 2n + 4, 2n, 2n - 4, \dots, n - 3, \dots, 6, 2$ , which includes all  $t \in T_{n+4}$ .

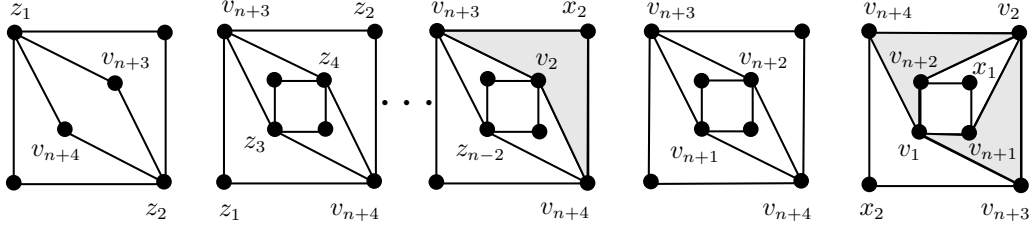


Figure 11

The final result  $\Phi_{n+4}$  is a quadrangular embedding of  $K_{n+4} - E_2$  where  $E_2 = \{v_{n+1}v_{n+2}, v_{n+3}v_{n+4}\}$ . If we define a diagonal set using the last two reserved squares, i.e.,

$$\mathcal{D}_{n+4}^+ = \{di(v_{n+1}v_{n+3}, v_{n+1}v_2v_{n+3}v_1), di(v_{n+2}v_{n+4}, v_{n+2}v_1v_{n+4}v_2)\}$$

then  $\Phi_{n+4}$  has a full diagonal set  $\mathcal{D}_{n+4} = \mathcal{D}_n \cup \mathcal{D}_{n+4}^+$  with  $v_2$ -nearly-perfect and  $v_1$ -nearly-perfect subsets  $\mathcal{D}'_{n+4} = \mathcal{D}'_n \cup \mathcal{D}_{n+4}^+$  and  $\mathcal{D}''_{n+4} = \mathcal{D}''_n \cup \mathcal{D}_{n+4}^+$ , respectively.

**(4.2) Stage 3.** Suppose  $p = n + 6$ . Since  $n + 6 \equiv 3 \pmod{8}$  we have  $T_{n+6} = \{t \mid 0 \leq t \leq (n + 6) - 4, t \equiv 1 \pmod{4}\} = \{1, 5, \dots, n\}$ .

Starting from  $\Phi_{n+4}$  with  $\mathcal{D}''_{n+4}$ , we can employ a disc addition to add vertices  $v_{n+5}$  and  $v_{n+6}$ , then  $\frac{n+1}{2}$  handle additions of Type I, then a handle addition of Type III. See Figure 12. The last four handle additions create and use up a number of reserved squares; the net effect is that the two reserved squares from the Type II addition in Stage 2 are used up, and two new reserved squares  $v_{n+1}v_{n+5}v_{n+3}v_1$  and  $v_{n+2}v_{n+6}v_{n+4}v_2$  are created for use in Stage 4 below. After the initial disc addition we have  $\binom{n+4}{2} + 2 = \binom{n+6}{2} - (2n + 7)$  edges. So this process creates embeddings  $Q(n + 6, t)$  for  $t = 2n + 7, 2n + 3, \dots, n, \dots, 5, 1$ , which includes all  $t \in T_{n+6}$ .

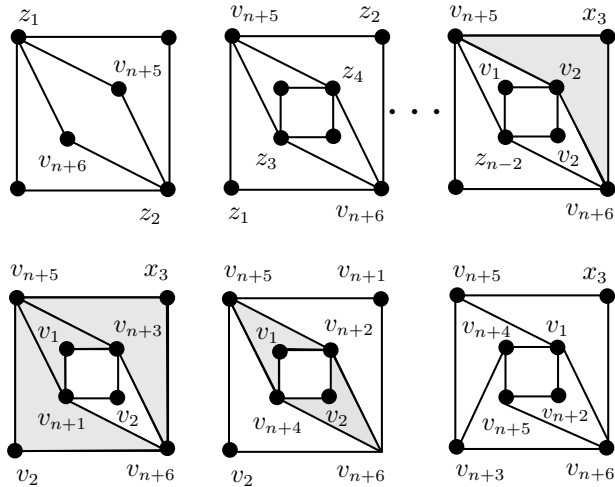


Figure 12

The final result  $\Phi_{n+6}$  is a quadrangular embedding of  $K_{n+6} - E_1$  where  $E_1 = \{v_{n+1}v_{n+2}\}$ . If we

define a diagonal set containing the two unused reserved squares,

$$\mathcal{D}_{n+6}^+ = \{di(v_{n+1}v_{n+3}, v_{n+1}v_{n+5}v_{n+3}v_1), di(v_{n+2}v_{n+4}, v_{n+2}v_{n+6}v_{n+4}v_2), di(v_{n+5}v_{n+6})\}$$

then  $\Phi_{n+6}$  has a full diagonal set  $\mathcal{D}_{n+6} = \mathcal{D}_n \cup \mathcal{D}_{n+6}^+$  with  $v_2$ -nearly-perfect and  $v_1$ -nearly-perfect subsets  $\mathcal{D}'_{n+6} = \mathcal{D}'_n \cup \mathcal{D}_{n+6}^+$  and  $\mathcal{D}''_{n+6} = \mathcal{D}''_n \cup \mathcal{D}_{n+6}^+$ , respectively.

**(4.2) Stage 4.** Suppose  $p = n + 8$ . Since  $n + 8 \equiv 5 \pmod{8}$  we have  $T_{n+8} = \{t \mid 0 \leq t \leq (n + 8) - 4, t \equiv 0 \pmod{4}\} = \{0, 4, \dots, n + 3\}$ .

Starting from  $\Phi_{n+6}$  with  $\mathcal{D}''_{n+6}$ , we can employ a disc addition to add vertices  $v_{n+7}$  and  $v_{n+8}$ . Then we use  $\frac{n+3}{2}$  handle additions of Type I using diagonals in the order  $di(z_1z_2)$ ,  $di(z_3z_4)$ ,  $\dots$ ,  $di(z_{n-2}v_2)$ ,  $di(v_{n+5}v_{n+6})$  and lastly, using the two reserved squares from Stage 3,  $di(v_{n+1}v_{n+3})$  and  $di(v_{n+2}v_{n+4})$ . We finish with a handle addition of Type III. See Figure 13. The last three handle additions create and use up three additional reserved squares. After the initial disc addition we have  $\binom{n+6}{2} + 3 = \binom{n+8}{2} - (2n + 10)$  edges. So this process creates embeddings  $Q(n + 8, t)$  for  $t = 2n + 10, 2n + 6, \dots, n + 3, \dots, 4, 0$ , which includes all  $t \in T_{n+8}$ .

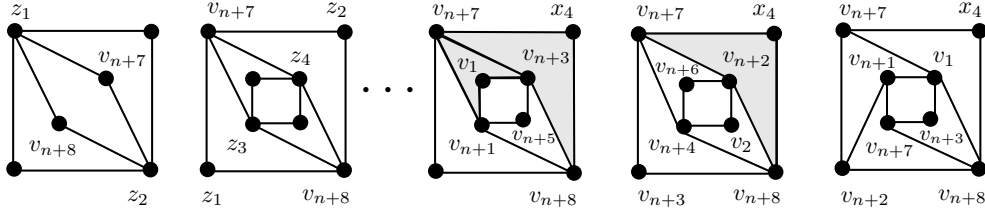


Figure 13

The final result  $\Phi_{n+8}$  is a quadrangular embedding of  $K_{n+8}$ . If we define a diagonal set

$$\mathcal{D}_{n+8}^+ = \{di(v_{n+1}v_{n+3}), di(v_{n+2}v_{n+4}), di(v_{n+5}v_{n+6}), di(v_{n+7}v_{n+8})\}$$

then  $\Phi_{n+8}$  is a  $Q(n + 8, 0)$  with a full diagonal set  $\mathcal{D}_{n+8} = \mathcal{D}_n \cup \mathcal{D}_{n+8}^+$  having  $v_2$ -nearly-perfect and  $v_1$ -nearly-perfect subsets  $\mathcal{D}'_n \cup \mathcal{D}_{n+8}^+$  and  $\mathcal{D}''_n \cup \mathcal{D}_{n+8}^+$ , respectively.

This completes the proof of the induction step. Now the basis and the induction step together imply Lemma 4.2.  $\square$

## Nonorientable surfaces

Let  $\tilde{Q}(n, t)$  denote a nonorientable quadrangular embedding of a simple graph with  $n$  vertices and  $\binom{n}{2} - t$  edges, for any integers  $n \geq 1$  and  $t$  with  $0 \leq t \leq \binom{n}{2}$ . Let  $\tilde{T}_n = \{t \mid 0 \leq t \leq n - 4, t \equiv \frac{1}{2}n(n - 5) \pmod{2}\}$  for each integer  $n \geq 4$ . The elements of  $\tilde{T}_n$  form an arithmetic progression with difference 2. The condition  $t \equiv \frac{1}{2}n(n - 5) \pmod{2}$  is equivalent to  $4 \mid n(n - 5) - 2t$  and so we must consider the value of  $n \pmod{4}$  to determine  $\tilde{T}_n$ . We must show that there exist  $\tilde{Q}(n, t)$  for each  $n \geq 4$  and  $t \in \tilde{T}_n$  except when  $(n, t) = (5, 0)$ , and that there exists a  $\tilde{Q}(6, 3)$  in  $N_2$ . We divide the proof into the cases where  $n \leq 6$ , where  $n \geq 8$  is even, and where  $n \geq 7$  is odd.

When working with nonorientable surfaces we may use a square  $v_i v_j v_k v_\ell$  in its reverse order  $v_i v_\ell v_k v_j$  whenever convenient.

**Lemma 4.3.** *Suppose that  $4 \leq n \leq 6$ . Then there exist  $\tilde{Q}(n, t)$  for each  $t \in \tilde{T}_n$ , except when  $(n, t) = (5, 0)$ . There also exists a  $\tilde{Q}(6, 3)$  in  $N_2$ .*

*Proof.* We have  $\tilde{T}_4 = \{0\}$ . The complete graph  $K_4$  admits a quadrangular embedding with three squares (every 4-cycle in  $K_4$  bounds a face) in the projective plane  $N_1$ , which is a  $\tilde{Q}(4, 0)$ .

We have  $\tilde{T}_5 = \{0\}$ . If  $(n, t) = (5, 0)$  then the graph is  $K_5$ , but by Theorem 1.1 there is no nonorientable quadrangular embedding of  $K_5$ , so no embedding exists for this case.

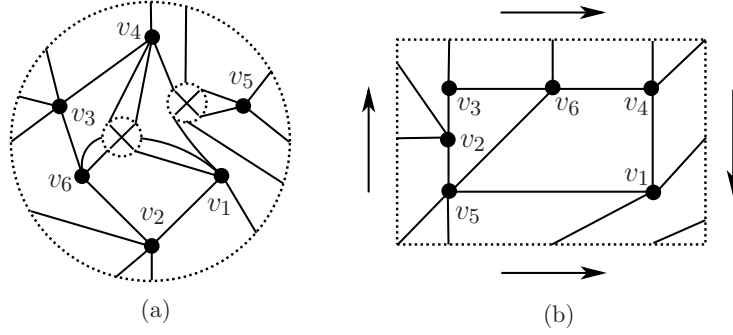


Figure 14

We have  $\tilde{T}_6 = \{1\}$ . Figure 14(a) shows that  $K_6 - E_1$  has a quadrangular embedding in  $N_3$  with  $E_1 = \{v_5v_6\}$ , which is a  $\tilde{Q}(6, 1)$ . Also, Figure 14(b) shows that  $K_6 - E_3$  has a quadrangular embedding  $\Phi_6^-$  in  $N_2$  with  $E_3 = \{v_1v_3, v_2v_6, v_3v_4\}$ , which is a  $\tilde{Q}(6, 3)$  in  $N_2$ .  $\square$

**Lemma 4.4.** *There exists a  $\tilde{Q}(n, t)$  for each even  $n \geq 8$  and  $t \in \tilde{T}_n$ .*

*Proof.* We proceed inductively.

**Basis.** There exist  $\tilde{Q}(8, t)$  for all  $t \in \tilde{T}_8$ . In particular, there exists a  $\tilde{Q}(8, 2)$  that embeds a graph  $K_8 - \{u_1u_2, u_3u_4\}$  and has a perfect diagonal set  $\mathcal{D}_8$  in which  $u_1, u_2, u_3$  and  $u_4$  belong to four distinct diagonals.

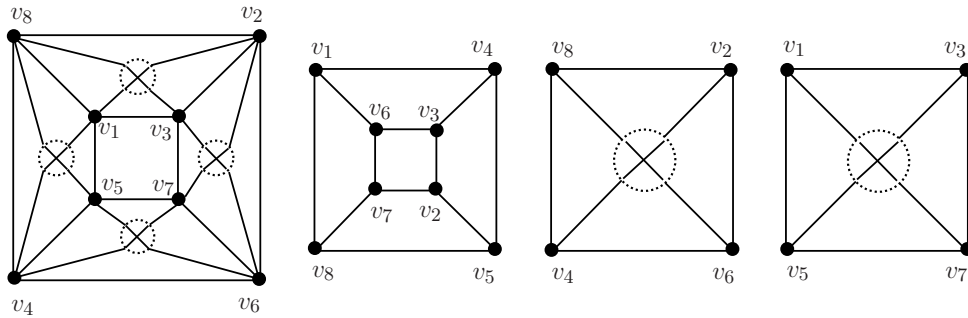


Figure 15

We have  $\tilde{T}_8 = \{0, 2, 4\}$ . At left in Figure 15 is a  $\tilde{Q}(8, 8)$  obtained by four crosscap additions from the usual spherical (planar) embedding of a cube, to which we apply a handle addition of Type IV

followed by two crosscap additions. The result is a quadrangular embedding of  $K_8$  in  $N_8$ , and along the way we construct embeddings  $\tilde{Q}(8, t)$  with  $t = 8, 4, 2, 0$ , which includes all  $t \in \tilde{T}_8$ .

We examine the  $\tilde{Q}(8, 2)$  obtained by performing the handle addition and the first crosscap addition, but not the second crosscap addition. This is a quadrangular embedding  $\Phi_8^-$  of  $K_8 - \{v_1v_7, v_3v_5\}$  in  $N_7$ . There is a perfect diagonal set

$$\mathcal{D}_8 = \{di(v_8v_1, v_8v_4v_1v_5), di(v_2v_3, v_2v_8v_3v_1), di(v_4v_5, v_4v_6v_5v_7), di(v_6v_7, v_6v_2v_7v_3)\}.$$

Taking  $u_1 = v_1$ ,  $u_2 = v_7$ ,  $u_3 = v_3$  and  $u_4 = v_5$ , we see that the conditions for the particular  $\tilde{Q}(8, 2)$  are satisfied.

**Induction step.** Suppose  $n = 4k$ ,  $k \geq 2$ , and we are given a  $\tilde{Q}(n, 2)$  that embeds a graph  $K_n - \{u_1u_2, u_3u_4\}$  and has a perfect diagonal set  $\mathcal{D}_n$  in which  $u_1$ ,  $u_2$ ,  $u_3$  and  $u_4$  belong to four distinct diagonals. Then there exist  $\tilde{Q}(p, t)$  for all  $p \in \{n+2, n+4\}$  and  $t \in \tilde{T}_p$ . In particular, there exists a  $\tilde{Q}(n+4, 2)$  that embeds a graph  $K_{n+4} - \{u'_1u'_2, u'_3u'_4\}$  and has a perfect diagonal set  $\mathcal{D}_{n+4}$  in which  $u'_1$ ,  $u'_2$ ,  $u'_3$  and  $u'_4$  belong to four distinct diagonals.

Suppose that  $n = 4k$ ,  $k \geq 2$ , and there is a  $\tilde{Q}(n, 2)$ , denoted by  $\Phi_n^-$ , as described above. We may assume without loss of generality that

$$\mathcal{D}_n = \{di(v_1v_2), di(v_3v_4), di(v_5v_6), di(v_7v_8), \dots, di(v_{n-1}v_n)\}$$

and that  $u_1 = v_1$ ,  $u_2 = v_3$ ,  $u_3 = v_5$  and  $u_4 = v_7$ . Thus,  $\Phi_n^-$  is an embedding of  $K_n - E_2$  with  $E_2 = \{v_1v_3, v_5v_7\}$ . We construct the necessary embeddings in two stages.

**(4.4) Stage 1.** Suppose  $p = n + 2$ . Since  $n + 2 \equiv 2 \pmod{4}$  we have  $\tilde{T}_{n+2} = \{t \mid 0 \leq t \leq (n+2) - 4, t \equiv 1 \pmod{2}\} = \{1, 3, 5, \dots, n-3\}$ .

We start with  $\Phi_n^-$ . First, by applying a disc addition, we add two vertices  $v_{n+1}$  and  $v_{n+2}$  into the square with diagonal  $di(v_1v_2)$  from  $\mathcal{D}_n$ . Then we emply a handle addition of Type I with  $di(v_3v_4)$ , creating two reserved squares, one of which,  $v_3v_{n+2}v_4x_1$  is for use in Stage 2 below. The other reserved square,  $v_{n+1}v_3v_{n+2}v_1$ , we use immediately for a crosscap addition. We then perform  $\frac{n}{2} - 2$  handle additions of Type I using diagonals  $di(v_5v_6)$ ,  $di(v_7v_8)$ ,  $\dots$ ,  $di(v_{n-1}v_n)$  in that order. The first of these creates another reserved square  $v_5v_{n+1}v_6x_2$  for use in Stage 2. See Figure 16.

After the initial disc addition we have  $\binom{n}{2} + 2 = \binom{n+2}{2} - (2n-1)$  edges. Since handle additions add four edges and crosscap additions add two edges, our process creates embeddings  $\tilde{Q}(n+2, t)$  for  $t = 2n-1, 2n-5, 2n-7, 2n-11, 2n-15, \dots, 5, 1$ . However, we do not use the squares created by the crosscap addition in any later handle addition, so we can just omit the crosscap addition. This creates embeddings  $\tilde{Q}(n+2, t)$  for  $t = 2n-1, 2n-5, 2n-9, \dots, 7, 3$ . Combining these, we have  $\tilde{Q}(n+2, t)$  for  $t = 2n-1, 2n-5, 2n-7, 2n-9, \dots, n-3, \dots, 3, 1$ , i.e., for all odd  $t$  with  $1 \leq t \leq 2n-1$  except  $t = 2n-3$ . Since  $n \geq 8$ , this includes all  $t \in \tilde{T}_{n+2}$ .

The final result  $\Phi_{n+2}$  is an embedding of  $K_{n+2} - E_1$  where  $E_1 = \{v_5v_7\}$ . Using the reserved squares containing  $x_1$  and  $x_2$ , we see that  $\Phi_{n+2}$  has a perfect diagonal set

$$\mathcal{D}_{n+2} = \{di(v_1v_2), di(v_3v_4, v_3v_{n+2}v_4x_1), di(v_5v_6, v_5v_{n+1}v_6x_2), di(v_7v_8), di(v_9v_{10}), \dots, di(v_{n+1}v_{n+2})\}.$$

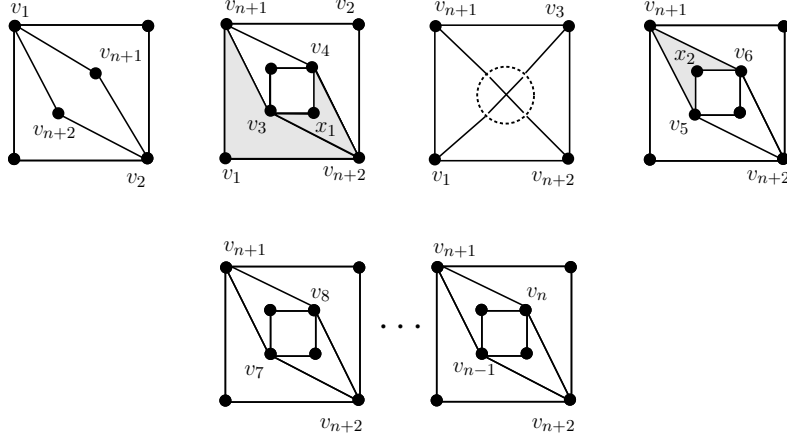


Figure 16

(4.4) **Stage 2.** Suppose  $p = n + 4$ . Since  $n + 4 \equiv 0 \pmod{4}$  we have  $\tilde{T}_{n+4} = \{t \mid 0 \leq t \leq (n + 4) - 4, t \equiv 0 \pmod{2}\} = \{0, 2, 4, \dots, n\}$ .

Starting from  $\Phi_{n+2}$  with  $\mathcal{D}_{n+2}$  we can employ a disc addition to add vertices  $v_{n+3}$  and  $v_{n+4}$ , using the reserved underlying square for  $di(v_5v_6)$  as the outer square. This creates a reserved square  $v_{n+1}v_6v_{n+3}v_5$  which we will use to create a new diagonal later. Then we perform a handle addition of Type I using  $di(v_7v_8)$ , creating a reserved square  $v_{n+3}v_7v_{n+4}v_5$ , which we use immediately for a crosscap addition. Next we employ a handle addition of Type I, using the reserved underlying square for  $di(v_3v_4)$  as the inner square. This creates a new reserved square  $v_{n+2}v_4v_{n+4}v_3$ , which we will also use to create a new diagonal later. Finally we perform  $\frac{n}{2} - 2$  handle additions of Type I using all remaining diagonals from  $\mathcal{D}_{n+2}$ . See Figure 17.

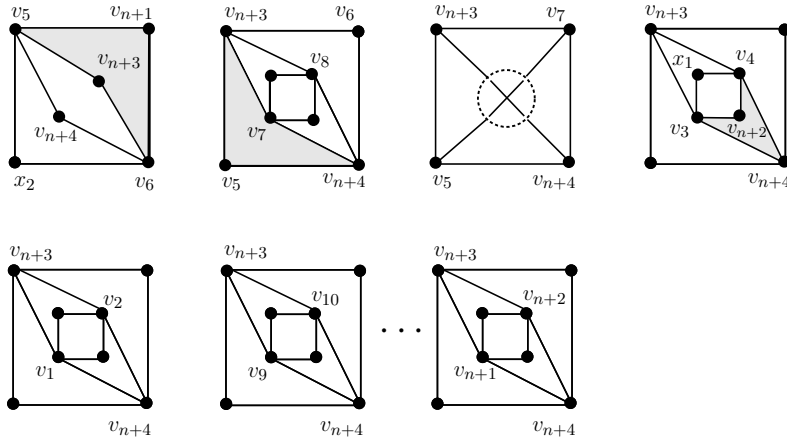


Figure 17

After the initial disc addition we have  $\binom{n+2}{2} + 3 = \binom{n+4}{2} - (2n + 2)$  edges. Since handle additions add four edges and crosscap additions add two edges, our process creates embeddings  $\tilde{Q}(n + 4, t)$  for  $t = 2n + 2, 2n - 2, 2n - 4, 2n - 8, 2n - 12, \dots, 4, 0$ . However, we do not use the square created

by the crosscap addition in any later handle addition, so we can just omit the crosscap addition. This produces  $\tilde{Q}(n+4, t)$  for  $t = 2n+2, 2n-2, 2n-6, 2n-10, \dots, 6, 2$ . Combining these, we have  $\tilde{Q}(n+4, t)$  for  $t = 2n+2, 2n-2, 2n-4, 2n-6, 2n-8, \dots, n, \dots, 2, 0$ , i.e., for all even  $t$  with  $0 \leq t \leq 2n+2$  except  $t = 2n$ . Since  $n \geq 8$ , this includes all  $t \in \tilde{T}_{n+4}$ .

If we perform all handle additions but omit the crosscap addition we obtain an embedding  $\Phi_{n+4}^-$  of  $K_{n+4} - E_2$  where  $E_2 = \{v_5v_7, v_{n+3}v_{n+4}\}$ . This is a  $\tilde{Q}(n+4, 2)$ . Using two reserved squares to create new diagonals, we see that  $\Phi_{n+4}^-$  has a perfect diagonal set

$$\mathcal{D}_{n+4} = \{di(v_1v_2), di(v_3v_4), di(v_5v_6), di(v_7v_8), \dots, di(v_{n-1}v_n), \\ di(v_{n+1}v_{n+3}, v_{n+1}v_6v_{n+3}v_5), di(v_{n+2}v_{n+4}, v_{n+2}v_4v_{n+4}v_3)\}.$$

Taking  $u'_1 = v_5$ ,  $u'_2 = v_7$ ,  $u'_3 = v_{n+3}$ ,  $u'_4 = v_{n+4}$ , we see that the required properties for a particular  $\tilde{Q}(n+4, 2)$  hold.

This completes the proof of the induction step. Now the basis and the induction step together imply Lemma 4.4.  $\square$

**Lemma 4.5.** *There exists a  $\tilde{Q}(n, t)$  for each odd  $n \geq 7$  and  $t \in \tilde{T}_n$ .*

*Proof.* We proceed inductively. Our argument requires a slightly technical induction hypothesis, so we make the following definition.

**Definition.** A  $\tilde{Q}(n, 3)$ , where  $n \equiv 3 \pmod{4}$ , is said to have *Property P* if the following conditions (a), (b) and (c) hold.

- (a) The graph is  $K_n - E_3$  where  $E_3 = \{v_1v_{n-1}, v_1v_n, v_{n-3}v_{n-2}\}$ .
- (b) There is a full diagonal set

$$\mathcal{D}_n = \{di(v_1v_3), di(v_2v_3), di(y_1y_2), di(y_3y_4), \dots, di(y_{n-8}y_{n-7}), di(v_{n-3}v_{n-1}), di(v_{n-2}v_n)\}$$

where  $\{y_1, y_2, \dots, y_{n-7}\} = \{v_4, v_5, \dots, v_{n-4}\}$ .

(c) There is a square  $v_2v_{n-1}x_1v_n$  (the exact identity of  $x_1$  does not matter) that is not an underlying square for  $\mathcal{D}_n$ . (We reserve this square for later use.)

**Basis.** There exist  $\tilde{Q}(7, t)$  for all  $t \in \tilde{T}_7$ . In particular, there exists a  $\tilde{Q}(7, 3)$  with Property P.

We have  $\tilde{T}_7 = \{1, 3\}$ . Figure 18 shows a quadrangular embedding  $\Phi_7$  of  $K_7 - E_1$  in  $N_5$  with  $E_1 = \{v_1v_6\}$ . This is the required  $\tilde{Q}(7, 1)$ . If we delete the two edges  $v_1v_7$  and  $v_4v_5$  of  $\Phi_7$ , we create a face, bounded by a 4-cycle and containing a crosscap, which we can remove and replace by a disc (this is the inverse of a crosscap addition). We obtain a quadrangular embedding  $\Phi_7^-$  of  $K_7 - E_3$  in  $N_4$ , which is a  $\tilde{Q}(7, 3)$ . We verify that  $\Phi_7^-$  also has Property P. (a) The missing edges form  $E_3 = \{v_1v_6, v_1v_7, v_4v_5\}$ , as required. (b) There is a full diagonal set

$$\mathcal{D}_7 = \{di(v_1v_3, v_1v_2v_3v_5), di(v_2v_3, v_2v_1v_3v_5), di(v_4v_6, v_4v_2v_6v_3), di(v_5v_7, v_5v_2v_7v_6)\}$$

of the required form. (c) There is a square  $v_2v_6v_5v_7$  that is not an underlying square for  $\mathcal{D}_7$ , as required. Thus,  $\Phi_7^-$  has Property P.

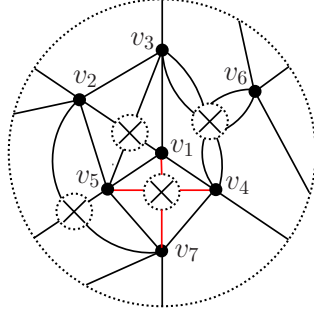


Figure 18

**Induction step.** Given a  $\tilde{Q}(n, 3)$  with Property P, where  $n = 4k + 3$ ,  $k \geq 1$ , there exist  $\tilde{Q}(p, t)$  for all  $p \in \{n + 2, n + 4\}$  and  $t \in \tilde{T}_p$ . In particular, there exists a  $\tilde{Q}(n + 4, 3)$  with Property P.

Suppose that  $n = 4k + 3$ ,  $k \geq 1$ , and there is a  $\tilde{Q}(n, 3)$ , denoted by  $\Phi_n^-$ , satisfying Property P. We construct the necessary embeddings in two stages.

**(4.5) Stage 1.** Suppose  $p = n + 2$ . Since  $n + 2 \equiv 1 \pmod{4}$  we have  $\tilde{T}_{n+2} = \{t \mid 0 \leq t \leq (n + 2) - 4, t \equiv 0 \pmod{2}\} = \{0, 2, 4, \dots, n - 3\}$ .

We start with  $\Phi_n^-$ . First, by applying a disc addition, we add two vertices  $v_{n+1}$  and  $v_{n+2}$  into the square with  $di(v_{n-3}v_{n-1})$  from  $\mathcal{D}_n$ , creating a reserved square  $v_{n-3}v_{n+1}v_{n-1}x_2$  for use in Stage 2 below. Then we employ a handle addition of Type I with  $di(v_{n-2}v_n)$ , creating two reserved squares, one of which,  $v_{n-2}v_{n+2}v_nx_3$ , is for use in Stage 2. The other reserved square,  $v_{n+1}v_{n-3}v_{n+2}v_{n-2}$ , we use immediately for a crosscap addition. We then perform  $\frac{n-5}{2}$  handle additions of Type I using diagonals  $di(y_1y_2)$ ,  $di(y_3y_4)$ ,  $\dots$ ,  $di(y_{n-8}y_{n-7})$ ,  $di(v_1v_3)$ , in that order. The last handle addition creates a reserved square  $v_{n+1}v_1v_{n+2}x_4$ , which we then use, along with the reserved square  $v_2v_{n-1}x_1v_n$  from Property P(c), in a handle addition of Type II, which creates a further reserved square  $v_{n+1}v_2v_{n+2}x_4$  for use in Stage 2. See Figure 19.

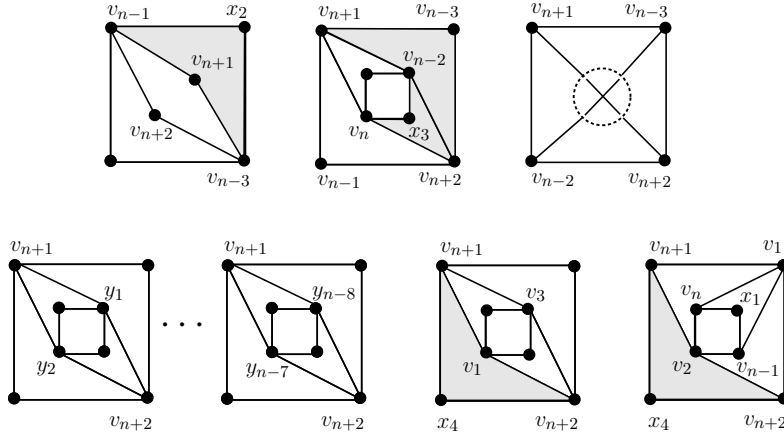


Figure 19

After the initial disc addition we have  $\binom{n}{2} + 1 = \binom{n+2}{2} - 2n$  edges. Since handle additions add four edges and crosscap additions add two edges, our process creates embeddings  $\tilde{Q}(n+2, t)$  for  $t = 2n, 2n-4, 2n-6, 2n-10, 2n-14, \dots, 4, 0$ . However, we do not use the squares created by the crosscap addition in any later handle addition, so we can just omit the crosscap addition. This creates embeddings  $\tilde{Q}(n+2, t)$  for  $t = 2n, 2n-4, 2n-8, \dots, 6, 2$ . Combining these, we have  $\tilde{Q}(n+2, t)$  for  $t = 2n, 2n-4, 2n-6, 2n-8, \dots, n-3, \dots, 2, 0$ , i.e., for all even  $t$  with  $0 \leq t \leq 2n$  except  $t = 2n-2$ . Since  $n \geq 7$ , this includes all  $t \in \tilde{T}_{n+2}$ .

If we perform all handle additions but omit the crosscap addition, we obtain an embedding  $\Phi_{n+2}^-$  of  $K_{n+2} - E_2$  where  $E_2 = \{v_{n-2}v_{n-3}, v_{n+1}v_{n+2}\}$  (the edges of the omitted crosscap addition). Using the reserved squares containing  $x_2, x_3$  and  $x_4$ , we see that  $\Phi_{n+2}^-$  has a full diagonal set

$$\mathcal{D}_{n+2} = \{di(v_1v_3), di(v_2v_3), di(y_1y_2), di(y_3y_4), \dots, di(y_{n-8}y_{n-7}), di(v_{n-3}v_{n-1}, v_{n-3}v_{n+1}v_{n-1}x_2), di(v_{n-2}v_n, v_{n-2}v_{n+2}v_nx_3), di(v_{n+1}v_{n+2}, v_{n+1}v_2v_{n+2}x_4)\}.$$

**(4.5) Stage 2.** Suppose  $p = n+4$ . Since  $n+4 \equiv 3 \pmod{4}$  we have  $\tilde{T}_{n+4} = \{t \mid 0 \leq t \leq (n+4) - 4, t \equiv 1 \pmod{2}\} = \{1, 3, 5, \dots, n\}$ .

Starting from  $\Phi_{n+2}^-$  with  $\mathcal{D}_{n+2}$ , we can employ a disc addition to add vertices  $v_{n+3}$  and  $v_{n+4}$ , using the reserved underlying square for  $di(v_{n-3}v_{n-1})$  as the outer square. This creates a reserved square  $v_{n+1}v_{n-1}v_{n+3}v_{n-3}$  which we will use to satisfy Property P(b). Then we perform a handle addition of Type I, using the reserved underlying square for  $di(v_{n-2}v_n)$  as the inner square. This creates two reserved squares, one of which,  $v_{n+2}v_nv_{n+4}v_{n-2}$ , we will use to satisfy Property P(b). The other,  $v_{n+3}v_{n-2}v_{n+4}v_{n-3}$ , we use immediately for a crosscap addition. Next we employ a handle addition of Type I, using the reserved underlying square for  $di(v_{n+1}v_{n+2})$  as the inner square. This creates a reserved square  $v_{n+2}v_2v_{n+1}v_{n+4}$ , which we use immediately for a crosscap addition. We then perform  $\frac{n-5}{2}$  handle additions of Type I using diagonals  $di(y_1y_2), di(y_3y_4), \dots, di(y_{n-8}y_{n-7}), di(v_1v_3)$ , in that order. The last handle addition creates a reserved square  $v_1v_{n+4}x_5v_{n+3}$ , which we will use to satisfy Property P(c). See Figure 20.

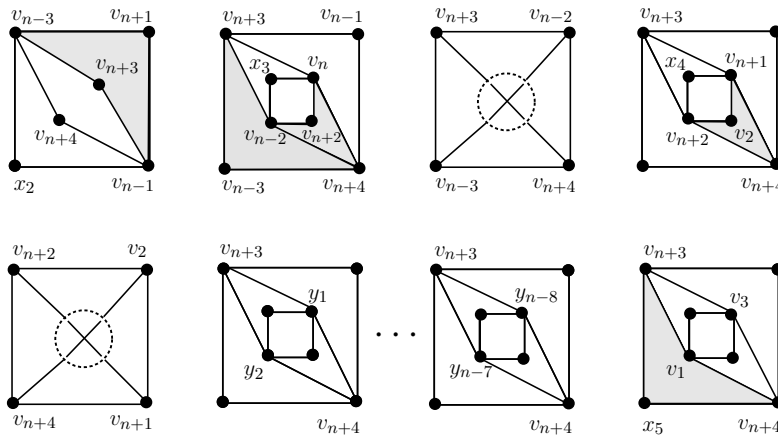


Figure 20

After the initial disc addition we have  $\binom{n+2}{2} + 2 = \binom{n+4}{2} - (2n+3)$  edges. Since handle additions

add four edges and crosscap additions add two edges, our process creates embeddings  $\tilde{Q}(n+4, t)$  for  $t = 2n+3, 2n-1, 2n-3, 2n-7, 2n-9, 2n-13, 2n-17, \dots, 5, 1$ . However, we do not use the squares created by either crosscap addition in any later handle addition, so we can omit one or both crosscap additions. If we omit the first crosscap addition, we obtain embeddings  $\tilde{Q}(n+4, t)$  for  $t = 2n+3, 2n-1, 2n-5, 2n-7, 2n-11, 2n-15, \dots, 7, 3$ . Combining these, we have  $\tilde{Q}(n+4, t)$  for  $t = 2n+3, 2n-1, 2n-3, 2n-5, \dots, n, \dots, 3, 1$ , i.e., for all odd  $t$  with  $1 \leq t \leq 2n+3$  except  $t = 2n+1$ . Since  $n \geq 7$ , this includes all  $t \in \tilde{T}_{n+4}$ .

If we perform all operations we obtain an embedding  $\Phi_{n+4}$  of  $K_{n+4} - E_1$  where  $E_1 = \{v_2v_{n+3}\}$ . If we perform all handle additions and the first crosscap addition but omit the second crosscap addition, we obtain an embedding  $\Phi_{n+4}^-$  of  $K_{n+4} - E_3$ , i.e., a  $\tilde{Q}(n+4, 3)$ . Observe the following. (a')  $E_3 = \{v_2v_{n+3}, v_2v_{n+4}, v_{n+1}v_{n+2}\}$  ( $E_1$  and the edges of the omitted crosscap addition). (b') We see that  $\Phi_{n+4}^-$  has a full diagonal set

$$\mathcal{D}_{n+4} = \{di(v_1v_3), di(v_2v_3), di(y_1y_2), di(y_3y_4), \dots, di(y_{n-8}y_{n-7}), di(v_{n-3}v_{n-1}), di(v_{n-2}v_n), di(v_{n+1}v_{n+3}, v_{n+1}v_{n-1}v_{n+3}v_{n-3}), di(v_{n+2}v_{n+4}, v_{n+2}v_nv_{n+4}v_{n-2})\}.$$

(c') There is a reserved square  $v_1v_{n+4}x_5v_{n+3}$ , or equivalently  $v_1v_{n+3}x_5v_{n+4}$ , that is not an underlying square for  $\mathcal{D}_{n+4}$ . Conditions (a'), (b'), (c') are almost what we need to say that  $\Phi_{n+4}^-$  has Property P. Condition (b') is correct but (a') has  $v_2$  where it should have  $v_1$ , and (c') has  $v_1$  where it should have  $v_2$ . However, renaming  $v_1$  as  $v_2$  and  $v_2$  as  $v_1$  does not affect (b'), and puts (a') and (c') into the correct form for Property P. Thus, after this renaming we have a  $\tilde{Q}(n+4, 3)$  with Property P.

This completes the proof of the induction step. Now the basis and the induction step together imply Lemma 4.5.  $\square$

Theorem 1.2 now follows from Lemmas 4.1, 4.2, 4.3, 4.4 and 4.5.

## 5 Conclusion

Face-simple quadrangulations are of interest because they are somewhere in between closed 2-cell embeddings and polyhedral embeddings. An embedding is *closed 2-cell* if every face is bounded by a cycle, so that a face does not 'self-touch' (equivalently, a 2-representative embedding of a 2-connected graph), and *polyhedral* if it is also true that two distinct faces touch at most once, meaning that the intersection of their boundaries is empty, a single vertex, or a single edge (equivalently, a 3-representative embedding of a 3-connected graph). Every quadrangulation is closed 2-cell by definition, but the following lemma shows that minimal quadrangulations cannot be polyhedral.

**Lemma 5.1.** *If  $\Phi$  is a quadrangular embedding of an  $n$ -vertex  $m$ -edge graph with  $m > \frac{1}{2}\binom{n}{2}$  then  $\Phi$  is not polyhedral.*

*Proof.* Let  $E$  be the edge set of the underlying (simple) graph of  $\Phi$ , where each edge is considered as a vertex pair, and let  $D$  be the multiset of diagonals of squares of  $\Phi$ , i.e., all vertex pairs  $\{u, v\}$  that occur as diagonals, counted by the number of squares in which each diagonal occurs. Then  $|E| = |D| = m$  and since  $|E| + |D| > \binom{n}{2}$  there is some pair that either occurs twice in  $D$ , or occurs once in  $E$  and once in  $D$ . This means that there are two faces that touch more than once.  $\square$

Therefore, it is natural to consider a weakening of polyhedral to closed 2-cell and face-simple, where two faces can touch more than once, but not along two edges. In the orientable case, minimal quadrangulations of  $S_g$ ,  $g \geq 1$ , are automatically face-simple by Observation 2.3, giving Theorem 1.4. However, in general our nonorientable minimal quadrangulations are not face-simple, since we often use crosscap additions, which create two faces that touch along two edges.

**Question 5.2.** *Does  $N_q$  have a minimal quadrangulation that is also face-simple, so that  $n'(N_q) = n(N_q)$ , for all but a few small values of  $q$ ?*

We think that the answer is probably ‘yes.’ It should be possible to prove this by adapting the techniques in this paper. However, even if we avoid crosscap additions, some care is needed. A handle addition of Type I (or a disc addition followed by a suitable handle addition of Type I) preserves face-simplicity, but for nonorientable embeddings handle additions of Types II, III and IV may create squares that touch along two edges.

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