The orientable genus of some joins of complete graphs with large edgeless graphs

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Abstract

In an earlier paper the authors showed that with one exception the nonorientable genus of the graph $\overline{K_m} + K_n$ with $m \geq n-1$, the join of a complete graph with a large edgeless graph, is the same as the nonorientable genus of the spanning subgraph $\overline{K_m}+\overline{K_n} = K_{m,n}$. The orientable genus problem for $\overline{K_m} + K_n$ with $m \geq n-1$ seems to be more difficult, but in this paper we find the orientable genus of some of these graphs. In particular, we determine the genus of $\overline{K_m} + K_n$ when n is even and $m \geq n$, the genus of $\overline{K_m} + K_n$ when $n = 2^p + 2$ for $p \geq 3$ and $m \geq n-1$, and the genus of $\overline{K_m} + K_n$ when $n = 2^p + 1$ for $p \geq 3$ and $m \geq n+1$. In all of these cases the genus is the same as the genus of $K_{m,n}$, namely $\lceil (m-2)(n-2)/4 \rceil$.

1 Introduction

In 1965 Ringel [14, 15] determined the orientable and nonorientable genus of complete bipartite graphs $K_{m,n}$. The embeddings that attain the minimum genus have roughly $mn/2$ faces, almost all of degree 4. Therefore, roughly $mn/2$ edges can be added to the graph and the embedding without altering the surface. So, it is natural to ask what graphs obtained by adding edges to $K_{m,n}$ have the same (orientable or nonorientable) genus as $K_{m,n}$. One of us (Ellingham) surveyed recent results on this question at the Wuhan International Conference on Graph Structure Theory in Wuhan, China, in July 2005. In this paper we describe some of those results.

In the nonorientable case, it has recently been shown that there are two natural classes of graphs obtained by adding edges to $K_{m,n}$ that have the same nonorientable genus as $K_{m,n}$. First, Kawarabayashi, Zha and the authors, in a series of papers [5, 6, 9], determined the nonorientable genus of all complete tripartite graphs K_{m,n_1,n_2}, with $m \geq n_1 \geq n_2$, showing that with three exceptions this is the same as the nonorientable genus of K_{m,n_1+n_2}. Second, the authors [4] showed that for $m \geq n-1$ the nonorientable genus of $\overline{K_m} + K_n$ (where '+' denotes the join of two graphs) is with one exception the same as that of $K_{m,n}$. We need $m \geq n-1$ here or there is not enough room in the minimum genus embedding of $K_{m,n}$ to add the extra edges.

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The orientable counterparts of these results seem to be more difficult to prove, in the same way that the orientable genus of K_n was significantly more difficult to determine than the nonorientable genus (see [16]). In this paper we determine some situations in which the orientable genus of $\overline{K_m} + K_n$ with $m \geq n-1$ is the same as the orientable genus of $K_{m,n}$, namely $\lceil (m-2)(n-2)/4 \rceil$. We use a reduction technique known as the 'diamond sum', which was also used for the nonorientable results above. For the initial cases we give two constructions. One uses *transition graphs*, which were introduced in [6]. The other uses a doubling construction.

We are aware of only two sources of previous results on the orientable genus of $\overline{K_m} + K_n$ with $m \geq n-1$. Jungerman [8] used computer investigations to show that there are some small cases, specifically $\overline{K_5} + K_6$ and $\overline{K_6} + K_7$, where $\overline{K_m} + K_n$ with $m \geq n-1$ does not have the same genus as $K_{m,n}$. Craft, in [2, Theorem 5.3] and [3, Theorem 1], has verified that $\overline{K_m} + K_n$ has the same genus as $K_{m,n}$ when n is even and $m \geq 2n-4$. His method involves a doubling construction starting from a 'graphical surface' constructed of 'tubes' and 'spheres', and a surgical construction called 'crowning' which accomplishes some of the same ends as our diamond sum, below.

There are also results on the orientable genus of $\overline{K_m} + K_n$ with $m \leq n-2$ (in particular, Korzhik [10] has many results on this question). However, when $m \leq n-2$ the minimum genus embeddings cannot be considered as minimum genus embeddings of $K_{m,n}$ with edges added, and the techniques we use (particularly the diamond sum) cannot be applied. For more information on this situation, see [4].

For background in topological graph theory we refer the reader to [7] and [13]. A *surface* is a compact 2-manifold without boundary. For $h \geq 0$, the surface S_h is the orientable surface obtained by adding h handles to a sphere. The *orientable genus* (or just *genus*) $g(G)$ of the graph G is the minimum h such that G can be embedded on S_h. The genus of a planar graph is 0. For brevity we will usually refer to the facial walks of an embedding as just the 'faces' of the embedding.

2 The diamond sum and reduction to small cases

In this section we briefly describe the construction we can use to reduce the genus question for $K_{m,n}$ for arbitrary $m \geq n-1$ to a few small values of m for each n.

Our reduction procedure was introduced in a dual form by Bouchet [1], who used it to obtain a new inductive proof for the genus of complete bipartite graphs. Magajna, Mohar and Pisanski reinterpreted Bouchet's construction in the context of quadrangular embeddings [11], and more details were given by Mohar, Parsons, and Pisanski [12]. The general version here is due to Kawarabayashi, Stephens and Zha [9]. We call this the 'diamond sum' because of the notation for (the dual of) Bouchet's construction used in Mohar and Thomassen [13, page 118].

Construction 2.1 (Diamond Sum) Suppose Ψ_1 is an embedding of a simple graph G_1 on surface Σ_1 and Ψ_2 is an embedding of a simple graph G_2 on Σ_2. Let u be a vertex of degree $k \geq 1$ in G_1, with neighbors $u_0, u_1, \ldots, u_{k-1}$ in cyclic order around u. Suppose there is a vertex v of degree k in G_2, with neighbors $v_0, v_1, \ldots, v_{k-1}$ in cyclic order around v. We can find a closed disk D_1 that intersects G_1 in u and the edges uu_0, uu_1, ..., uu_{k-1}, with $\partial D_1 \cap G_1 = \{u_0, u_1, \ldots, u_{k-1}\}$. Similarly, we can find a closed disk D_2 that intersects G_2 in v and the edges vv_0, vv_1, ..., vv_{k-1}, with $\partial D_2 \cap G_2 = \{v_0, v_1, \ldots, v_{k-1}\}$. Remove the interiors of D_1 and D_2, and identify ∂D_1 with ∂D_2 so that u_i is identified with v_i for $0 \leq i \leq k-1$. The result is called a *diamond sum* $\Psi_1 \diamondsuit \Psi_2$, and it embeds a graph we denote by $G_1 \diamondsuit G_2$ on the surface $\Sigma_1 \# \Sigma_2$, where # denotes the connected sum of two surfaces. Note that $\Psi_1 \diamondsuit \Psi_2$ will be orientable if both Ψ_1 and Ψ_2 are orientable.

The diamond sum is not unique; it depends on u and v, and how we match the neighbors of u to the neighbors of v. When $k = 1$ or 2, this does not determine the direction in which we identify ∂D_1 with ∂D_2, and we also need to choose that direction. However, when we apply the diamond

sum we will have $k \geq 3$, and every permutation of the neighbors of u will be an automorphism of G_1, so given u and v the graph $G_1 \diamondsuit G_2$ will be unique up to isomorphism.

In particular, we will let $n \geq 3$, and take G_1 to be $\overline{K_p} + K_n$ with u one of the vertices of the $\overline{K_p}$, and G_2 to be $K_{q,n} = \overline{K_q} + \overline{K_n}$ with v one of the vertices of the $\overline{K_q}$. Then the graph $(\overline{K_p} + K_n) \diamondsuit K_{q,n}$ is $\overline{K_{p+q-2}} + K_n$.

If $g(\overline{K_p} + K_n) = g(K_{p,n}) = \lceil (p-2)(n-2)/4 \rceil$ we may take Ψ_1 to be an embedding of $\overline{K_p} + K_n$ on the orientable surface of this genus. We may take Ψ_2 to be an embedding of $K_{q,n}$ on the orientable surface of genus $g(K_{q,n}) = \lceil (q-2)(n-2)/4 \rceil$. The diamond sum yields an embedding of $\overline{K_{p+q-2}} + K_n$ on the orientable surface of genus

$$\left\lceil \frac{(p-2)(n-2)}{4} \right\rceil + \left\lceil \frac{(q-2)(n-2)}{4} \right\rceil$$

which we would like to be the same as

$$g(K_{p+q-2,n}) = \left\lceil \frac{(p+q-4)(n-2)}{4} \right\rceil .$$

These expressions will certainly be equal if either $(p-2)(n-2)$ or $(q-2)(n-2)$ is divisible by 4. Therefore, supposing that $m \geq p$ and letting $q = m - p + 2$ we obtain the following.

Lemma 2.2 *Suppose $n \geq 3$, $p \geq n-1$, $g(\overline{K_p} + K_n) = g(K_{p,n})$ and $4 \mid (p-2)(n-2)$. Then $g(\overline{K_m} + K_n) = g(K_{m,n})$ for all $m \geq p$.*

This has the following consequence.

Theorem 2.3 *Given $n \geq 3$, in order to prove that $g(\overline{K_m} + K_n) = g(K_{m,n})$ for all $m \geq n-1$ it suffices to prove this for only the following values of m:*

(0) *$m = n-1$ and n, if $n \equiv 0 \pmod 4$;*

(1) *$m = n-1$, n and $n+1$, if $n \equiv 1 \pmod 4$;*

(2) *$m = n-1$, if $n \equiv 2 \pmod 4$; and*

(3) *$m = n-1$, if $n \equiv 3 \pmod 4$.*

Proof Let $m_0(n) = \min\{m \mid m \geq n-1 \text{ and } 4 \mid (m-2)(n-2)\}$. If the result is true for $m = m_0(n)$, it is true for all $m \geq m_0(n)$ by Lemma 2.2. So it suffices to prove the result for m such that $n-1 \leq m \leq m_0(n)$. The theorem follows. ∎

The case $m = n+1$ for (1) above can also be reduced to proving the case $m = n-1$ for (2) above, as follows.

Lemma 2.4 *Given $n \geq 3$, in order to prove that $g(\overline{K_{n+1}} + K_n) = g(K_{n+1,n})$ it suffices to prove that $g(\overline{K_{n'-1}} + K_{n'}) = g(K_{n'-1,n'})$ where $n' = n+1$, i.e., that $g(\overline{K_n} + K_{n+1}) = g(K_{n,n+1})$.*

Proof Note that $\overline{K_{n+1}} + K_n$ can be regarded as a spanning subgraph of $\overline{K_n} + K_{n+1}$. Therefore if $g(\overline{K_n} + K_{n+1}) = g(K_{n,n+1})$, we have

$$g(K_{n+1,n}) \leq g(\overline{K_{n+1}} + K_n) \leq g(\overline{K_n} + K_{n+1}) = g(K_{n,n+1}).$$

Since the first and last terms here are equal, all terms are equal. ∎

3 Transition graphs for even n

In this section we show that when n is even we can use algebraic constructions known as 'transition graphs' to show that $g(\overline{K_n} + K_n) = g(K_{n,n})$, which can then be used with Lemma 2.2 to get more general results. Transition graphs are an alternative representation of embedded voltage graphs. They were introduced in [6], which describes the equivalence between transition graphs and general embedded voltage graphs. While equivalent to embedded voltage graphs, transition graphs are more convenient in certain circumstances.

In this paper we will use only a special class of transition graphs, the orientable cyclic n-transition graphs. We therefore give a self-contained presentation of these, which can be used to describe certain orientable embeddings of symmetric complete bipartite graphs $K_{n,n}$.

Suppose $n \geq 3$. An *orientable cyclic n-transition graph* consists of n vertices labelled by the distinct elements of the group $\mathbf{Z}_n = \{0, 1, 2, \ldots, n-1\}$, together with two sets of n directed edges, the solid edges and the dashed edges, such that the solid edges form a directed hamilton cycle and the dashed edges also form a directed hamilton cycle. Write $j \Rightarrow k$ to represent a solid edge from j to k, and $j \rightarrow k$ to represent a dashed edge from j to k. The solid and dashed edges do not restrict each other, so that we may possibly have both $j \Rightarrow k$ and $j \rightarrow k$, or both $j \Rightarrow k$ and $k \rightarrow j$. For examples of such transition graphs, see Figure 1 and the other figures in this section.

Every orientable cyclic n-transition graph T describes an orientable embedding of $K_{n,n}$, as follows. The embedding is *cellular*, i.e., the interior of every face is homeomorphic to an open disk. As is well known, an orientable cellular embedding can be described by specifying a cyclic ordering, known as the *clockwise rotation*, of the edges incident with each vertex. Suppose $K_{n,n}$ has bipartition (X, Y) where $X = \{x_0, x_1, \ldots, x_{n-1}\}$ and $Y = \{y_0, y_1, \ldots, y_{n-1}\}$: the vertices of X and Y are indexed by elements of the group $\mathbf{Z}_n$. The vertex labelled j in T corresponds to all edges of the form $x_i y_{i+j}$, which we will call edges of *slope* j. Then $j \Rightarrow k$ in T indicates that an edge of slope k follows an edge of slope j in the clockwise rotation around every vertex in X, and $k \rightarrow l$ in T indicates that an edge of slope l follows an edge of slope k in the clockwise rotation around every vertex in Y. Thus, T describes all clockwise rotations and hence determines an embedding.

The faces of the embedding may be traced using closed walks in T that alternate between solid and dashed edges, all traversed in the forward direction. Suppose $W = k_1 \rightarrow k_2 \Rightarrow k_3 \ldots \Rightarrow k_{2p-1} \rightarrow k_{2p} \Rightarrow k_1$ is such a walk. Starting from x_i we follow an edge of slope k_1 to y_{i+k_1}, then since $k_1 \rightarrow k_2$ we follow an edge of slope k_2 to $x_{i+k_1-k_2}$, then since $k_2 \Rightarrow k_3$ we follow an edge of slope k_3 to $y_{i+k_1-k_2+k_3}$, and so on, until we arrive at $x_{i+k_1-k_2+\ldots+k_{2p-1}-k_{2p}}$. If t is the order in $\mathbf{Z}_n$ of the alternating sum $k_1 - k_2 + \ldots + k_{2p-1} - k_{2p}$ then we must repeat W an additional $t-1$ times to complete the tracing of the face. Altogether the embedding will contain a total of n/t faces derived from W. The edges of T may be partitioned into edge-disjoint closed alternating directed walks, and each such walk generates a family of faces.

To build embeddings with specified types of faces, we actually begin with alternating closed directed walks, and try to assemble T from a collection of these chosen so that the solid and dashed edges both form hamilton cycles. We will need two types of such walks. A *type X* walk in T is a 4-cycle of the form $j \rightarrow j+d \Rightarrow k+d \rightarrow k \Rightarrow j$ for some d, j and k. In the embedding it generates n faces of degree 4 of the form $x_i y_{i+j} x_{i-d} y_{i+k} x_i$. For example, $3 \rightarrow 4 \Rightarrow 1 \rightarrow 0 \Rightarrow 3$ is a type X walk for $n = 8$ in Figure 1. For even n, a *type I* walk in T is a 2-cycle of the form $j \rightarrow j + (n/2) \Rightarrow j$. In the embedding it generates $n/2$ faces of degree 4 of the form $x_i y_{i+j} x_{i+(n/2)} y_{i+j+(n/2)} x_i$. For example, $7 \rightarrow 0 \Rightarrow 7$ is a type I walk for $n = 14$ in Figure 3.

Theorem 3.1 *Suppose n is even and $n \geq 2$. Then $g(\overline{K_m} + K_n) = g(K_{m,n}) = \lceil (m-2)(n-2)/4 \rceil$ for all $m \geq n$.*

Proof When $n = 2$ all graphs $\overline{K_m} + K_2$ with $m \geq 2$ are planar as required, so we may assume that $n \geq 4$. By Lemma 2.2, it will suffice to prove the result in the case $m = n$, because $4 \mid (n-2)(n-2)$

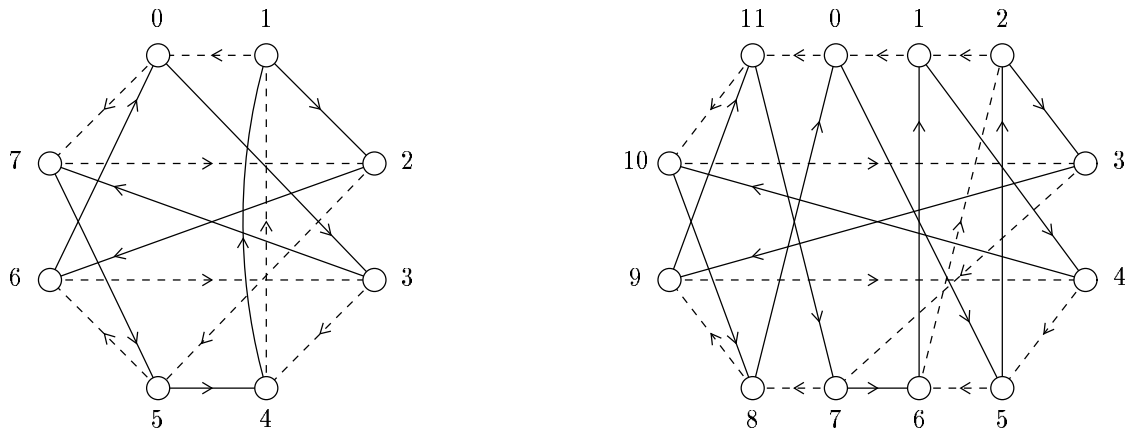

Figure 1: Transition graphs with $n = 8$ (left) and $n = 12$ (right)

when n is even. The proof for $m = n$ involves using a transition graph to construct a minimum genus embedding of $K_{n,n}$ that can have edges added to get an embedding of $\overline{K_n} + K_n$.

We need special arguments for $n = 4$, 6 and 10. For $n = 4$ and $n = 6$, the following sets of faces provide minimum genus embeddings of $K_{n,n}$ that allow us to insert edges between every y_i and y_j, as required.

4: $(x_0y_0x_1y_1)$, $(x_0y_1x_2y_2)$, $(x_0y_2x_1y_3)$, $(x_0y_3x_3y_0)$, $(x_1y_0x_2y_3)$, $(x_1y_2x_3y_1)$, $(x_2y_0x_3y_2)$, $(x_2y_1x_3y_3)$.

6: $(x_0y_0x_1y_1)$, $(x_0y_1x_2y_2)$, $(x_0y_2x_1y_3)$, $(x_0y_3x_2y_4)$, $(x_0y_4x_1y_5)$, $(x_0y_5x_3y_0)$, $(x_1y_0x_4y_3)$, $(x_1y_2x_4y_5)$, $(x_1y_4x_3y_1)$, $(x_2y_0x_3y_2)$, $(x_2y_1x_4y_4)$, $(x_2y_3x_3y_5)$, $(x_2y_5x_5y_0)$, $(x_3y_3x_5y_1)$, $(x_3y_4x_5y_2)$, $(x_4y_0x_5y_4)$, $(x_4y_1x_5y_5)$, $(x_4y_2x_5y_3)$.

The case $n = 10$ is dealt with by Theorem 4.4 in the next section. So, we may assume that $n = 8$ or $n \geq 12$.

Let T be an orientable cyclic n-transition graph. Assume T is constructed using walks of type X and type I, so that all faces in the embedding have degree 4. Counting edges in faces and using Euler's formula, we see that we get a minimum genus embedding of $K_{n,n}$. Define the *length* of a solid edge $j \Rightarrow k$ in T to be $k - j$. Suppose every nonzero element of $\mathbf{Z}_n$ occurs as the length of at least one solid edge. Then we claim that for every distinct i and j the resulting embedding has at least one face containing both y_i and y_j. For there is a solid edge $a \Rightarrow b$ whose length $b - a$ equals $j - i$. Tracing the face that uses the edge of slope a incident with y_i in the YX direction, we see that it goes from y_i to x_{i-a}, and then since $a \Rightarrow b$ it goes to $y_{i-a+b} = y_j$. Thus, for every i and j we may insert the edge y_iy_j in a face of degree 4 containing y_i and y_j, which converts the embedding of $K_{n,n}$ to an embedding of $\overline{K_n} + K_n$, verifying that $g(\overline{K_n} + K_n) = g(K_{n,n})$.

So now it suffices to find suitable transition graphs, constructed from walks of type X and type I and using every possible solid edge length. There are two constructions, depending on the value of n modulo 4.

Suppose $n = 4t$, with $t \geq 2$. Then we use four families of type X walks, as follows:

(a) $t - 1 \Rightarrow t \to 2t + 1 \Rightarrow 2t \to t - 1$ (lengths ± 1),

(b) $3t + i \Rightarrow 3t - i \to 3t + 1 - i \Rightarrow 3t + 1 + i \to 3t + i$, $1 \leq i \leq t - 1$ (lengths ± 2, ± 4, ..., $\pm(2t - 2)$),

(c) $t + 1 + i \Rightarrow t - i \to t - 1 - i \Rightarrow t + i \to t + 1 + i$, $1 \leq i \leq t - 1$ (lengths ± 3, ± 5, ..., $\pm(2t - 1)$), and

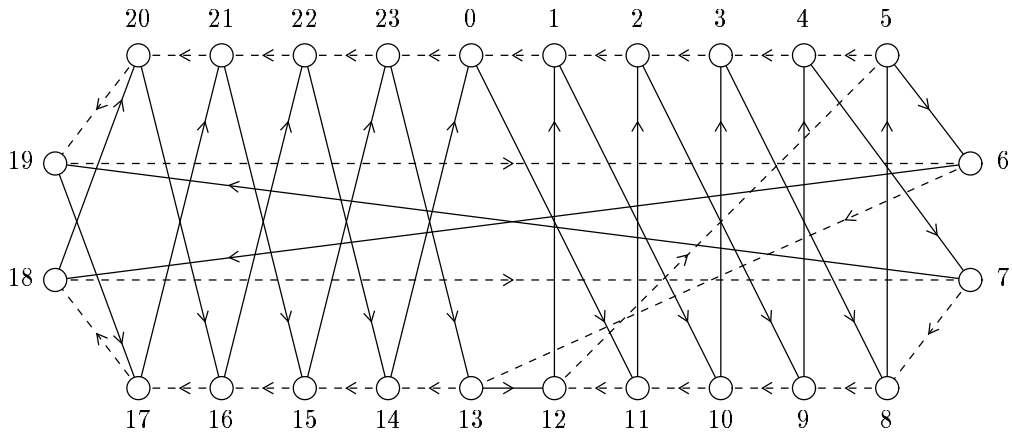

Figure 2: Transition graph with $n = 24$

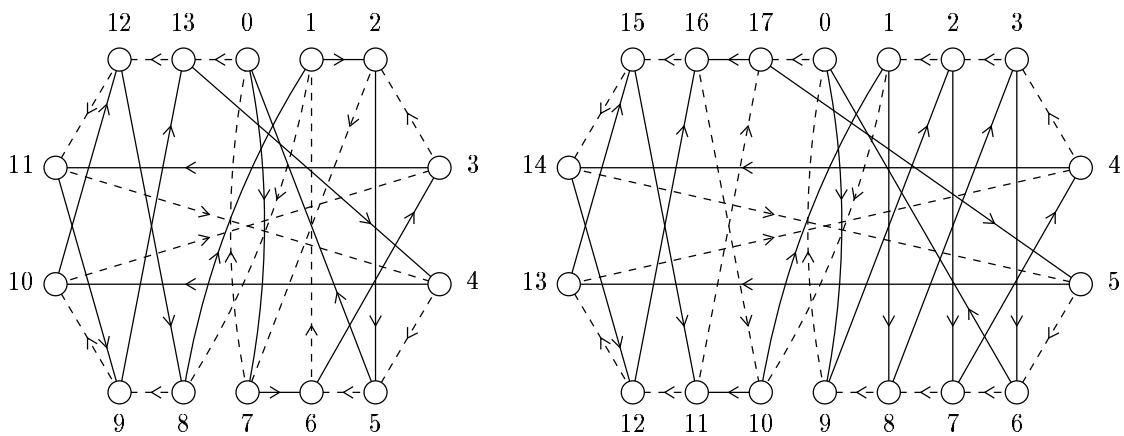

Figure 3: Transition graphs with $n = 14$ (left) and $n = 18$ (right)

(d) $t \Rightarrow 3t \to t+1 \Rightarrow 3t+1 \to t$ (length $2t$, twice).

The structure of the solid hamilton cycle is slightly different depending on whether n is 0 or 4 mod 8, but in both cases it is not difficult to verify that the solid and dashed edges each form a hamilton cycle, so that we have a transition graph with the required properties. In Figure 1 we give the two smallest examples, with $n = 8$ and $n = 12$, and in Figure 2 we give a larger example, $n = 24$, that shows the general pattern more clearly.

Suppose $n = 4t + 2$, with $t \geq 3$. The construction is a bit more complicated. We use use two type I walks:

(a) $0 \Rightarrow 2t+1 \to 0$ and $2t+2 \Rightarrow 1 \to 2t+2$ (both length $2t+1$)

and four families of type X walks:

(b) $3t+1+i \Rightarrow 3t+1-i \to 3t+2-i \Rightarrow 3t+2+i \to 3t+1+i, \quad 1 \leq i \leq t-1$ (lengths ± 2, ± 4, …, $\pm(2t-2)$),

(c) $t-i \Rightarrow t+1+i \to t+2+i \Rightarrow t+1-i \to t-i, \quad 1 \leq i \leq t-1$ (lengths ± 3, ± 5, …, $\pm(2t-1)$),

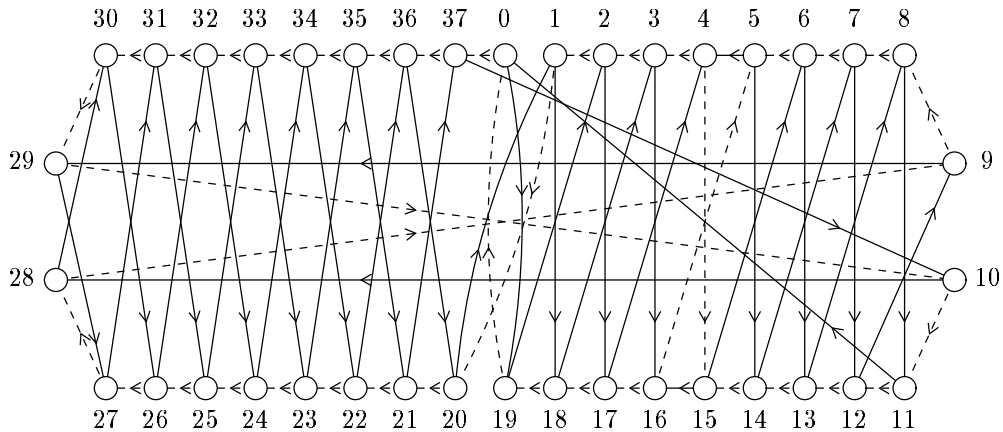

Figure 4: Transition graph with $n = 38$

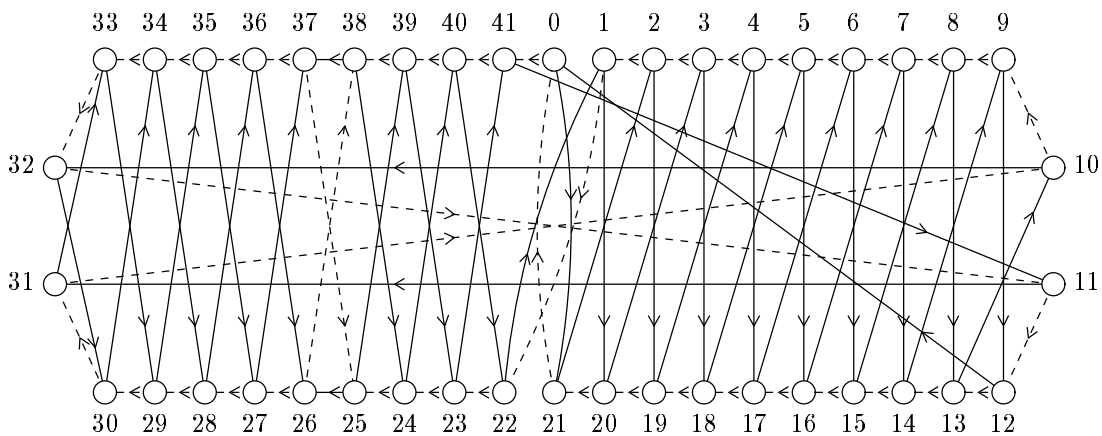

Figure 5: Transition graph with $n = 42$

(d) $t \Rightarrow 3t+2 \rightarrow t+1 \Rightarrow 3t+1 \rightarrow t$ (lengths $\pm 2t$), and

(e) $4t+1 \Rightarrow t+1 \rightarrow t+2 \Rightarrow 0 \rightarrow 4t+1$ (lengths $\pm(t+2)$).

But notice that we do not have any solid edges of length ± 1 and we use the edges of length $\pm(t+2)$ twice. We remedy this with one additional step.

(f) Choose the type X walk in either (b) or (c) that contains solid edges of lengths $\pm(t+2)$, and exchange solid and dashed edges in that walk. This removes the excess solid edges of length $\pm(t+2)$ and creates solid edges of length ± 1.

The structure of the hamilton cycles actually varies depending on whether n is 2, 6, 10 or 14 mod 16, but in every case it is not difficult to verify that the solid and dashed edges now form hamilton cycles. Thus, we have the required transition graph. In Figure 3 we give the two smallest cases, $n = 14$ and $n = 18$. In Figures 4 and 5 we give larger examples, with $n = 38$ and $n = 42$, that show the general pattern more clearly.

This concludes the proof. ∎

We have the following corollary for joins of arbitrary graphs with large edgeless graphs.

Corollary 3.2 *Suppose G is a simple graph with $|V(G)| = n$ where n is even and $n \geq 2$. Then for every $m \geq n$ we have $g(\overline{K_m} + G) = g(K_{m,n}) = \lceil (m-2)(n-2)/4 \rceil$.*

Proof We have $K_{m,n} = \overline{K_m} + \overline{K_n} \subseteq \overline{K_m} + G \subseteq \overline{K_m} + K_n$ and the first and last graphs both have the same genus, so $\overline{K_m} + G$ also has that genus. ∎

4 A doubling construction

The constructions in the previous section do not give any results for $\overline{K_m} + K_n$ when $m = n-1$ or when n is odd. In this section we obtain some results of these kinds, although only for rather special values of n. These results are based on a recursive construction for deriving new minimum genus embeddings of graphs of the form $\overline{K_{4s+1}} + K_{4s+2}$ from old ones. Rather than working with joins, it will be easier to work with *hamilton cycle* embeddings, embeddings where every face is a hamilton cycle. The following lemma allows us to do this.

Lemma 4.1 *Suppose G is an n-vertex r-regular simple graph, where $r \geq 2$. Then the following are equivalent.*

(i) *$\overline{K_r} + G$ has an orientable triangular embedding.*

(ii) *$g(\overline{K_r} + G) = g(K_{r,n})$ and $4 \mid (r-2)(n-2)$.*

(iii) *G has an orientable hamilton cycle embedding.*

Proof We show that (i) $\Leftrightarrow$ (ii) and (i) $\Leftrightarrow$ (iii). Note that $\overline{K_r} + G$ has $3rn/2$ edges. Since G is simple, every face of an embedding of $\overline{K_r} + G$ has length at least 3. By counting edges in faces, we see that an embedding of $\overline{K_r} + G$ is triangular if and only if it has exactly rn faces.

Suppose (i) holds, so $\overline{K_r} + G$ has a triangular embedding, which has rn faces. Euler's formula shows that the genus of the embedding is $(r-2)(n-2)/4$. Therefore $(r-2)(n-2)/4$ is an integer, and hence is the genus of $K_{r,n}$, so (ii) follows. Also, each vertex v of $\overline{K_r}$ is incident with n of the rn faces, and removing v from the embedding leaves a face that is a hamilton cycle in G. Removing all vertices of $\overline{K_r}$ accounts for all rn of the original triangular faces, leaving an embedding of G in which all faces are hamilton cycles, so (iii) also follows.

Suppose (ii) holds, so $g(\overline{K_r}+G) = g(K_{r,n}) = (r-2)(n-2)/4$. From Euler's formula the minimum genus embedding of $\overline{K_r} + G$ has exactly rn faces and (i) follows.

Suppose (iii) holds, so G has a hamilton cycle embedding. This must have exactly r faces, and we may insert a vertex of $\overline{K_r}$ in each one to obtain a triangular embedding of $\overline{K_r} + G$, so (i) follows. ∎

One ingredient of our construction will be hamilton cycle embeddings of symmetric complete bipartite graphs, which are known to exist.

Observation 4.2 *For $n \geq 2$, $K_{n,n}$ has an orientable hamilton cycle embedding.*

Proof This is a consequence of Lemma 4.1 and the fact that Ringel and Youngs [17] have shown that $g(K_{n,n,n} = \overline{K_n} + K_{n,n})$ is equal to $(n-1)(n-2)/2 = g(K_{n,2n})$. However, it is simple to verify directly. For each $i \in \mathbf{Z}_n$ construct the cycle $C_i = (x_0 y_i x_1 y_{i+1} x_2 y_{i+2} \ldots x_{n-1} y_{i+n-1})$ in $K_{n,n}$ (with the usual vertex set). The cycles C_i, $i \in \mathbf{Z}_n$, form the faces of the required embedding. (This construction is also very easy to represent using either an embedded voltage graph or a transition graph.) ∎

Now we can state our construction, which takes a hamilton cycle embedding and produces a new one on roughly double the number of vertices.

Theorem 4.3 *Suppose $s \geq 1$ and K_{4s+2} has an orientable hamilton cycle embedding, or equivalently (by Lemma 4.1) suppose $g(\overline{K_{4s+1}} + K_{4s+2}) = g(K_{4s+1,4s+2})$. Then K_{8s+2} also has an orientable hamilton cycle embedding, or equivalently $g(\overline{K_{8s+1}} + K_{8s+2}) = g(K_{8s+1,8s+2})$.*

Proof Suppose K_{4s+2} has an orientable hamilton cycle embedding. We regard the faces of the embedding as directed hamilton cycles, each directed anticlockwise in the natural way obtained by following the clockwise rotations around each vertex. (To see why clockwise rotations at vertices yield anticlockwise face directions it may be helpful to consider planar graphs.)

Take one copy of this embedding, call its graph G, label any vertex of G as x_∞ and label the remaining vertices x_0, x_1, x_2, ..., x_{4s} in their clockwise order around x_∞, with subscripts from the group $\mathbf{Z}_{4s+1}$. For each $i \in \mathbf{Z}_{4s+1}$, let A_i denote the face that uses the path $x_i x_\infty x_{i+1}$ as it passes through x_∞.

Take a second copy of the embedding, call its graph H, label any vertex of H as y_∞, and label the remaining vertices $y_0, y_{4s-1}, y_{4s-3}, \ldots, y_3, y_1, y_{4s}, y_{4s-2}, \ldots, y_4, y_2$ in clockwise order around y_∞ (the subscript decreasing by 2 each time). For each $i \in \mathbf{Z}_{4s+1}$ let B_i denote the face that uses the path $y_{i+2} y_\infty y_i$ as it passes through y_∞.

Let $G' = G - x_\infty$ (on vertex set $X = \{x_i \mid i \in \mathbf{Z}_{4s+1}\}$) and let $H' = H - y_\infty$ (on vertex set $Y = \{y_i \mid i \in \mathbf{Z}_{4s+1}\}$). For each $i \in \mathbf{Z}_{4s+1}$ let A'_i be $A_i - x_\infty$, which is a directed path from x_{i+1} to x_i in G', and let B'_i be $B_i - y_\infty$, which is a directed path from y_i to y_{i+2} in H'. Let C_∞ be the directed cycle $(y_0 x_0 y_1 x_1 y_2 x_2 \ldots y_{4s} x_{4s})$, and let $\overline{C_\infty}$ denote the underlying undirected cycle. For each $i \in \mathbf{Z}_{4s+1}$ let C_i be the directed cycle $A'_i \cup B'_i \cup \{x_i y_i, y_{i+2} x_{i+1}\}$. Both new directed edges $x_i y_i$ and $y_{i+2} x_{i+1}$ of C_i are the reverse of edges $y_i x_i$ and $x_{i+1} y_{i+2}$ of C_∞. Therefore, if we take the collection of directed cycles consisting of C_∞ and C_i, $i \in Z_{4s+1}$, we cover every edge of the graph $J_1 = G' \cup H' \cup \overline{C_\infty}$ (with vertex set $X \cup Y$) once in each direction. In fact, this gives an orientable hamilton cycle embedding of J_1. The clockwise rotations are the same as in G or H, except that at each x_i the edge $x_i x_\infty$ is replaced by $x_i y_i$ and $x_i y_{i+1}$ in that order, and at each y_i the edge $y_i y_\infty$ is replaced by $y_i x_{i-1}$ and $y_i x_i$ in that order.

By Observation 4.2 there is an orientable hamilton cycle embedding of $J_2 = K_{4s+1,4s+1}$, with $4s+1$ faces $D_0, D_1, \ldots, D_{4s}$. We can label the vertices of J_2 using $X \cup Y$ so that D_{4s} becomes the reverse of C_∞. Because the vertices of C_∞ alternate between X and Y, the bipartition of J_2 is then (X, Y).

Now delete the interior of the face C_∞ from the embedding of J_1 to get an embedding with boundary curve $\overline{C_\infty}$. Also delete the interior of the face D_{4s} from the embedding of J_2 to get another embedding with boundary curve $\overline{C_\infty}$. The two embeddings share no edges other than those in $\overline{C_\infty}$. Glue the two embeddings together by identifying the copies of $\overline{C_\infty}$. The result is an orientable embedding of $J_1 \cup J_2$ with faces C_0, C_1, ..., C_{4s} and D_0, D_1, ..., D_{4s-1}. Since G' is a complete graph on X, H' is a complete graph on Y, and J_2 is a complete bipartite graph with bipartition (X, Y), $J_1 \cup J_2$ is just the complete graph on vertex set $X \cup Y$. Hence we have an orientable hamilton cycle embedding of K_{8s+2}, as required. ∎

Unfortunately we cannot apply Theorem 4.3 for $s = 1$, because Jungerman [8] showed that $\overline{K_5} + K_6$ has no orientable triangular embedding, so that K_6 has no orientable hamilton cycle embedding. However, we can apply it for $s = 2$ and then repeat. As a consequence, we can determine the genus of all joins $\overline{K_m} + K_n$ when $m \geq n - 1$ and n has the form $2^p + 2$ for $p \geq 3$.

Theorem 4.4 *For $p \geq 3$ and $m \geq 2^p + 1$, $g(\overline{K_m} + K_{2^p+2}) = g(K_{m,2^p+2}) = 2^{p-2}(m-2)$.*

Proof By Lemma 2.2, it suffices to prove the result in the case where $m = 2^p + 1$. For $p = 3$, taking $V(K_{10}) = \{0, 1, 2, \ldots, 9\}$, the following set of faces found by computer provide an orientable hamilton cycle embedding of K_{10}, showing that $g(\overline{K_9} + K_{10}) = g(K_{9,10})$:

10 : (0123456789), (0213547698), (0314258796), (0415268397), (0573816492), (0629185374), (0719328465), (0827594361), (0951724863).

The result now follows by induction on p using Theorem 4.3. ∎

We also obtain a result for $\overline{K_m} + K_n$ when $m \geq n+1$ and n has the form $2^p + 1$ for $p \geq 3$.

Corollary 4.5 *For $p \geq 3$ and $m \geq 2^p + 2$, $g(\overline{K_m} + K_{2^p+1}) = g(K_{m,2^p+1}) = \lceil (m-2)(2^p-1)/4 \rceil$.*

Proof Let $n = 2^p + 1$. For $m = 2^p + 2 = n + 1$, apply Lemma 2.4 to the result from Theorem 4.4. For larger values of m, apply Lemma 2.2. ∎

We get results for more general joins as well, in the same way as for Corollary 3.2.

Corollary 4.6 *Suppose G is a simple graph with $|V(G)| = 2^p + 2$ where $p \geq 3$. Then for every $m \geq 2^p + 1$ we have $g(\overline{K_m} + G) = g(K_{m,2^p+2}) = 2^{p-2}(m-2)$.*

Also, suppose H is a simple graph with $|V(H)| = 2^p + 1$ where $p \geq 3$. Then for every $m \geq 2^p + 2$ we have $g(\overline{K_m} + H) = g(K_{m,2^p+1}) = \lceil (m-2)(2^p-1)/4 \rceil$.

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