Toughness and spanning trees in K_4-minor-free graphs

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Abstract

For an integer k, a *k-tree* is a tree with maximum degree at most k. More generally, if f is an integer-valued function on vertices, an *f-tree* is a tree in which each vertex v has degree at most $f(v)$. Let $c(G)$ denote the number of components of a graph G. We show that if G is a connected K_4-minor-free graph and

$$c(G - S) \le \sum_{v \in S} (f(v) - 1) \quad \text{for all } S \subseteq V(G) \text{ with } S \neq \emptyset$$

then G has a spanning f-tree. Consequently, if G is a $\frac{1}{k-1}$-tough K_4-minor-free graph, then G has a spanning k-tree. These results are stronger than results for general graphs due to Win (for k-trees) and Ellingham, Nam and Voss (for f-trees). The K_4-minor-free graphs form a subclass of planar graphs, and are identical to graphs of treewidth at most 2, and also to graphs whose blocks are series-parallel. We provide examples to show that the inequality above cannot be relaxed by adding 1 to the right-hand side, and also to show that our result does not hold for general planar graphs. Our proof uses a technique where we incorporate toughness-related information into weights associated with vertices and cutsets.

Keywords: toughness, spanning tree, K_4-minor-free, series-parallel, treewidth.

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1 Introduction

All graphs considered are simple and finite. Let G be a graph. We denote by $d_G(v)$ the degree of vertex v in G. For $S \subseteq V(G)$ the subgraph induced on $V(G) - S$ is denoted by $G - S$; we abbreviate $G - \{v\}$ to $G - v$. The number of components of G is denoted by $c(G)$. The graph is said to be *t-tough* for a real number $t \geq 0$ if $|S| \geq t \cdot c(G - S)$ for each $S \subseteq V(G)$ with $c(G - S) \geq 2$. The *toughness* $\tau(G)$ is the largest real number t for which G is t-tough, or ∞ if G is complete. Positive toughness implies that G is connected. If G has a hamiltonian cycle it is well known that G is 1-tough.

In 1973, Chvátal [3] conjectured that for some constant t_0, every t_0-tough graph is hamiltonian. Thomassen (see [2, p. 132]) showed that there are nonhamiltonian graphs with toughness greater than 3/2. Enomoto, Jackson, Katerinis and Saito [8] showed that every 2-tough graph has a 2-factor (2-regular spanning subgraph), but also constructed $(2 - \varepsilon)$-tough graphs with no 2-factor, and hence no hamiltonian cycle, for every $\varepsilon > 0$. Bauer, Broersma and Veldman [1] constructed $(\frac{9}{4} - \varepsilon)$-tough nonhamiltonian graphs for every $\varepsilon > 0$. Thus, any such t_0 is at least $\frac{9}{4}$.

There have been a number of papers on toughness conditions that guarantee the existence of more general spanning structures in a graph. A *k-tree* is a tree with maximum degree at most k, and a *k-walk* is a closed walk with each vertex repeated at most k times. Note that a spanning 2-tree is a hamiltonian path and a spanning 1-walk is a hamiltonian cycle. Jackson and Wormald [10] showed that on a given vertex set a k-walk can be obtained from a k-tree; conversely, a $(k + 1)$-tree can be obtained from a k-walk. More generally, if $f : V(G) \to \mathbb{Z}$ then an *f-tree* is a tree with $d_T(v) \leq f(v)$ for all $v \in V(T)$, and an *f-walk* is a closed walk that uses every vertex v at most $f(v)$ times.

The first toughness result for spanning trees of bounded degree was by Win.

Theorem 1.1 (Win [13]). *Suppose G is a connected graph, $k \geq 2$, and*

$$c(G - S) \quad \leq \quad (k - 2)|S| + 2 \quad \textit{for all } S \subseteq V(G).$$

Then G has a spanning k-tree.

Win's result implies that for $k \geq 3$, $\frac{1}{k-2}$-tough graphs have a k-tree (and hence a k-walk). Ellingham, Nam and Voss showed that the bound on the degrees need not be constant.

Theorem 1.2 (Ellingham, Nam and Voss [7]). *Suppose G is a connected graph, $f : V(G) \to \mathbb{Z}$ with $f(v) \geq 2$ for all $v \in V(G)$, and*

$$c(G - S) \quad \leq \quad \sum_{v \in S}(f(v) - 2) + 2 \quad \textit{for all } S \subseteq V(G). \tag{1.1}$$

Then G has a spanning f-tree.

While these conditions are sufficient, and sharp in the sense that the right-hand side of the inequality cannot be increased by 1, they are not necessary. For a necessary condition, a graph with a spanning k-walk (and hence a graph with a spanning k-tree) must be $\frac{1}{k}$-tough. However, a stronger necessary condition, which also applies for non-constant degree bounds, can be obtained by counting components in a tree T: for any $S \subseteq V(T)$, $c(T-S) \leq \sum_{v \in S}(d_T(v)-1)+1$. Applying this to a spanning f-tree of a graph G gives the following.

Observation 1.3. *Suppose G is a graph with a spanning f-tree. Then*

$$c(G-S) \quad \leq \quad \sum_{v \in S}(f(v)-1)+1 \quad \textit{for all } S \subseteq V(G). \tag{1.2}$$

When $f(v) = k$ for all v, this condition is slightly weaker than being $\frac{1}{k-1}$-tough.

Recently some stronger versions of Theorem 1.2 have been posted by Hasanvand [9, Subsection 4.2], although they do not seem to change the condition (1.1) in a substantial way. Hasanvand does, however, appear to have made a significant advance for f-walks. For $k \geq 3$, as noted above, $\frac{1}{k-2}$-tough graphs have a spanning k-tree and hence a spanning k-walk. Jackson and Wormald [10] conjectured that in fact $\frac{1}{k-1}$-tough graphs have a spanning k-walk. Hasanvand used a clever argument to combine a stronger form of Theorem 1.2 with a result of Kano, Katona and Szabó [11] on the existence of spanning subgraphs with parity conditions, to show the following.

Theorem 1.4 (Hasanvand, [9, Theorem 5.5])**.** *Suppose G is a graph, M is a matching in G, $f : V(G) \to \mathbb{Z}$ with $f(v) \geq 1$ for all $v \in V(G)$, and*

$$c(G-S) \quad \leq \quad \sum_{v \in S}(f(v)-1)+1 \quad \textit{for all } S \subseteq V(G). \qquad \text{(1.2) again}$$

Then G has a spanning f-walk that uses the edges of M.

With $f(v) = k$ for all v, Theorem 1.4 verifies Jackson and Wormald's conjecture.

It is unknown whether Theorem 1.2 can be improved for graphs in general by weakening condition (1.1). It is therefore of interest to see whether this can be done for special classes of graphs. In this paper we focus on K_4-minor-free graphs.

A graph H is a *minor* of a graph G if a graph isomorphic to H can be obtained from G by edge contractions, edge deletions and vertex deletions; if not, G is *H-minor-free*. Since both $K_{3,3}$ and K_5 contain a K_4 minor, K_4-minor-free graphs are $K_{3,3}$-minor-free and K_5-minor-free, i.e., planar. The class of K_4-minor-free graphs includes all *series-parallel graphs*, constructed by series and parallel compositions starting from copies of K_2. Duffin [5] gave three characterizations for series-parallel graphs; in particular, he showed that a graph with no cutvertex is K_4-minor-free if and only if it is series-parallel. Wald and Colbourn [12] showed that K_4-minor-free graphs are also identical to graphs of treewidth at most 2.

Let G be a K_4-minor-free graph with toughness greater than $\frac{2}{3}$. The toughness implies that G has no cutvertex, and in conjunction with K_4-minor-freeness, also implies that G is $K_{2,3}$-minor-free (see Lemma 3.2). Thus, G is either K_2 or a 2-connected outerplanar graph. Thus, a K_4-minor-free graph with at least three vertices and toughness greater than $\frac{2}{3}$ is hamiltonian. Dvořák, Král' and Teska [6] showed that every K_4-minor-free graph with toughness greater than $\frac{4}{7}$ has a spanning 2-walk, and they constructed a $\frac{4}{7}$-tough K_4-minor-free graph with no spanning 2-walk.

Our main result is a sufficient condition for a connected K_4-minor-free graph to have an f-tree. We show that in fact a very slight strengthening of the necessary condition (1.2) suffices, subtracting 1 from the right-hand side. For vertices x, v in a graph, define $\delta_x(v)$ to be 1 if $x = v$, and 0 otherwise.

Theorem 1.5. *Let G be a connected K_4-minor-free graph, and $f : V(G) \to \mathbb{Z}$. Suppose that $z \in V(G)$ and*

$$c(G - S) \quad \leq \quad \sum_{v \in S}(f(v) - 1) \quad \textit{for all } S \subseteq V(G) \textit{ with } S \neq \emptyset. \tag{CC}$$

Then G has a spanning $(f - \delta_z)$-tree, i.e., a spanning f-tree T such that $d_T(z) \leq f(z) - 1$.

The condition in Theorem 1.5 is something we refer to frequently; we call it the *Component Condition* and refer to it as (CC). Condition (CC) with $S = \{v\}$ implies that $f(v) \geq 1$ if $V(G) = \{v\}$, and that $f(v) \geq 2$ for all $v \in V(G)$ if $|V(G)| \geq 2$. Therefore, we do not need to explicitly specify that f is nonnegative. If we take $f(v) = k \geq 2$ for all the vertices, then (CC) just means that G is $\frac{1}{k-1}$-tough.

Corollary 1.6. *Let G be a K_4-minor-free graph, let $k \geq 2$ be an integer, and let $z \in V(G)$. If G is $\frac{1}{k-1}$-tough, then G has a spanning k-tree T such that $d_T(z) \leq k - 1$.*

Examples show that we cannot weaken (CC) in Theorem 1.5 by adding 1 to the right-hand side, which gives (1.2). Let $f(x) \geq 2, f(y) \geq 2, f(z) \geq 2$ be three integers, and let G be the K_4-minor-free graph obtained from a triangle (xyz) by adding $f(v) - 1$ pendant edges at each $v \in \{x, y, z\}$. Set $f(v) = 2$ for every $v \in V(G) - \{x, y, z\}$. Then (1.2) is easily verified, but every spanning tree T of G has a vertex $v \in \{x, y, z\}$ with $d_T(v) = f(v) + 1$.

Also, note that there is no upper bound on the values we can assign to f. If we take a specified set of vertices X and let $f(v) = |V(G)| + 1$ for all $v \in V(G) - X$, (CC) is automatically satisfied for sets S containing a vertex of $V(G) - X$ (we use this trick in some of our proofs). We can therefore bound the degrees of just the specified vertices.

Corollary 1.7. *Let G be a connected K_4-minor-free graph, $X \subseteq V(G)$, and $f : X \to \mathbb{Z}$. Suppose that $z \in X$ and*

$$c(G - S) \quad \leq \quad \sum_{v \in S}(f(v) - 1) \quad \textit{for every } S \subseteq X \textit{ with } S \neq \emptyset.$$

Then G has a spanning tree T such that $d_T(v) \leq f(v)$ for all $v \in X$, and $d_T(z) \leq f(z) - 1$.

Toughness is an awkward parameter to deal with in arguments. It is hard to control the toughness of a subgraph, or of a graph obtained by some kind of reduction from an original graph. This makes it difficult to prove results based on toughness conditions using induction. However, in this paper we provide a way of doing induction using a toughness-related condition, by first, in Section 2, transforming the toughness information into weights associated with vertices and cutsets. This approach seems to be new, and of interest apart from our results for K_4-minor-free graphs.

Another idea in this paper is dealing with situations where we delete independent sets of vertices separately from situations where we delete sets that may have adjacencies. Loosely, we expect to get more components when we delete an independent set. The concept of 'fundamental' graphs in Section 2 is used to implement this distinction. In Section 3 we prove a result for fundamental graphs first, then extend it to more general graphs.

2 Structure of nontrivial 2-cuts

In this section we show how to convert a toughness-related condition into weights associated with vertices and certain cutsets. The results in this section apply to all graphs, not just to K_4-minor-free graphs.

If G is a graph and $H \subseteq G$ (H may just be a set of vertices) then a *bridge of H* or *H-bridge* in G is a subgraph of G that is either an edge not in H joining two vertices of H (a *trivial* bridge), or a component of $G - V(H)$ together with all edges joining it to $V(H)$ (a *nontrivial* bridge). The *set of attachments* of an H-bridge B is $V(B) \cap V(H)$. For $S \subseteq V(G)$, when no confusion will result, we also refer to the set of attachments of a component of $G - S$, meaning that of the corresponding S-bridge. A *k-cut* in G is a set S of k vertices for which $G - S$ is disconnected. If $F = \{u, v\}$ is a set of two vertices in G, let $c(G, F)$ or $c(G, uv)$ denote the number of nontrivial F-bridges that contain both vertices of F. The notation $c(G, uv)$ does not imply that $uv \in E(G)$. A *nontrivial 2-cut* or *N2C* in G is a set F of two vertices with $c(G, F) \geq 3$.

A *block* is a connected graph with no cutvertex, and a *block of G* is a maximal subgraph of G that is itself a block. Let $\mathcal{B}$ be the set of blocks and $\mathcal{C}$ the set of cutvertices of G. The *block-cutvertex tree* of a connected graph G has vertex set $\mathcal{B} \cup \mathcal{C}$, and $c \in \mathcal{C}$ is adjacent to $B \in \mathcal{B}$ if and only if the block B contains the cutvertex c. A block of G is a *leaf block* if it is a leaf of the block-cutvertex tree. If we choose a particular *root edge* e_0 of G, then the block B_0 containing e_0 is the *root block* which we treat as the root vertex of the block-cutvertex tree. Every block other than B_0 has a unique *root vertex*, namely its parent in the rooted block-cutvertex tree.

In what follows we try to develop information to help us construct a spanning tree with restricted degrees, by allocating the bridges of each nontrivial 2-cut $\{u, v\}$ between u and v. We begin with some basic properties of nontrivial 2-cuts.

Lemma 2.1. *Let $\{u, v\}$ be an N2C in a graph G.*
(a) *The graph G has three internally disjoint uv-paths.*
(b) *If $S \subseteq V(G) - \{u, v\}$ and $|S| \leq 2$ then u and v lie in the same component of $G - S$.*
(c) *If $\{x, y\}$ is an N2C of G distinct from $\{u, v\}$, then u and v lie in a unique $\{x, y\}$-bridge.*

Proof. Since $c(G, uv) \geq 3$, there are three different $\{u, v\}$-bridges that attach at both u and v and we can take a path through each, proving (a). Then (b) follows immediately. For (c) we may assume that $u \notin \{x, y\}$. Then the unique $\{x, y\}$-bridge containing u also contains v, either because $v \in \{x, y\}$, or by (b) otherwise. □

Let G be a connected graph, $\mathcal{F}$ the set of N2Cs of G, and $W = \bigcup_{F \in \mathcal{F}} F$. The graph with vertex set W and edge set $\mathcal{F}$ is called the *N2C graph of* G. If W is an independent set in G, we call G *fundamental*.

The following observations come from comparing the components of $G - S$ and of $G' - S$ for some subgraph (block or union of $\{x, y\}$-bridges) G' of G, and using Lemma 2.1.

Observation 2.2. *Suppose G is a connected graph with $|V(G)| \geq 2$. Let $u, v \in V(G)$. If u and v are not in a common block of G, then $c(G, uv) = 1$; if they belong to a common block B then $c(B, uv) = c(G, uv)$.*

Thus, the N2Cs of a block B are the N2Cs $\{u, v\}$ of G with $\{u, v\} \subseteq V(B)$, and if G is fundamental then B is also fundamental. Conversely, the set of N2Cs of G is the disjoint union of the sets of N2Cs of its blocks, and if each block is fundamental then G is fundamental.

Observation 2.3. *Suppose $\{x, y\}$ is an N2C of a 2-connected graph G. Let G' be the union of two or more nontrivial $\{x, y\}$-bridges of G (possibly $G' = G$). Then G' is 2-connected. If u and v are not in a common $\{x, y\}$-bridge in G', then $c(G, uv) = 1$; if they belong to a common $\{x, y\}$-bridge in G' and $\{u, v\} \neq \{x, y\}$ then $c(G', uv) = c(G, uv)$.*

Thus, the N2Cs of G' are the N2Cs $\{u, v\}$ of G with $\{u, v\} \subseteq V(D)$ for some $\{x, y\}$-bridge D of G', except that $\{x, y\}$ is not necessarily an N2C of G'. If G is fundamental then G' is also fundamental.

Observation 2.3 is not necessarily true if we take G' to be a single nontrivial $\{x, y\}$-bridge; then G' may not be 2-connected, and we have $c(G', uv) > c(G, uv)$ if x and y are in different components of $G' - \{u, v\}$.

Lemma 2.4. *Suppose J is a connected subgraph of the N2C graph H of a graph G.*
(a) *If $S \subseteq V(G)$, $|S| \leq 2$ and $S \cap V(J) = \emptyset$, then $V(J)$ lies in a single component of $G - S$.*
(b) *If J' is a connected subgraph of H vertex-disjoint from J, then $V(J)$ lies in a single component of $G - V(J')$.*
(c) *If $u, v \in V(J)$ are separated in G by a cutvertex x of G, then x is also a cutvertex of J separating u and v in J.*

Proof. (a) By Lemma 2.1(b), for each $uv \in E(J)$, u and v are in the same component of $G - S$. Applying this repeatedly, all of $V(J)$ is in a single component of $G - S$.

(b) Let $uv \in E(J)$. By (a), all of $V(J')$ lies in the same component of $G - \{u, v\}$, say C. Since $\{u, v\}$ is an N2C of G, there is a path from u to v in $G - V(C) \subseteq G - V(J')$, so u and v are in the same component of $G - V(J')$. Applying this repeatedly, all of $V(J)$ is in a single component of $G - V(J')$.

(c) If there is a uv-path in J that does not contain x, then by (a) this path lies in a single component of $G - x$, contradicting the choice of u and v. Thus, every uv-path in J contains x, so x is a cutvertex separating u and v in J. □

Our overall strategy now is to use our toughness-related condition to assign weights associated with (N2C, vertex) ordered pairs. We show that we can either (a) assign weights satisfying certain conditions, or else (b) find a set that violates our toughness-related condition. The following proposition, giving a lower bound on $c(G - U)$ for certain subsets U, will be used to demonstrate (b). The statement of this result and many of our computations involve terms of the form $c(G - u) - 1$ and $c(G, uv) - 2$. The reader may wonder why we do not simplify these; the answer is that we often think of the graph as having a 'main' part consisting of one of the bridges of a cutvertex, or two of the bridges of a 2-cut, and these terms count the number of 'extra' bridges outside the 'main' part.

Proposition 2.5. *Let J be a connected subgraph of the N2C graph of a connected fundamental graph G, and $U = V(J)$. Then*

$$c(G - U) \geq \sum_{uv \in E(J)} (c(G, uv) - 2) + \sum_{w \in U} (c(G - w) - 1) + |U|. \tag{2.1}$$

Proof. The proof is by induction on $|E(J)|$. It is easy to check that the conclusion is true if $|E(J)| = 0$, when $|U| = |V(J)| = 1$. So we assume that $|E(J)| \geq 1$. We consider three cases.

Case 1: Suppose G has no cutvertex. Since $|E(J)| \geq 1$, G has at least five vertices, so G is 2-connected. For the 2-connected case (2.1) simplifies to

$$c(G - U) \geq \sum_{uv \in E(J)} (c(G, uv) - 2) + |U|. \tag{2.2}$$

Let $xy \in E(J)$ and let the $\{x, y\}$-bridges in G be $D_1, D_2, \ldots, D_k$, where $k = c(G, uv) \geq 3$. For each i let $U_i = U \cap V(D_i) \supseteq \{x, y\}$ and $J_i = J[U_i] - xy$, so that $V(J_i) = U_i$.

Let $G_i = D_i \cup xa_iy$, where a_i is a new vertex. Each G_i is a minor of G, and hence K_4-minor-free. We may apply Observation 2.3 to both G_i and G, considered as subgraphs of $G \cup xa_iy$. We see that G_i is 2-connected. Suppose that $\{u, v\} \subseteq V(G_i)$ and $\{u, v\} \neq \{x, y\}$. If $a_i \in \{u, v\}$, then $c(G_i, uv) = 1$. If $a_i \notin \{u, v\}$ then $c(G_i, uv) = c(G \cup xa_iy, uv) = c(G, uv)$. Thus, $\{u, v\} \neq \{x, y\}$ is an N2C of G_i if and only if it is an N2C of G with $\{u, v\} \subseteq V(G_i)$.

It follows that G_i is fundamental. Moreover, if H_i is the N2C graph of G_i, we have $E(J_i) \subseteq E(H_i)$ and $V(J_i) \subseteq V(H_i) \cup \{x, y\}$ (each of x and y is in J_i, but could possibly be in no N2C of G_i, in which case it is an isolated vertex of J_i and not a vertex of H_i). Since $xy \notin E(J_i)$, $|E(J_i)| < |E(J)|$ for all i. (In future we will provide less detail when applying Observation 2.3, but the reasoning will be similar to the reasoning here.)

Claim. *We have* $c(G_i - U_i) \geq \sum_{uv \in E(J_i)} (c(G_i, uv) - 2) + |U_i|$ *for each* $i = 1, 2, \dots k$.

Proof of claim. First suppose that J_i is connected. Then x and y are incident with edges of J_i, so they are contained in N2Cs of G_i, and so $J_i \subseteq H_i$. Thus, the claim follows because we can apply (2.2) to $G_i - U_i$ and J_i by induction.

Now suppose that J_i is disconnected. Since J is connected, J_i must have exactly two components J_i^x and J_i^y, containing x and y respectively. Let $U_i^x = V(J_i^x)$ and $U_i^y = V(J_i^y)$. Consider the components of $G_i - U_i$ and divide them into three groups: $\mathcal{C}_x$ are those all of whose attachments belong to U_i^x, $\mathcal{C}_y$ are those all of whose attachments belong to U_i^y, and $\mathcal{C}_{xy}$ are those that have attachments in both U_i^x and U_i^y.

We claim that $|\mathcal{C}_{xy}| \geq 2$. Obviously $a_i \in \mathcal{C}_{xy}$. Take an xy-path $v_0 v_1 \dots v_t$ in D_i. There is a subpath $P = v_i v_{i+1} \dots v_j$ such that $v_i \in U_i^x$, $v_j \in U_i^y$ and the internal vertices (if any) of P belong to neither U_i^x nor U_i^y. Since G is fundamental, P does have internal vertices, which belong to a second component in $\mathcal{C}_{xy}$.

Now we claim that

$$|\mathcal{C}_x| \geq \sum_{uv \in E(J_i^x)} (c(G_i, uv) - 2) + |U_i^x| - 1. \tag{2.3}$$

If $|U_i^x| = 1$ this just says that $|\mathcal{C}_x| \geq 0$, which is trivially true. So we may suppose that $|U_i^x| \geq 2$, which means that x is in an N2C of G_i and $J_i^x \subseteq H_i$. Then U_i^y is in a single component of $G_i - U_i^x$, either trivially if $|U_i^y| = 1$, or by Lemma 2.4(b) if $|U_i^y| \geq 2$ so that J_i^y is a subgraph of H_i. The component of $G_i - U_i^x$ containing U_i^y must also contain all components in $\mathcal{C}_y \cup \mathcal{C}_{xy}$. On the other hand, each $C \in \mathcal{C}_x$ is a separate component of $G_i - U_i^x$. Therefore, $c(G_i - U_i^x) = |\mathcal{C}_x| + 1$. Using this and applying (2.2) to $G_i - U_i^x$ and J_i^x by induction gives (2.3).

We have an inequality similar to (2.3) for $\mathcal{C}_y$. Therefore,

$$\begin{aligned} c(G_i - U_i) &= |\mathcal{C}_x| + |\mathcal{C}_y| + |\mathcal{C}_{xy}| \\ &\geq \sum_{uv \in E(J_i^x)} (c(G_i, uv) - 2) + |U_i^x| - 1 + \\ &\qquad \sum_{uv \in E(J_i^y)} (c(G_i, uv) - 2) + |U_i^y| - 1 + 2 \\ &= \sum_{uv \in E(J_i)} (c(G_i, uv) - 2) + |U_i| \end{aligned}$$

which proves the claim. ■

Now we prove (2.2) and hence (2.1). For each i, $c(D_i - U_i) = c(G_i - U_i) - 1$ since we lose the component a_i. Thus,

$$\begin{aligned}
c(G-U) &= \sum_{i=1}^{k} c(D_i - U_i) \;=\; \sum_{i=1}^{k} (c(G_i - U_i) - 1) \\
&\geq \sum_{i=1}^{k} \left(\sum_{uv \in E(J_i)} (c(G,uv) - 2) + |U_i| - 1 \right) \\
&\qquad\qquad \text{by the claim and since } c(G_i, uv) = c(G, uv) \\
&= \sum_{uv \in E(J-xy)} (c(G,uv) - 2) + |U| + 2(k-1) - k \\
&\qquad\qquad \text{since } x \text{ and } y \text{ are overcounted } k-1 \text{ times each} \\
&= \sum_{uv \in E(J-xy)} (c(G,uv) - 2) + |U| + c(G,xy) - 2 \\
&= \sum_{uv \in E(J)} (c(G,uv) - 2) + |U|.
\end{aligned}$$

Case 2: Suppose G has a cutvertex but all vertices of J lie in a single block B of G. If $S \subseteq U$ then each component of $G - S$ is either (a) a component C of $B - S$ together with all B-bridges in G that attach at a vertex of C, or (b) a component of $G - w$ not containing $B - w$, for some $w \in S$. Thus,

$$c(G-U) = c(B-U) + \sum_{w \in U} (c(G-w) - 1). \tag{2.4}$$

By Observation 2.2, B is fundamental, J is a subgraph of the N2C graph of B, and

$$c(G,uv) = c(B,uv) \quad \text{for all } uv \in E(J). \tag{2.5}$$

Applying Case 1 to B, we know from (2.2) that

$$c(B-U) \geq \sum_{uv \in E(J)} (c(B,uv) - 2) + |U|. \tag{2.6}$$

Combining (2.4), (2.5) and (2.6) gives (2.1).

Case 3: Suppose the vertices of J do not all lie in a single block of G. Then there is a cutvertex x of G and vertices u, v of J in distinct components of $G - x$. Let G_1 be the x-bridge in G containing u, and G_2 the union of all other x-bridges in G, which contains v.

By Lemma 2.4(c), x is also a cutvertex of J separating u and v, and it follows from Lemma 2.4(a) that the vertex set of each x-bridge in J lies in some x-bridge in G. So for $i = 1, 2$ let J_i be the union of the x-bridges in J whose vertex sets lie in G_i. We have

$u \in V(J_1)$, $v \in V(J_2)$, $V(J_1) \cap V(J_2) = \{x\}$, and $J = J_1 \cup J_2$. Since $|V(J_1)|, |V(J_2)| \geq 2$, we have $|E(J_1)|, |E(J_2)| > 0$, and so $|E(J_1)|, |E(J_2)| < |E(J)|$.

Let $U_i = V(J_i)$ for $i = 1, 2$. Thinking of G, G_1 and G_2 as the unions of their blocks, it follows from Observation 2.2 that the N2Cs of G_i are the N2Cs $\{u, v\}$ of G with $\{u, v\} \subseteq V(G_i)$, J_i is a subgraph of the N2C graph of G_i, $c(G_i, uv) = c(G, uv)$ for $uv \in E(J_i)$, and G_i is fundamental. Applying (2.1) to $G_i - U_i$ and J_i by induction for each i,

$$\begin{aligned}
c(G-U) &= \sum_{i=1}^{2} c(G_i - U_i) \\
&\geq \sum_{i=1}^{2} \left(\sum_{uv \in E(J_i)} (c(G_i, uv) - 2) + \sum_{w \in U_i} (c(G_i - w) - 1) + |U_i| \right) \\
&= \sum_{uv \in E(J)} (c(G, uv) - 2) + \sum_{w \in U} (c(G - w) - 1) - 1 + |U| + 1 \\
&\qquad \text{since } c(G_1 - x) + c(G_2 - x) = c(G - x) \text{ and } |U_1| + |U_2| = |U| + 1 \\
&= \sum_{uv \in E(J)} (c(G, uv) - 2) + \sum_{w \in U} (c(G - w) - 1) + |U|,
\end{aligned}$$

proving (2.1). □

Theorem 2.6. *Let $\mathcal{F}$ be the set of all N2Cs in a connected fundamental graph G, and let $W = \bigcup \mathcal{F}$ be the set of vertices used by $\mathcal{F}$. For each $v \in V(G)$, let $\mathcal{F}(v) = \{F \in \mathcal{F} \mid v \in F\}$. If $f : V(G) \to \mathbb{Z}$ satisfies*

$$c(G - S) \;\leq\; \sum_{v \in S} (f(v) - 1) \quad \text{for all } S \subseteq V(G) \text{ with } S \neq \emptyset \tag{CC}$$

then there is a nonnegative integer function ω on ordered pairs (F, u) with $F \in \mathcal{F}$ and $u \in F$ such that

$$\omega(F, u) + \omega(F, v) \;=\; c(G, F) - 2 \quad \text{for all } F = \{u, v\} \in \mathcal{F}, \text{ and} \tag{2.7}$$

$$\sum_{F \in \mathcal{F}(u)} \omega(F, u) \;\leq\; f(u) - c(G - u) - 1 \quad \text{for all } u \in V(G). \tag{2.8}$$

Proof. If $v \in V(G) - W$, then $\mathcal{F}(v) = \emptyset$ so $\sum_{F \in \mathcal{F}(v)} \omega(F, v) = 0$. So (2.8) is equivalent to $c(G - v) \leq f(v) - 1$ which follows from (CC) by taking $S = \{v\}$. Moreover, (2.7) does not involve any vertices not in W. Thus, in constructing ω, the only vertices we need to be concerned with are those in W.

We associate with G a network N. Its vertex set is $V(N) = \{s, t\} \cup \mathcal{F} \cup W$ where s, t are new vertices. Its arc set $A(N)$ consists of three subsets: $A_1 = \{sF \mid F \in \mathcal{F}\}$, $A_2 = \{Fu \mid F \in \mathcal{F}, u \in W, u \in F\}$, and $A_3 = \{ut \mid u \in W\}$. Each arc a has a capacity $\gamma(a)$

defined as follows:

$$\begin{array}{lll}\gamma(sF) = c(G,F) - 2 & \text{if } sF \in A_1, & \\ \gamma(Fu) = \infty & \text{if } Fu \in A_2, & \text{and} \\ \gamma(ut) = f(u) - c(G-u) - 1 & \text{if } ut \in A_3. & \end{array}$$

We claim that a maximum st-flow φ of value $\Phi = \sum_{sF \in A_1} \gamma(sF)$ in N gives a desired way of distributing the weights on N2Cs to the vertices, by taking $\omega(F,u) = \varphi(Fu)$ for all $Fu \in A_2$. All arcs in A_1 must be saturated by such a flow, so flow conservation at a vertex $F \in \mathcal{F}$, where $F = \{u,v\}$, gives

$$\omega(F,u) + \omega(F,v) = \varphi(Fu) + \varphi(Fv) = \varphi(sF) = \gamma(sF) = c(G,F) - 2$$

which verifies (2.7), and flow conservation at a vertex $u \in W$ gives

$$\sum_{F \in \mathcal{F}(u)} \omega(F,u) = \sum_{Fu \text{ enters } u} \varphi(Fu) = \varphi(ut) \leq \gamma(ut) = f(u) - c(G-u) - 1$$

which verifies (2.8).

So assume that N does not have a maximum flow of value Φ; we will show that this gives a contradiction. By the Max-Flow Min-Cut Theorem, N has an st-cut

$$[S,T] = [\{s\} \cup \mathcal{F}_1 \cup W_1, \mathcal{F}_2 \cup W_2 \cup \{t\}] = [\{s\}, \mathcal{F}_2] \cup [\mathcal{F}_1, W_2] \cup [W_1, t]$$

such that $\gamma(S,T) < \Phi$. Here $[Q,R]$ denotes all arcs from Q to R, $\mathcal{F}_1 \cup \mathcal{F}_2 = \mathcal{F}$, $\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$, $W_1 \cup W_2 = W$, and $W_1 \cap W_2 = \emptyset$.

If $\mathcal{F}_2 = \mathcal{F}$ then $\gamma(S,T) \geq \Phi$ which is a contradiction, so $\mathcal{F}_2 \neq \mathcal{F}$ and $\mathcal{F}_1 \neq \emptyset$. Since arcs in A_2 have infinite capacity and $\gamma(S,T) < \infty$, we must have $[\mathcal{F}_1, W_2] = \emptyset$. Therefore,

$$\gamma(S,T) = \sum_{F \in \mathcal{F}_2} \gamma(sF) + \sum_{w \in W_1} \gamma(wt) = \sum_{\{u,v\} \in \mathcal{F}_2} (c(G,uv) - 2) + \sum_{w \in W_1} (f(w) - c(G-w) - 1).$$

Since $\gamma(S,T) < \Phi = \sum_{sF \in A_1} \gamma(sF) = \sum_{\{u,v\} \in \mathcal{F}} (c(G,uv) - 2)$, we obtain

$$\sum_{w \in W_1} (f(w) - c(G-w) - 1) < \sum_{\{u,v\} \in \mathcal{F}_1} (c(G,uv) - 2),$$

or

$$\sum_{\{u,v\} \in \mathcal{F}_1} (c(G,uv) - 2) + \sum_{w \in W_1} (c(G-w) - 1) > \sum_{w \in W_1} (f(w) - 2).$$

Since $[\mathcal{F}_1, W_2] = \emptyset$, $\bigcup \mathcal{F}_1 \subseteq W_1$. So we may consider the graph H_1 with vertex set W_1 and edge set $\mathcal{F}_1$. By the Pigeonhole Principle, there is a component J of H_1 with vertex set $U \subseteq W_1$ such that

$$\sum_{uv \in E(J)} (c(G,uv) - 2) + \sum_{w \in U} (c(G-w) - 1) > \sum_{w \in U} (f(w) - 2).$$

Combining this with Proposition 2.5, we get

$$\begin{aligned} c(G-U) &\geq \sum_{uv\in E(J)} (c(G,uv)-2) + \sum_{w\in U} (c(G-w)-1) + |U| \\ &> \sum_{w\in U} (f(w)-2) + |U| = \sum_{w\in U} (f(w)-1), \end{aligned}$$

giving the required contradiction to (CC). □

3 Proof of Theorem 1.5

In this section we use Theorem 2.6 to prove Theorem 1.5. We start with some preliminary results.

Lemma 3.1. *Suppose H is a subgraph of a K_4-minor-free graph G. Then every N2C of H is also an N2C of G.*

Proof. Suppose $\{u, v\}$ is an N2C of H that is not an N2C of G. Then $c(G, uv) < c(H, uv)$ so there must be a path P in $G - \{u, v\}$ between two components C_1, C_2 of $H - \{u, v\}$ that attach at both u and v, where we may assume that the internal vertices of P (if any) are not vertices of H. Let C_3 be a third component of $H - \{u, v\}$ attaching at both u and v. The subgraph $H[\{u, v\} \cup V(C_1 \cup C_2 \cup C_3)] \cup P$ of G contains a K_4 minor, which is a contradiction. □

Lemma 3.2. *If G is K_4-minor-free and G has no N2C, then G is outerplanar.*

Proof. A graph is outerplanar if and only if it is K_4-minor-free and $K_{2,3}$-minor-free. So assume that G has a $K_{2,3}$ minor. Since $K_{2,3}$ has maximum degree 3, the existence of a $K_{2,3}$ minor implies that G contains a subdivision N of $K_{2,3}$, consisting of two vertices s_1, s_2 of degree 3 and three internally disjoint s_1s_2-paths of length at least 2. But now $\{s_1, s_2\}$ is an N2C of N and hence, by Lemma 3.1, an N2C of G, which is a contradiction. □

Lemma 3.3. *For every non-isolated vertex x of a graph G, and for every block B of G that contains x, there is $xy \in E(B)$ such that $c(B, xy) = c(G, xy) \leq 1$.*

Proof. For each $uv \in E(G)$, let $\mathcal{A}(uv)$ be the set of $\{u, v\}$-bridges of G that attach at both u and v. Suppose that $c(G, xy) \geq 2$ for every $xy \in E(B)$. Let $D_0 \in \mathcal{A}(xy_0)$, $xy_0 \in E(B)$, attain the minimum $\min\{|V(D)| \mid D \in \mathcal{A}(xy),\ xy \in E(B)\}$. Let xy_1 be an edge of D_0 incident with x. There is a path from y_1 to y_0 in $D_0 - x$, so there is a cycle containing xy_0 and xy_1, showing that $xy_1 \in E(B)$ also. Hence $c(G, xy_1) \geq 2$, and we can choose $D_1 \in \mathcal{A}(xy_1)$ such that $y_0 \notin V(D_1)$. Now for each $z \in V(D_1) - \{x, y_1\}$ there is a path in $D_1 - x$ from z to y_1; this path avoids y_0 so it is also a path in $G - \{x, y_0\}$ from z to $y_1 \in V(D_0)$, showing that $z \in V(D_0)$ also. Thus, $V(D_1) \subseteq V(D_0) - \{y_0\}$, contradicting the minimality of $|V(D_0)|$. □

3.1 Trees in fundamental graphs

We will first show the existence of an f-tree in a fundamental K_4-minor-free graph, which will serve as a base case when we find f-trees in general K_4-minor-free graphs by induction on the number of vertices.

The following fairly technical result is used to translate our weight function from Theorem 2.6 into spanning trees. Recall the definition of root edge, root block and root vertex from Section 2. Note that the special edges $r_0s_0, r_1s_1, \ldots, r_ks_k$ below always exist, by Lemma 3.3.

Theorem 3.4. *Let $\mathcal{F}$ be the set of all N2Cs in a connected fundamental K_4-minor-free graph G with $|V(G)| \geq 2$. For each $v \in V(G)$, let $\mathcal{F}(v) = \{F \in \mathcal{F} \mid v \in F\}$. Suppose there are $f : V(G) \to \mathbb{Z}$ and a nonnegative integer function ω on ordered pairs (F, u) with $F \in \mathcal{F}$ and $u \in F$ such that*

$$\omega(F,u) + \omega(F,v) = c(G,F) - 2 \quad \text{for all } F = \{u,v\} \in \mathcal{F}, \text{ and} \tag{3.1}$$

$$\sum_{F \in \mathcal{F}(u)} \omega(F,u) + c(G-u) - 1 \leq f(u) - 2 \quad \text{for all } u \in V(G). \tag{3.2}$$

Choose a root edge $r_0s_0 \in E(G)$ such that $c(G, r_0s_0) \leq 1$. Let the blocks of G be $B_0, B_1, B_2, \ldots, B_k$, where $r_0s_0 \in E(B_0)$. For each $i = 1, 2, \ldots, k$, let r_i be the root vertex of B_i, and choose $r_is_i \in E(B_i)$ with $c(G, r_is_i) \leq 1$.

Then G has a spanning tree T such that $d_T(v) \leq f(v)$ for all $v \in V(G)$ and $d_T(v) \leq f(v) - 1$ for all $v \in \{r_0\} \cup \{s_0, s_1, \ldots, s_k\}$. Furthermore, for $0 \leq i \leq k$,

$$r_is_i \in E(T) \quad \text{if } c(G, r_is_i) = 0, \qquad \text{and} \qquad r_is_i \notin E(T) \quad \text{if } c(G, r_is_i) = 1. \tag{3.3}$$

Proof. The proof is by induction on $|V(G)|$. For the basis, if $|V(G)| = 2$ then $G = K_2$, $\mathcal{F}$ is empty, conditions (3.1) and (3.2) are trivially satisfied, r_0s_0 is the single edge, and we can take $T = G$. So we may assume that $|V(G)| \geq 3$. There are two cases.

Case 1: G has no N2C.

By Observation 2.2 and Lemma 3.2 each block of G is outerplanar and $c(G, r_is_i) = c(B_i, r_is_i) \leq 1$ for $i = 0, 1, \ldots, k$. If $c(G, r_is_i) = 0$ then r_is_i is a cutedge of G, so $B_i = r_is_i$, and we take $T_i = B_i$. Otherwise, $c(G, r_is_i) = c(B_i, r_is_i) = 1$, so B_i contains a cycle using r_is_i. Thus, B_i is a 2-connected outerplanar graph, which we may embed in the plane with a hamiltonian cycle C_i as its outer face. Since $c(B_i, r_is_i) = 1$, $r_is_i \in E(C_i)$, so we take $T_i = C_i - r_is_i$. In either case, T_i is a hamiltonian path and a spanning tree in B_i, and hence $T = \bigcup_{i=0}^{k} T_i$ is a spanning tree of G.

Let us examine degrees in T. Since G has no N2C, $\mathcal{F} = \emptyset$, and inequality (3.2) just says that $c(G - v) - 1 \leq f(v) - 2$ for every $v \in V(G)$. Suppose first that $u \in V(G) - \{r_0, s_0, r_1, s_1, \ldots, r_k, s_k\}$. The cutvertices of G are $r_1, r_2, \ldots, r_k$, so u is not a cutvertex of G. Hence u lies in a single block, so by construction of T, $d_T(u) = 2 \leq f(u)$, as required. Suppose next that $u \in \{r_1, \ldots, r_k\} - (\{r_0\} \cup \{s_0, s_1, \ldots, s_k\})$. By construction of T, u has two incident edges of T from its parent block, and one incident edge from each child block, so $d_T(u) = 2 + (c(G-u) - 1) \leq f(u)$, as required. Finally, suppose that $u \in \{r_0\} \cup \{s_0, s_1, \ldots, s_k\}$. Then u has one incident edge of T from each block to which it belongs, so $d_T(u) = c(G-u) \leq f(u) - 1$, as required.

Furthermore, by construction of T, for $0 \leq i \leq k$ we have $r_i s_i \in E(T)$ if $c(G, r_i s_i) = 0$ and $r_i s_i \notin E(T)$ if $c(G, r_i s_i) = 1$.

Case 2: G contains an N2C.

Let $\{x, y\}$ be an N2C of G, with $\{x, y\}$ contained in a block B_m. Since $c(G, xy) \geq 3$, we can choose a nontrivial $\{x, y\}$-bridge D that attaches at both x and y so that $r_0 s_0, r_m s_m \notin E(D)$. Let $G_1 = G - (V(D) - \{x, y\})$, so G_1 is the union of all $\{x, y\}$-bridges other than D, including all bridges that attach only at x or only at y. We may assume that the blocks of G_1 are $B_0, B_1, \ldots, B_{m-1}$ and the new block B'_m consisting of all $\{x, y\}$-bridges of B_m other than $D \cap B_m$. Let $G_2 = D \cup xay$ where a is a new vertex; G_2 is a minor of G, so it is K_4-minor-free. Neither x nor y is a cutvertex of G_2, and $\{x, y\}$ is not an N2C of G_2 since $c(G_2, xy) = 2$. The blocks of G_2 are all the blocks of G included in D, which are $B_{m+1}, B_{m+2}, \ldots, B_k$, and the new block $B''_m = (D \cap B_m) \cup xay$. No edge of $B_{m+1}, B_{m+2}, \ldots, B_k$ is incident with x or y.

We can apply Observations 2.2 and 2.3, breaking G_1, G_2 and G up into blocks and into $\{x, y\}$-bridges in B'_m and B''_m, which are both subgraphs of $B_m \cup xay$. It follows that G_1 and G_2 are both fundamental. Also, if $\mathcal{F}_j$, $j = 1$ or 2, is the set of N2Cs of G_j, then $\mathcal{F}_1$ and $\mathcal{F}_2$ are disjoint and $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \{\{x, y\}\}$. Possibly $\{x, y\} \in \mathcal{F}_1$, but $\{x, y\} \notin \mathcal{F}_2$. If $\{u, v\} \in \mathcal{F}_j$ and $\{u, v\} \neq \{x, y\}$ then $c(G, uv) = c(G_j, uv)$.

As $\{x, y\}$ is an N2C, $\omega(\{x, y\}, x) \geq 1$ or $\omega(\{x, y\}, y) \geq 1$; without loss of generality, assume that $\omega(\{x, y\}, x) \geq 1$.

For $F \in \mathcal{F}_1 \cup \{\{x, y\}\}$ and $u \in F$, recall that $\delta_x(x) = 1$ and $\delta_x(y) = 0$, and define

$$\omega_1(F, u) = \begin{cases} \omega(\{x, y\}, u) - \delta_x(u) & \text{if } F = \{x, y\}, \\ \omega(F, u) & \text{otherwise.} \end{cases}$$

Then $\omega_1(F, u) \geq 0$ whenever it is defined. For $F \in \mathcal{F}_2$ and $u \in F$ define $\omega_2(F, u) = \omega(F, u)$.

For $u \in V(G_1)$ define

$$f_1(u) = \begin{cases} f(u) - \delta_x(u) - \sum_{F \in \mathcal{F}_2(u)} \omega(F, u) & \text{if } u \in \{x, y\}, \\ f(u) & \text{otherwise.} \end{cases}$$

For $u \in V(G_2)$ define

$$f_2(u) = \begin{cases} \sum_{F \in \mathcal{F}_2(u)} \omega(F, u) + 2 & \text{if } u \in \{x, y\}, \\ 2 & \text{if } u = a, \\ f(u) & \text{otherwise.} \end{cases}$$

Claim. *For $j = 1$ and 2, (3.1) and (3.2) hold for f_j and ω_j in G_j.*

Proof of claim. If $F \in \mathcal{F}_j$ and $F \neq \{x, y\}$, then (3.1) for F in G_j is just a rewriting of (3.1) for F in G. If $u \in V(G_j)$ and $u \notin \{x, y, a\}$ then (3.2) for u in G_j is just a rewriting of (3.2) for u in G.

If $F = \{x, y\} \in \mathcal{F}_1$ then (3.1) holds for $\{x, y\}$ in G_1 because

$$\omega_1(F, x) + \omega_1(F, y) = \omega(F, x) + \omega(F, y) - 1 \leq c(G, F) - 2 - 1 = c(G_1, F) - 2.$$

Suppose $u \in \{x, y\}$. If $\{x, y\} \in \mathcal{F}_1$ then $\mathcal{F}(u) = \mathcal{F}_1(u) \cup \mathcal{F}_2(u)$ and (3.2) for u in G can be rewritten as

$$\sum_{F \in \mathcal{F}_1(u)} \omega_1(F, u) + \delta_x(u) + \sum_{F \in \mathcal{F}_2(u)} \omega(F, u) + c(G_1 - u) - 1 \leq f_1(u) + \delta_x(u) + \sum_{F \in \mathcal{F}_2(u)} \omega(F, u) - 2$$

which gives (3.2) for u in G_1 after cancelling $\delta_x(u) + \sum_{F \in \mathcal{F}_2(u)} \omega(F, u)$ on both sides. If $\{x, y\} \notin \mathcal{F}_1$ then $\{x, y\}$ is included in $\mathcal{F}(u)$ but not in $\mathcal{F}_1(u)$, so this inequality is modified by adding a nonnegative term $\omega_1(\{x, y\}, u)$ on the left, which can be deleted, so (3.2) still holds for u in G_1. Also, (3.2) holds (with equality) for u in G_2 by definition of $f_2(u)$ and because ω_2 is just a restriction of ω and $c(G_2, u) = 1$.

For $u = a$, $\mathcal{F}_2(a) = \emptyset$, $c(G_2 - a) = 1$, and $f_2(a) = 2$, so (3.2) holds for a in G_2. ■

In G_1 we can still choose $r_0 s_0$ as the root edge, and $r_1 s_1, r_2 s_2, \ldots r_m s_m$ as the other special edges (where $r_m s_m$ is now in B'_m instead of B_m). By the claim we can apply induction to find a spanning tree T_1. In G_2 we choose $ax \in E(B''_m)$ as the root edge, and we can still choose $r_{m+1} s_{m+1}, \ldots, r_k s_k$ as the other special edges. By the claim we can apply induction to find a spanning tree T_2. Since $c(G_2, ax) = 1$, we have $ax \notin E(T_2)$ by (3.3), so $ay \in E(T_2)$. Thus, $T'_2 = T_2 - ay$ is a spanning tree of $D = G_2 - a$.

Now T_1 and T'_2 both contain xy-paths. So $T_1 \cup T'_2$ is a spanning subgraph of G containing a single cycle C through x and y. Let yz be the edge of $C \cap G_2$ incident with y. Then $T = (T_1 \cup T'_2) - yz$ is a spanning tree of G.

For all $v \in V(G)$, define $\sigma(v)$ to be 1 if $v \in \{r_0\} \cup \{s_0, s_1, \ldots, s_k\}$, and 0 otherwise. If $u \in V(G) - \{x, y\}$, then $u \in V(G_j) - \{x, y\}$ for $j = 1$ or 2, and $d_T(u) \leq d_{T_j}(u) \leq f_j(u) - \sigma(u) = f(u) - \sigma(u)$ as required.

We also want to show that $d_T(u) \leq f(u) - \sigma(u)$ when $u \in \{x, y\}$. We know that no edge of $B_{m+1}, B_{m+2}, \ldots, B_k$ is incident with u, so if $\sigma(u) = 1$, it is because u is incident with an edge $r_i s_i$ for $0 \leq i \leq m$; this edge belongs to G_1. Thus, $d_{T_1}(u) \leq f_1(u) - \sigma(u)$ by construction of T_1. Since ax is the root edge of G_2, $d_{T'_2}(x) = d_{T_2}(x) \leq f_2(x) - 1$, so

$$d_T(x) = d_{T_1}(x) + d_{T'_2}(x) \leq (f_1(x) - \sigma(x)) + (f_2(x) - 1) = f(x) + 1 - \delta_x(x) - \sigma(x) = f(x) - \sigma(x).$$

Since we delete two edges ay and yz from $T_1 \cup T_2$ when forming T,

$$d_T(y) = d_{T_1}(y) + d_{T_2}(y) - 2 \leq (f_1(y) - \sigma(y)) + f_2(y) - 2 = f(y) - \delta_x(y) - \sigma(y) = f(y) - \sigma(y).$$

Finally we verify (3.3). For each i, $0 \le i \le k$, $r_i s_i \in E(G_j)$ for a unique $j \in \{1, 2\}$. If $c(G, r_i s_i) = 0$ then $r_i s_i$ is a cutedge of G and so must belong to the spanning tree T. Otherwise, $c(G, r_i s_i) = c(G_j, r_i s_i) = 1$ and so $r_i s_i \notin E(T_j)$, and hence $r_i s_i \notin E(T)$. □

Combining Theorems 2.6 and 3.4 we obtain our result on spanning trees in fundamental K_4-minor-free graphs. The hypotheses (3.1) and (3.2) of Theorem 3.4 are just slightly rewritten versions of the conclusions (2.7) and (2.8) of Theorem 2.6. Note that Theorem 3.4 does not handle the case where $|V(G)| = 1$, but that is trivial.

Theorem 3.5. *Let G be a fundamental connected K_4-minor-free graph. Suppose that $f : V(G) \to \mathbb{Z}$ satisfies*

$$c(G - S) \;\le\; \sum_{v \in S} (f(v) - 1) \quad \text{for all } S \subseteq V(G) \text{ with } S \ne \emptyset. \tag{CC}$$

Choose a root edge $r_0 s_0 \in E(G)$ such that $c(G, r_0 s_0) \le 1$. Let the blocks of G be $B_0, B_1, B_2, \ldots, B_k$, where $r_0 s_0 \in E(B_0)$. For each $i = 1, 2, \ldots, k$, let r_i be the root vertex of B_i, and choose $r_i s_i \in E(B_i)$ with $c(G, r_i s_i) \le 1$.

Then G has a spanning tree T such that $d_T(v) \le f(v)$ for all $v \in V(G)$ and $d_T(v) \le f(v) - 1$ for all $v \in \{r_0\} \cup \{s_0, s_1, \ldots, s_k\}$. Furthermore, for $0 \le i \le k$, $r_i s_i \in E(T)$ if $c(G, r_i s_i) = 0$, and $r_i s_i \notin E(T)$ if $c(G, r_i s_i) = 1$.

This theorem is quite technical, and only applies to fundamental K_4-minor-free graphs, but it gives strong results in that situation, particularly for graphs with many cutedges. For example, neither the general result Theorem 1.2 nor our main result Theorem 1.5 can prove that when G is a tree, G has a spanning f-tree where $f = d_G$ is just the degree function in G. However, we can get this result from Theorem 3.5 with $f(v) = d_G(v) + 1$ for all v, $r_0 s_0$ an arbitrary edge, and $\{s_1, s_2, \ldots, s_k\} = V(G) - \{r_0, s_0\}$.

3.2 Trees in general graphs

In this subsection, we find spanning f-trees in general K_4-minor-free graphs. We will use the Component Condition (CC), as stated in Theorem 1.5, often.

Lemma 3.6. *Let G be a 2-connected graph and $f : V(G) \to \mathbb{Z}$. If G and f satisfy (CC), then for every 2-cut $\{u, v\}$ of G with $uv \in E(G)$, $G - uv$ and f also satisfy (CC).*

Proof. Let $G' = G - uv$, and let $S \subseteq V(G')$ with $S \ne \emptyset$. If $\{u, v\} \cap S \ne \emptyset$ then $G' - S = G - S$, and if u and v are in the same component of $G' - S$ then $c(G' - S) = c((G' - S) \cup uv) = c(G - S)$; in either case (CC) holds for G', f and S. So we may assume that $u, v \notin S$ and u and v are in distinct components of $G' - S$.

Let $D_1, D_2, \ldots, D_t$ be the $\{u, v\}$-bridges in G', and let $S_i = S \cap V(D_i)$ for each i. Then $S = \bigcup_{i=1}^{t} S_i$. As $\{u, v\}$ is a 2-cut, $t \ge 2$. Now S_i must separate u and v in D_i, so the set of components of $D_i - S_i$ can be written as $\mathcal{A}_i \cup \{C_i^u, C_i^v\}$ where $u \in V(C_i^u)$, $v \in V(C_i^v)$, and $\mathcal{A}_i$ contains all other components. The set of components of $G - S_i$ is then $\mathcal{A}_i \cup \{C_i^u \cup C_i^v \cup \bigcup_{j \neq i} D_j\}$, so $c(G - S_i) = |\mathcal{A}_i| + 1$. The set of components of $G' - S$ is $\bigcup_{i=1}^t \mathcal{A}_i \cup \{\bigcup_{i=1}^t C_i^u, \bigcup_{i=1}^t C_i^v\}$. Since $t \geq 2$ and (CC) holds for G and f, we have

$$\begin{aligned} c(G' - S) &= \sum_{i=1}^t |\mathcal{A}_i| + 2 = \sum_{i=1}^t (c(G - S_i) - 1) + 2 \\ &\leq \sum_{i=1}^t \sum_{v \in S_i} (f(v) - 1) - t + 2 = \sum_{v \in S} (f(v) - 1) - t + 2 \leq \sum_{v \in S} (f(v) - 1) \end{aligned}$$

and so (CC) holds for G', f and S. □

If every 2-cut $\{u, v\}$ of a graph G satisfies $uv \notin E(G)$, we say that G is *2-cut-reduced*. If G is 2-connected and $\{u, v\}$ is a 2-cut of G with $uv \in E(G)$, then $G - uv$ is still 2-connected. Therefore, we can apply Lemma 3.6 repeatedly. We may create new 2-cuts when we do, but we can only create a bounded number, so we obtain the following.

Corollary 3.7. *Let G be a 2-connected graph and $f : V(G) \to \mathbb{Z}$. Assume that G and f satisfy (CC). Then G has a 2-cut-reduced spanning subgraph G' such that G' and f also satisfy (CC).*

We write a uv-path P as $P[u, v]$ to emphasize its endvertices. Suppose the block-cutvertex tree of a graph H is a path. Then we write $H = v_0 B_1 v_1 B_2 v_2 \ldots v_{t-1} B_t v_t$, where each B_i is a block of H, $v_0 \in V(B_1) - \{v_1\}$, $v_t \in V(B_t) - \{v_{t-1}\}$, and each v_i, $i = 1, \ldots, t-1$, is a cutvertex of H with $v_i \in V(B_i) \cap V(B_{i+1})$. (If H has only one vertex we take $t = 0$ so that $H = v_0$.) We say H is a *chain of blocks from v_0 to v_t*. If $t \geq 2$ we say the chain of blocks is *nontrivial*.

Lemma 3.8. *Let G be a 2-connected graph and $xy \in E(G)$.*
(a) *Then $G - xy$ is a chain of blocks from x to y.*
(b) *Moreover, if G is a 2-cut-reduced K_4-minor-free graph, then $G - xy$ is a nontrivial chain of blocks.*

Proof. (a) Since G is 2-connected, if $G - xy$ has a cutvertex then every leaf block of $G - xy$ contains x or y as a non-cutvertex vertex. Hence, $G - xy$ is either a single block or has only two leaf blocks, and is a chain of blocks from x to y.

(b) If $G - xy$ is 2-connected, then there are internally disjoint xy-paths P_1 and P_2 in $G - xy$. Since G is 2-cut-reduced, $\{x, y\}$ is not a 2-cut of G, so there is a path P in $G - \{x, y\}$ connecting P_1 to P_2, where we may assume that $|V(P) \cap V(P_i)| = 1$, $i = 1, 2$. Then $P_1 \cup P_2 \cup P \cup xy$ contains a K_4 minor, which is a contradiction. Hence $G - xy$ is not 2-connected, so it is a nontrivial chain of blocks. □

If H has at least two blocks, B is a leaf block of H, and the cutvertex of H in B is v, then $H \ominus B$ denotes $H - (V(B) - \{v\})$.

Lemma 3.9. *Let G be a 2-cut-reduced 2-connected K_4-minor-free graph. If G is not fundamental and $u \in V(G)$, then there exist $xy \in E(G)$ and a leaf block B of $G - xy$ such that each of x and y is contained in an N2C of G, B is fundamental, and $u \in V((G - xy) \ominus B)$.*

Proof. Since G is not fundamental there is at least one edge $xy \in E(G)$ such that both x and y are contained in N2Cs of G. By Lemma 3.8, $G - xy$ is a nontrivial chain of blocks from x to y. For such an xy there is at least one leaf block B of $G - xy$ so that $u \in V((G - xy) \ominus B)$. Choose such an xy and such a B so that $|V((G - xy) \ominus B)|$ is as large as possible. We may assume that $G - xy = v_0 B_1 v_1 \dots v_{t-1} B_t v_t$ where $x = v_0$, $y = v_t$ and $B = B_1$.

We claim that B_1 is fundamental. Suppose not. Then there is $x_1 y_1 \in E(B_1)$ with both x_1 and y_1 in N2Cs of B_1. By Lemma 3.1, both x_1 and y_1 are also in N2Cs of G. Since B_1 has N2Cs it has at least five vertices and is 2-connected. By Lemma 3.8(a), $B_1 - x_1 y_1$ is a chain of blocks $w_0 D_1 w_1 \dots w_{s-1} D_s w_s$ (possibly $s = 1$, but $s \neq 0$). Choose h and k such that $h \leq k$, x and v_1 are vertices of $w_h D_{h+1} w_{h+1} \dots w_{k-1} D_k w_k$, and $k - h$ is as small as possible. Since $x \neq v_1$ we have $h < k$. We may assume that $x_1 = w_0$, $y_1 = w_s$, $x \in D_{h+1} - \{w_{h+1}\}$ and $v_1 \in D_k - \{w_{k-1}\}$.

By Lemma 3.8, $G - x_1 y_1$ is a nontrivial chain of blocks, which must be $w_0 D_1 \dots w_{h-1} D_h w_h D^* w_k D_{k+1} \dots D_s w_s$ where D^* is a block $(x D_{h+1} w_{h+1} D_{h+2} \dots w_{k-1} D_k v_1 B_2 v_2 \dots v_{t-1} B_t y) \cup xy$. Since $G - x_1 y_1$ is nontrivial, either $h > 0$ or $k < s$ (hence $s \geq 2$). If $h > 0$ then D_1 is a leaf block of $G - x_1 y_1$, $u \in V((G - xy) \ominus B_1) = V(B_2 \cup \dots \cup B_t) \subset V(D^*) \subseteq V((G - x_1 y_1) \ominus D_1)$ and $|V((G - x_1 y_1) \ominus D_1)| \geq |V(D^*)| > |V((G - xy) \ominus B_1)|$. Thus, $x_1 y_1$ and D_1 contradict the choice of xy and $B = B_1$. We obtain a similar contradiction from $x_1 y_1$ and D_s if $k < s$.

Hence, $B = B_1$ is fundamental, as claimed, which proves the lemma. □

We are now ready to prove Theorem 1.5, which we restate for convenience.

Theorem 1.5. *Let G be a connected K_4-minor-free graph, and $f : V(G) \to \mathbb{Z}$. Suppose that $z \in V(G)$ and*

$$c(G - S) \;\leq\; \sum_{v \in S} (f(v) - 1) \quad \text{for all } S \subseteq V(G) \text{ with } S \neq \emptyset. \qquad \text{(CC)}$$

Then G has a spanning $(f - \delta_z)$-tree, i.e., a spanning f-tree T such that $d_T(z) \leq f(z) - 1$.

Proof. The proof is by induction on $|V(G)|$. The conclusion is true by Theorem 3.5 if $|V(G)| \leq 4$, since G has no N2Cs and is fundamental. So we assume that $|V(G)| \geq 5$. We consider two cases.

Case 1: G has a cutvertex x.

Let G_1 be one x-bridge, and let G_2 be the union of the remaining x-bridges. If $z \in V(G_1) - \{x\}$ let $z_1 = z$ and $z_2 = x$, if $z \in V(G_2) - \{x\}$ let $z_1 = x$ and $z_2 = z$, and if $z = x$ let $z_1 = z_2 = x$. For every $v \in V(G_i) - \{x\}$, $i = 1, 2$, let $f_i(v) = f(v)$. Then each G_i and f_i, $i = 1$ or 2, satisfy (CC) for all S that do not contain x.

Let $f_2(x)$ be the minimum integer such that G_2 and f_2 satisfy (CC) for all S containing x, and let $f_1(x) = f(x) - f_2(x) + 1$. We claim that G_1 and f_1 also satisfy (CC) for sets S containing x. If not, there is $U \subseteq V(G_1)$ with $x \in U$ so that

$$c(G_1 - U) \geq \sum_{v \in U} (f_1(v) - 1) + 1 = \sum_{v \in U - \{x\}} (f(v) - 1) + f_1(x).$$

By minimality of $f_2(x)$ there is $W \subseteq V(G_2)$ with $x \in W$ so that

$$c(G_2 - W) = \sum_{v \in W} (f_2(v) - 1) = \sum_{v \in W - \{x\}} (f(v) - 1) + f_2(x) - 1.$$

Thus, if $S = U \cup W$ we have

$$\begin{aligned} c(G - S) &= c(G_1 - U) + c(G_2 - W) = \sum_{v \in S - \{x\}} (f(v) - 1) + f_1(x) + f_2(x) - 1 \\ &= \sum_{v \in S - \{x\}} (f(v) - 1) + f(x) > \sum_{v \in S} (f(v) - 1), \end{aligned}$$

contradicting (CC) for G and f.

Thus, for each $i = 1, 2$, G_i and f_i satisfy (CC), so by induction there is a spanning f_i-tree T_i of G_i with $d_{T_i}(z_i) \leq f_i(z_i) - 1$. Let $T = T_1 \cup T_2$, which is a spanning tree of G. If $v \notin \{z, x\}$ then $v \in V(G_i) - \{x\}$ for some i, and $d_T(v) = d_{T_i}(v) \leq f_i(v) = f(v)$. If $z \in V(G_i) - \{x\}$ then $d_T(z) = d_{T_i}(z) \leq f_i(z) - 1 = f(z) - 1$ since $z = z_i$, and $d_T(x) = d_{T_1}(x) + d_{T_2}(x) \leq f_1(x) + f_2(x) - 1 = f(x)$ since $x = z_{3-i}$. If $z = x$ then $d_T(x) \leq (f_1(x) - 1) + (f_2(x) - 1) = f(x) - 1$ since $x = z_1 = z_2$. Thus, T is a spanning f-tree with $d_T(z) \leq f(z) - 1$.

Case 2: G has no cutvertex; since $|V(G)| \geq 5$, G is 2-connected.

We may assume that G is not fundamental, by Theorem 3.5, and that G is 2-cut-reduced, by Corollary 3.7. By Lemmas 3.8 and 3.9, G contains an edge xy with both x and y in N2Cs of G such that some leaf block B of $G - xy$, which is a nontrivial chain of blocks from x to y, is fundamental and $z \in V((G - xy) \ominus B)$. Without loss of generality $x \in V(B)$ and so $G - xy = xB_1v_1B_2v_2 \ldots B_ty$ where $t \geq 2$ and $B = B_1$. Write $L_2 = (G - xy) \ominus B_1 = v_1B_2v_2 \ldots v_{t-1}B_tv_t$, so that $z \in V(L_2)$. Since x and y are in N2Cs, $d_G(x), d_G(y) \geq 3$ and hence $|V(B_1)|, |V(L_2)| \geq 3$.

Let $G_1 = B_1 \cup xa_1v_1$ where a_1 is a new vertex, and let $G_2 = L_2 \cup v_1y$. Both G_1 and G_2 are minors of G and hence K_4-minor-free. For $i = 1, 2$ define $f_i(v) = f(v)$ for $v \in V(G_i) - \{v_1, a_1\}$. Then for $S \subseteq V(G_2)$ with $v_1 \notin S$ we have $c(G - S) = c(G_2 - S)$ and

$f_2(v) = f(v)$ for all $v \in S$, so (CC) holds for G_2, f_2 and S. Choose $f_2(v_1)$ to be the minimum integer such that (CC) holds for G_2 and f_2, then by minimality there is $W \subseteq V(G_2)$ with $v_1 \in W$ so that

$$c(G_2 - W) = \sum_{v \in W} (f_2(v) - 1) = \sum_{v \in W - \{v_1\}} (f(v) - 1) + f_2(v_1) - 1.$$

Let $f_1(v_1) = f(v_1) - f_2(v_1) + 2$ and $f_1(a_1) = |V(G_1)| + 1$ (note that our theorems do not require that $f(v) \leq d_G(v)$). We claim that (CC) holds for G_1 and f_1. This is true for any $S \subseteq V(G_1)$ with $a_1 \in S_1$, by choice of $f_1(a_1)$. If we have nonempty $S \subseteq V(G_1)$ with $a_1, v_1 \notin S$,

$$c(G_1 - S) = c(G - S) \leq \sum_{v \in S} (f(v) - 1) = \sum_{v \in S} (f_1(v) - 1)$$

and so (CC) holds in this situation as well. So if (CC) does not hold for G_1 and f_1 there is $U \subseteq V(G_1)$ with $a_1 \notin U$, $v_1 \in U$ and

$$c(G_1 - U) \geq \sum_{v \in U} (f_1(v) - 1) + 1 = \sum_{v \in U - \{v_1\}} (f(v) - 1) + f_1(v_1).$$

Let $S = U \cup W$ then $a_1 \notin S$, $v_1 \in S$. Note that $c(G - xy - S) = c(B_1 - U) + c(L_2 - W)$. If $x \notin U$ then $c(G - S) \geq c(G - xy - S) - 1$ and $c(G_1 - U) = c(B_1 - U)$, so

$$c(G - S) \geq c(B_1 - U) + c(L_2 - W) - 1 = c(G_1 - U) + c(G_2 - W) - 1$$

and if $x \in U$ then $c(G - S) = c(G - xy - S)$ and $c(G_1 - U) = c(B_1 - U) + 1$, so

$$c(G - S) = c(B_1 - U) + c(L_2 - W) = c(G_1 - U) + c(G_2 - W) - 1.$$

Thus, in either case,

$$\begin{aligned} c(G - S) &\geq c(G_1 - U) + c(G_2 - W) - 1 \\ &\geq \sum_{v \in S - \{v_1\}} (f(v) - 1) + f_1(v_1) + f_2(v_1) - 1 - 1 \\ &= \sum_{v \in S - \{v_1\}} (f(v) - 1) + f(v_1) \; > \; \sum_{v \in S} (f(v) - 1) \end{aligned}$$

contradicting (CC) for G and f. Thus, G_1 and f_1 satisfy (CC).

Now a_1 does not belong to any N2C of G_1 since it has degree 2, so G_1 is fundamental. Choosing root edge $r_0 s_0 = a_1 x$ and applying Theorem 3.5, there is a spanning f_1-tree T_1 of G_1 with $d_{T_1}(x) \leq f_1(x) - 1$ and $a_1 x \notin E(T_1)$. Thus, $a_1 v_1 \in E(T_1)$ and $T_1' = T_1 - a_1$ is a spanning f_1-tree of $G_1 - a_1 = B_1$ with $d_{T_1'}(v) \leq f_1(v) - 1$ for $v \in \{x, v_1\}$.

Since $|V(B_1)| \geq 3$, $|V(G_2)| < |V(G)|$, so by induction there is a spanning $(f_2 - \delta_z)$-tree T_2 of G_2 (i.e., an f_2-tree with $d_{T_2}(z) \leq f_2(z) - 1$). Suppose that some such tree T_2 has

$v_1y \in E(T_2)$. Then $T = T_1' \cup (T_2 - v_1y) \cup xy$ has $d_T(v) \le f_1(v) = f(v)$ for $v \in V(B_1) - \{x, v_1\}$, $d_T(v) \le f_2(v) - \delta_z(v) = f(v) - \delta_z(v)$ for $v \in V(L_2) - \{v_1, y\}$, and

$$\begin{aligned} d_T(x) &\le (f_1(x) - 1) + 1 = f(x), \\ d_T(v_1) &\le (f_1(v_1) - 1) + (f_2(v_1) - \delta_z(v_1) - 1) = f(v_1) - \delta_z(v_1), \\ d_T(y) &\le f_2(y) - \delta_z(y) - 1 + 1 = f(y) - \delta_z(y). \end{aligned}$$

Thus, T is our desired spanning tree. We may henceforth assume that every $(f_2 - \delta_z)$-tree T_2 of G_2 has $v_1y \notin E(T_2)$, and so is a spanning tree of L_2.

Now define g_1 on $V(G_1)$ by $g_1(x) = f_1(x) + 1 = f(x) + 1$, $g_1(v_1) = f_1(v_1) - 1 = f(v_1) - f_2(v_1) + 1$, and $g_1(v) = f_1(v)$ for $v \in V(G_1) - \{x, v_1\}$. Suppose that G_1 and g_1 satisfy (CC). By similar reasoning to that for T_1' above, there is a spanning g_1-tree R_1' of B_1 with $d_{R_1'}(v) \le g_1(v) - 1$ for $v \in \{x, v_1\}$. Taking $T = R_1' \cup T_2$ (remembering that T_2 is now a spanning tree of L_2) we get a spanning tree of G with $d_T(v) \le g_1(v) = f(v)$ for $v \in V(B_1) - \{x, v_1\}$, $d_T(v) \le f_2(v) - \delta_z(v) = f(v) - \delta_z(v)$ for $v \in V(L_2) - \{v_1, y\}$, and

$$\begin{aligned} d_T(x) &\le (g_1(x) - 1) = f(x), \\ d_T(v_1) &\le (g_1(v_1) - 1) + (f_2(v_1) - \delta_z(v_1)) = f(v_1) - \delta_z(v_1), \\ d_T(y) &\le f_2(y) - \delta_z(y) = f(y) - \delta_z(y). \end{aligned}$$

Thus, T is our desired spanning tree. We may henceforth assume that G_1 and g_1 do not satisfy (CC).

Let $G_2^+ = L_2 \cup v_1a_2y$, where a_2 is a new vertex; G_2^+ is a minor of G and hence is K_4-minor-free. Extend f_2 to G_2^+ by defining $f_2(a_2) = |V(G_2^+)| + 1$. Suppose that G_2^+ and f_2 satisfy (CC). Since $|V(B_1)| \ge 3$, we have $|V(G_2^+)| < |V(G)|$, and so by induction there is a spanning $(f_2 - \delta_z)$-tree Q_2 of G_2^+. If Q_2 uses the path v_1a_2y then G_2 has a spanning $(f_2 - \delta_z)$-tree using v_1y, which we have assumed does not exist. So $d_{Q_2}(a_2) = 1$ and deleting whichever of v_1a_2, a_2y is an edge of Q_2 gives a spanning $(f_2 - \delta_z z)$-tree Q_2' of L_2 for which there is $w \in \{v_1, y\}$ such that $d_{Q_2'}(w) \le f_2(w) - \delta_z(w) - 1$. Let w' be the vertex of $\{v_1, y\}$ other than w. Now $T_1' \cup Q_2' \cup xy$ is a connected spanning subgraph containing a cycle through x, v_1 and y. Let $w't$ be an edge of this cycle incident with w' and let $T = (T_1' \cup Q_2' \cup xy) - w't$, which is a spanning tree in G. We have $d_T(v) \le f_1(v) = f(v)$ for $v \in V(B_1) - \{x, v_1\}$, $d_T(v) \le f_2(v) - \delta_z(v) = f(v) - \delta_z(v)$ for $v \in V(L_2) - \{v_1, y\}$, and

$$\begin{aligned} d_T(x) &\le (f_1(x) - 1) + 1 = f(x), \\ d_T(v_1) &\le (f_1(v_1) - 1) + (f_2(v_1) - \delta_z(v_1)) - 1 = f(v_1) - \delta_z(v_1), \\ d_T(y) &\le (f_2(y) - \delta_z(y)) + 1 - 1 = f(y) - \delta_z(y), \end{aligned}$$

where we add 1 to the degrees of x and y due to xy, but subtract 1 from the degrees of v_1 and y because $\{v_1, y\} = \{w, w'\}$. Thus, T is our desired spanning tree. We may henceforth assume that G_2^+ and f_2 do not satisfy (CC).

Now G_1 and g_1 do not satisfy (CC), although G_1 and f_1 do. Suppose (CC) fails for $U \subseteq V(G_1)$. By definition of g_1, we see that G_1, g_1 and a specific set $S \subseteq V(G_1)$ do satisfy (CC) provided $v_1 \notin S$, $a_1 \in S$, or $\{x, v_1\} \subseteq S$ (since $g_1(x) + g_1(v_1) = f_1(x) + f_1(v_1)$). Thus, $v_1 \in U$, $a_1 \notin U$ (so $U \subseteq V(B_1)$), $x \notin U$, and

$$c(B_1 - U) = c(G_1 - U) \geq \sum_{v \in U} (g_1(v) - 1) + 1 = \sum_{v \in U - \{v_1\}} (f(v) - 1) + f(v_1) - f_2(v_1) + 1.$$

Also G_2^+ and f_2 do not satisfy (CC), although G_2 and f_2 do. Suppose (CC) fails for $W \subseteq V(G_2)$. We see that G_2^+, f_2 and a specific set $S \subseteq V(G_2)$ do satisfy (CC) provided $a_2 \in S$, or $a_2 \notin S$ and $c(G_2^+ - S) = c(G_2 - S)$. The only situation where $a_2 \notin S$ and $c(G_2^+ - S) \neq c(G_2 - S)$ is when $\{v_1, y\} \subseteq S$. Thus, $a_2 \notin W$ (so $W \subseteq V(L_2)$), $\{v_1, y\} \subseteq W$, and

$$c(L_2 - W) = c(G_2^+ - W) - 1 \geq \sum_{v \in W} (f_2(v) - 1) = \sum_{v \in W - \{v_1\}} (f(v) - 1) + f_2(v_1) - 1.$$

Letting $S = U \cup W \subseteq V(G)$, we have $x \notin S$ but $\{v_1, y\} \subseteq S$, and therefore

$$\begin{aligned} c(G - S) &= c(B_1 - U) + c(L_2 - W) \\ &\geq \sum_{v \in S - \{v_1\}} (f(v) - 1) + f(v_1) - f_2(v_1) + 1 + f_2(v_1) - 1 \\ &= \sum_{v \in S - \{v_1\}} (f(v) - 1) + f(v_1) \; > \; \sum_{v \in S} (f(v) - 1) \end{aligned}$$

which contradicts (CC) for G and f. Thus, this final situation never happens, and we can always find the required spanning tree T. □

4 Concluding remarks

In this section, we show the existence of planar graphs G such that $c(G-S) \leq \sum_{v \in S}(f(v) - 1)$ for every $S \subseteq V(G)$ and $S \neq \emptyset$, but where G has no spanning tree T such that $d_T(v) \leq f(v)$ for all $v \in V(G)$, where there is an integer $f(v) \geq 2$ for each vertex.

Let G be a 1-tough planar graph with an integer $f(v) \geq 2$ for each vertex, and let G' be obtained by attaching $f(v) - 2$ pendant edges to each $v \in V(G)$. Let $f(v) = 2$ for $v \in V(G') - V(G)$. Since $c(G - S) \leq |S|$ for every $S \subseteq V(G)$ with $S \neq \emptyset$, it is not too hard to show that $c(G' - S') \leq \sum_{v \in S'} (f(v) - 1)$ for every $S' \subseteq V(G')$ with $S' \neq \emptyset$. If G has no spanning 2-tree then G' has no spanning tree T such that $d_T(v) \leq f(v)$ for all $v \in V(G')$. Thus, it suffices to show the existence of a 1-tough planar graph G with no spanning 2-tree, i.e., no hamiltonian path.

A vertex in a graph is called a *simplicial vertex* if the neighbors of this vertex form a clique in the graph. The graph T given in Figure 1, constructed by Dillencourt [4] is

1-tough, non-hamiltonian, and maximal planar. Proposition 5 of that paper states that any path in T connecting any two of the three vertices A, B, C must omit at least one simplicial vertex. In particular, T has no hamiltonian path with both ends in A, B, C.

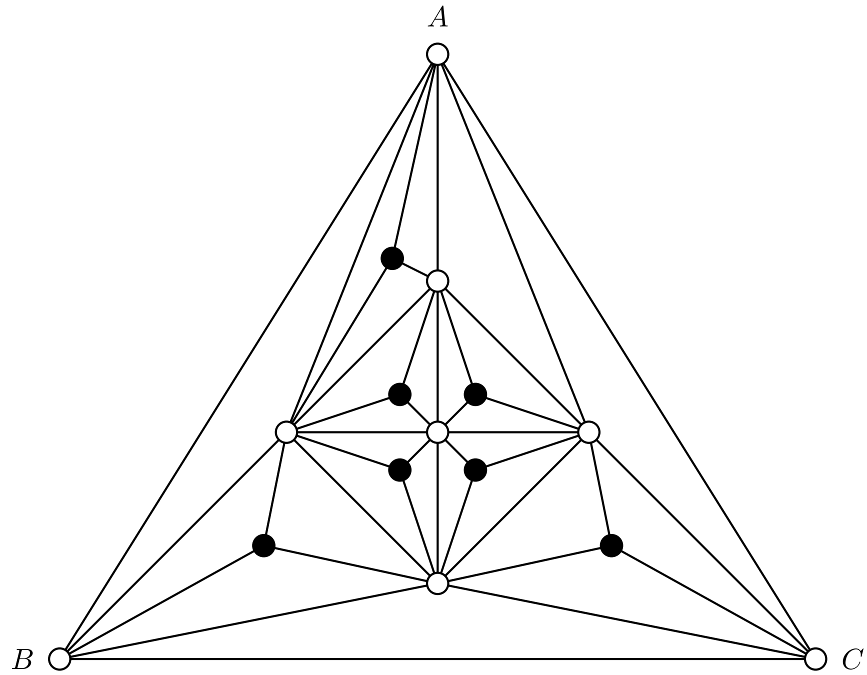

Figure 1: The 1-tough nonhamiltonian maximal planar graph T. The black vertices are the simplicial vertices.

Dillencourt constructed a sequence of 1-tough nonhamiltonian maximal planar graphs as follows. Let $G_1 = T$. For $n \geq 2$, let G_n be obtained from G_{n-1} by deleting every simplicial vertex and replacing it with a copy of T. More precisely, for each simplicial vertex u, let x, y, z be its neighbors. Delete u, insert a copy T_u (with vertices A_u, B_u, etc.) of T inside the triangle (xyz), and add the edges $A_u x, A_u y, B_u y, B_u z, C_u z, C_u x$.

Suppose that G_n, $n \geq 2$, has a hamiltonian path P. Since G_{n-1} has at least three simplicial vertices, for some such vertex u the copy T_u of T in G_n contains neither end of P. The structure of G_n guarantees that $P \cap T_u$ must have one of two forms: (1) a single hamiltonian path in T_u with ends being two of A_u, B_u, C_u, or (2) a union of two paths P_1 and P_2, where P_1 is a one-vertex path using one of A_u, B_u, C_u, and P_2 is a path between the other two of A_u, B_u, C_u using all other vertices of T_u. In case (2) we can join the vertex of P_1 to either end of P_2 to obtain a hamiltonian path in T_u with both ends in A_u, B_u, C_u. So either situation means T has a hamiltonian path with ends being two of A, B, C, which we know does not happen. Therefore, G_n is a 1-tough maximal planar graph with no hamiltonian path.

We conclude with a general question. Suppose G is a tree, i.e., a connected graph of treewidth at most 1. Then the sufficient condition (1.1), the necessary condition (1.2), and the even weaker condition

$$c(G-S) \quad \leq \quad \sum_{v \in S} f(v) \quad \text{for all } S \subseteq V(G) \text{ with } S \neq \emptyset \tag{4.1}$$

are all equivalent and all imply that $d_G(v) \leq f(v)$ for all $v \in V(G)$, i.e., that G has (in fact, is) a spanning f-tree. (When $f(v) = k$ for all v, (4.1) just says that G is $\frac{1}{k}$-tough.) If G has treewidth at most 2, we have a sufficient condition (CC) that is only slightly stronger than the necessary condition (1.2). It therefore seems natural to ask the following.

Question 4.1. Is there a natural strengthening of Theorem 1.2 involving treewidth? More specifically, suppose we have a connected graph G of treewidth h, and $f : V(G) \to \mathbb{Z}$. Is there a condition involving h and becoming stronger as h increases, similar to (1.2) or (CC) but weaker than (1.1), that implies that G has a spanning f-tree?

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