# The nonorientable genus of joins of complete graphs with large edgeless graphs

M. N. Ellingham*

*Department of Mathematics, 1326 Stevenson Center*

*Vanderbilt University, Nashville, TN 37240, U.S.A.*

`mne@math.vanderbilt.edu`

Chris Stephens†

*Department of Mathematical Sciences*

*Middle Tennessee State University, Murfreesboro, TN 37132, U.S.A.*

`cstephen@mtsu.edu`

3 September 2004

**Abstract**

We show that for $n = 4$ and $n \geq 6$, $K_n$ has a nonorientable embedding in which all the faces are hamilton cycles. Moreover, when $n$ is odd there is such an embedding that is 2-face-colorable. Using these results we consider the join of an edgeless graph with a complete graph, $\overline{K_m} + K_n = K_{m+n} - K_m$, and show that for $n \geq 3$ and $m \geq n - 1$ its nonorientable genus is $\lceil (m-2)(n-2)/2 \rceil$ except when $(m, n) = (4, 5)$. We then extend these results to find the nonorientable genus of all graphs $\overline{K_m} + G$ where $m \geq |V(G)| - 1$. We provide a result that applies in some cases with smaller $m$ when $G$ is disconnected. We also discuss some problems with a paper of Wei and Liu [*Util. Math.* **59** (2001) 237–251] that claims to provide embeddings of $K_n$ with hamilton cycle faces.

*Mathematics Subject Classification:* 05C10

* Supported by NSF grants DMS-0070613 and DMS-0215442, and NSA grant H98230-04-1-0110

† Supported by NSF grant DMS-0070613

## 1. Introduction

Recently Kawarabayashi, Zha, and the authors, in a series of papers [**5**, **6**, **13**], determined the nonorientable genus of all complete tripartite graphs $K_{l,m,n}$. If $l \geq m \geq n$, the nonorientable genus of $K_{l,m,n}$ is (with three exceptions) the same as that of the complete bipartite graph $K_{l,m+n}$, which is a subgraph of $K_{l,m,n}$. We were led to the question of when large sets of edges could be added to complete bipartite graphs without raising the genus (orientable or nonorientable). One natural case is when we try to add all possible edges on one side of the bipartition of $K_{m,n}$, to obtain the graph $\overline{K_m} + K_n$, the join of the edgeless graph $\overline{K_m}$ with the complete graph $K_n$ (which we can also think of as $K_{m+n} - K_m$, where we remove the edges of a subgraph $K_m$ of $K_{m+n}$). To do this, we require that $m \geq n-1$; otherwise, there are not enough faces in the embedding of $K_{m,n}$ for us to add the edges of the $K_n$. In this paper we therefore examine the case $m \geq n-1$. The 'diamond sum' technique used for complete tripartite graphs plays a crucial role. To approach the genus of $\overline{K_m} + K_n$ we need to consider embeddings of $K_n$ in which all the faces are hamilton cycles. Such embeddings were investigated by Wei and Liu [**24**], but there are some problems with their results, which we explain.

It is also natural to consider the genus of $\overline{K_m} + K_n$ when $m < n-1$. In fact, for small $m$ the problem has a long history, in association with work on the Map Color Theorem by Ringel, Youngs and others. We state a conjecture that covers all values of $m$.

The structure of this paper is therefore as follows. In the remainder of this section we give the basic definitions we need. In Section 2 we investigate embeddings of $K_n$ with all faces being hamilton cycles. In Section 3 we discuss the general conjecture for the genus of $\overline{K_m} + K_n = K_{m+n} - K_m$, mention previous results on this problem, and determine the nonorientable genus of $\overline{K_m} + K_n$ with $m \geq n-1$. In Section 4 we use this result to determine the nonorientable genus of $\overline{K_m} + G$ for certain general graphs $G$. In Section 5 we discuss the problems with Wei and Liu's results from [**24**]. Finally, in Section 6 we give some concluding remarks.

A *surface* is a compact 2-manifold without boundary. For $h \geq 0$, the surface $S_h$ is the sphere with $h$ handles added, which is an orientable surface. For $k \geq 0$, the surface $N_k$ is is the sphere with $k$ crosscaps added, which is nonorientable for $k \geq 1$. $N_0$ means the sphere. The *Euler genus* of $S_h$ is $2h$, and of $N_k$ is $k$. A graph is said to be *embeddable* on a surface if it can be drawn on that surface in such a way that no two edges cross. Such a drawing is referred to as an *embedding*. The *orientable genus* $g(G)$ of the graph $G$ is the minimum $h$ such that $G$ can be embedded on $S_h$. Likewise the *nonorientable genus* $\widetilde{g}(G)$ is the minimum $k$ such that $G$ can be embedded on $N_k$. By our definition the nonorientable genus of a planar graph is zero, which is convenient in various formulae.

As is commonly known, cellular embeddings of graphs on surfaces can be described in a purely combinatorial way. The most usual way to do this is to give a *rotation* at each vertex $v$, which is a cyclic permutation of the ends of edges incident with $v$, along with a *signature* for each edge, taking one of two possible values (see [**7**, **19**]). Another way is by means of edge-colored cubic graphs known as *gems* [**1**]. A third common way to describe cellular embeddings is by listing the facial walks, and that is the approach we use in this paper.

A graph, or a walk in a graph, will be called *trivial* if it has no edges. Let $\mathcal{F}$ be a multiset of nontrivial closed walks in a connected nontrivial graph $G$. It will be important how the faces of $\mathcal{F}$ travel through each vertex $v$. Assume each $F \in \mathcal{F}$ has an arbitrarily designated forward direction. An ordered pair $(\alpha, \beta)$ is a *transition of* $\mathcal{F}$ *at* $v$ if $\alpha$ and $\beta$ are edge ends incident with $v$, and $\beta$ immediately follows $\alpha$ in some $F \in \mathcal{F}$. Note that the same transition may appear twice in $\mathcal{F}$ (or even in the same $F$), and we consider transitions with their multiplicities.

To represent a cellular embedding, $\mathcal{F}$ must cover every edge exactly twice. However, this is not sufficient. If we glue 2-cells along each element of $\mathcal{F}$, there may be vertices of $G$ whose neighborhoods in the resulting topological space are not homeomorphic to a disk: we may have an embedding on a pseudosurface, rather than a surface. To prevent this, at each vertex $v$ there must be a cyclic permutation $\pi_v$ of the ends of edges incident with $v$ (the rotation at $v$) such that if $(\alpha, \beta)$ is a transition at $v$, then $\beta$ is either $\pi_v(\alpha)$ or $\pi_v^{-1}(\alpha)$.

Although it has been customary to describe rotations as permutations, it is sometimes easier to think of them as graphs. Given an arbitrary multiset $\mathcal{F}$ of nontrivial closed walks in $G$, define the *rotation graph* $R_v(\mathcal{F})$, or just $R_v$ if $\mathcal{F}$ is understood, to be an undirected graph with vertex set consisting of the ends of edges incident with $v$, and an edge $\alpha\beta$ for every transition $(\alpha, \beta)$ at $v$. Note that $R_v$ may have multiple edges and loops, even if $G$ is simple. Now $\mathcal{F}$ represents a cellular embedding of $G$ if and only if $R_v$ is a cycle (2-regular and connected) for every $v \in V(G)$. In that case, the rotations $\pi_v$ correspond to traversing $R_v$ in either direction, and the elements of $\mathcal{F}$ are called *facial walks*.

Regarding rotations as graphs rather than permutations is a minor change, but it does allow some results to be stated simply. Rotation graphs are particularly convenient when dealing with relative embeddings, embeddings where some faces are missing. For example, Škoviera and Širáň [**23**] characterize when a multiset of walks represents a relative embedding. Their results are stated in terms of words over an alphabet of edge ends. Using rotation graphs, their results can be summarized succinctly: a multiset $\mathcal{F}$ of nontrivial closed walks in $G$ can be extended to an embedding of $G$ if and only if each $R_v$ is a cycle or a union of vertex-disjoint paths. In the present paper, we sometimes build an embedding from two relative embeddings, each of which contributes a subgraph of the rotation graph. It is easier to assemble graphs from subgraphs than to describe how to assemble permutations from smaller objects.

If $G$ is a simple graph, each edge end incident with $v$ can be uniquely identified by specifying the other end of that edge. Therefore, we may regard the rotation graph $R_v$ as having vertex set $N(v)$ (the set of neighbors of $v$), with an edge $uw$ for each sequence $uvw$ in an element of $\mathcal{F}$. The graphs in this paper are simple, so for convenience we modify our definition of rotation graph in this way.

In this paper we construct nonorientable embeddings. An embedding represented by a multiset of nontrivial closed walks $\mathcal{F}$ is orientable if and only if there is an orientation of the elements of $\mathcal{F}$ such that every edge of $G$ is used exactly once in each direction. (In that case, it is also possible to assign orientations to the rotation graphs in a consistent manner.) To prove that our embeddings are nonorientable we shall demonstrate that $\mathcal{F}$ cannot be oriented in this way.

## 2. Hamilton cycle embeddings of complete graphs

In this section we prove the following theorem.

**Theorem 2.1.** *For $n = 4$ or $n \geq 6$, $K_n$ has a nonorientable embedding on $N_{(n-2)(n-3)/2}$ in which all faces are hamilton cycles. Moreover, when $n$ is odd there is such an embedding that is 2-face-colorable. For $n = 5$ there is no embedding (orientable or nonorientable) of $K_5$ in which all faces are hamilton cycles.*

A similar result is claimed by Wei and Liu [**24**]. They claim to show that $K_n$ has an embedding on $N_{(n-2)(n-3)/2}$ with all faces hamilton cycles, presumably for $n \geq 4$. This is not true for $n = 5$, the proof of their construction is incomplete for $n$ even, and their construction is incorrect for $n$ odd. See Section 5 for details.

The rest of this section describes the proof of Theorem 2.1. For $n \neq 5$ it is enough to show that there is a nonorientable embedding with all faces hamilton cycles; the genus of the surface is then easily verified by Euler's formula. We assume that the vertex set of $K_n$ is $\{0, 1, \ldots, n-2\} \cup \{\infty\}$, and we consider vertices 0, 1, $\ldots$, $n-2$ as elements of the group $\mathbf{Z}_{n-1}$. We denote edges of the graph by $[u, v]$, paths in the graph by $[v_1, \ldots, v_k]$, and cycles by $(v_1, v_2, \ldots, v_k)$.

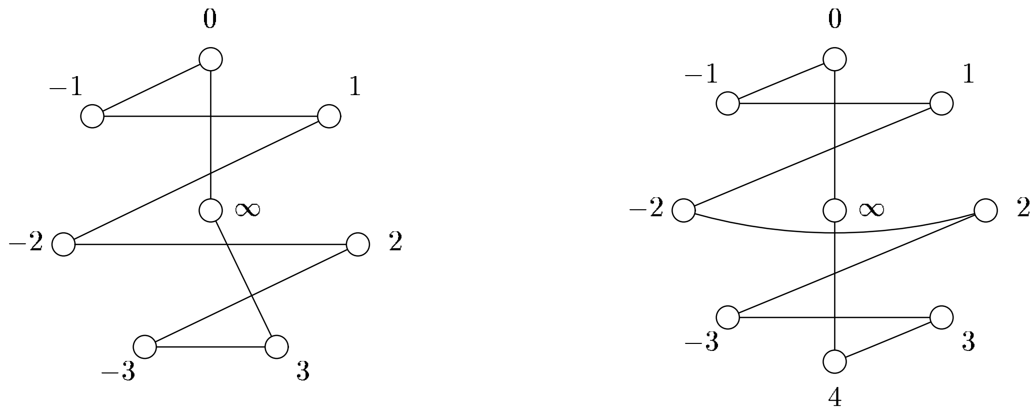
*2.1.1. $n$ is even, $n \geq 4$*

This is the easy case. We use a construction due to Wei and Liu [**24**], although we rediscovered it independently. As explained in Section 5, Wei and Liu do not give a complete proof that this construction is correct.

Write $n = 2k + 2$, so $V(K_n) = \mathbf{Z}_{2k+1} \cup \{\infty\}$. For each $i \in \mathbf{Z}_{2k+1}$, let $C_i$ be the hamilton cycle $(\infty, i, i-1, i+1, i-2, i+2 \ldots, i-k, i+k)$. The cycle $C_0$ for $n = 8$ is shown at left in Figure 2.1, and $C_i$ is just a rotation $i$ places clockwise. We claim that $\mathcal{F} = \{C_i | i \in \mathbf{Z}_{2k+1}\}$ represents a nonorientable embedding of $K_{2k+2}$.

Consider first $R_\infty$. Each $C_i$ contains the path $[i+k, \infty, i]$ and hence $R_\infty$ contains the edges $[i+k, i]$ for $i \in \mathbf{Z}_{2k+1}$. Since $gcd(k, 2k+1) = 1$, $R_\infty$ is a cycle, as desired.

Now consider $R_v$ for $v \in Z_{2k+1}$. In our construction all such $v$ are similar, so it suffices to consider $R_0$. First, $C_0$ contains $[\infty, 0, -1 = 2k]$ so $R_0$ contains $[\infty, 2k] = [2k, \infty]$. For $1 \leq i \leq k$, $C_i$ contains $[i + (i-1) = 2i-1, i - i = 0, i + i = 2i]$, so $R_0$ contains $[2i-1, 2i]$. Thus, $R_0$ has edges $[1, 2]$, $[3, 4]$, $\ldots$, $[2k-1, 2k]$. Now $C_{k+1}$ contains $[(k+1)-k = 1, (k+1)+k = 0, \infty]$ so $R_0$ contains $[1, \infty] = [\infty, 1]$. Finally, for $2 \leq i \leq k$, $C_{k+i}$ contains $[(k+i)-(k+1-i) = 2i-1, (k+i)+(k+1-i) = 2k+1 = 0, (k+i)-(k+2-i) = 2i-2]$




$C_0$ for $n = 8$ $\qquad$ $C_0$ for $n = 9$

Figure 2.1: Examples of cycle $C_0$

so $R_0$ contains $[2i-1, 2i-2] = [2i-2, 2i-1]$. Thus, $R_0$ has edges $[2,3]$, $[4,5]$, …, $[2k-2, 2k-1]$. Therefore, $R_0$ is a cycle, $(\infty, 1, 2, 3, \ldots, 2k-1, 2k)$, as required.

Suppose $\mathcal{F}$ has an orientation in which every edge is used once in each direction. We may assume that $C_0$ is oriented in the forwards direction $(\infty, 0, -1, 1, -2, \ldots, -k = k+1, k)$. Now $C_{k+1}$ contains a subpath $[0, \infty, k+1, k]$. So that $[0, \infty]$ is used in both directions, this subpath must be oriented forwards. However, then $[k+1, k]$ is used twice in the same direction, a contradiction. Therefore, $\mathcal{F}$ is nonorientable.

*2.1.2.* $n = 5$

Suppose $K_5$ has an embedding with 4 hamilton cycle faces. By Euler's formula this embedding is necessarily on $N_3$. By adding a vertex at the center of each face, joined to all vertices of the face, we obtain an embedding of $\overline{K_4} + K_5$ on $N_3$. Since $K_{4,4,1}$ is a subgraph of $\overline{K_4} + K_5$, there is then an embedding of $K_{4,4,1}$ on $N_3$. However, it was shown in [**5**] that such an embedding does not exist.

*2.1.3.* $n = 7$

Here we use an ad hoc construction. We depart from our usual convention, and take $V(K_7) = \{0, 1, 2, 3, 4, 5, 6\}$, rather than $\mathbf{Z}_6 \cup \{\infty\}$. Let $\mathcal{F} = \{C_0, C_1, C_2, D_0, D_1, D_2\}$, where

$$\begin{aligned} C_0 &= (0,1,2,3,4,5,6), & D_0 &= (0,1,5,6,3,2,4), \\ C_1 &= (0,2,4,6,1,3,5), & D_1 &= (0,2,6,1,4,5,3), \\ C_2 &= (0,3,6,2,5,1,4), & D_2 &= (0,6,4,3,1,2,5). \end{aligned}$$

This represents an embedding since the rotation graphs are all cycles:

$$\begin{aligned} R_0 &= (1,4,3,2,5,6), & R_3 &= (0,5,1,4,2,6), & R_6 &= (0,4,1,2,3,5). \\ R_1 &= (0,2,3,6,4,5), & R_4 &= (0,1,5,3,6,2), & & \\ R_2 &= (0,4,3,1,5,6), & R_5 &= (0,2,1,6,4,3), & & \end{aligned}$$

If $C_0$ is oriented forwards, then for $[6,0]$ to be used once in each direction $D_2$ must also be oriented forwards; but then $[1,2]$ is used twice in the same direction. Thus, the embedding is nonorientable. It is not hard to verify that $\mathcal{F}_1 = \{C_0, C_1, C_2\}$ covers every edge exactly once, as does $\mathcal{F}_2 = \{D_0, D_1, D_2\}$, so coloring elements of $\mathcal{F}_1$ white and elements of $\mathcal{F}_2$ black yields a 2-face-coloring of the embedding.

*2.1.4. $n$ is odd, $n \geq 9$*

Let $n = 2k+1$, and take $V(K_n) = \mathbf{Z}_{2k} \cup \{\infty\}$. For $i \in Z_{2k}$, let $C_i = (\infty, i, i-1, i+1, i-2, i+2, \ldots, i-(k-1), i+(k-1), i-k)$. The cycle $C_0$ for $n = 9$ is shown at right in Figure 2.1, and $C_i$ is just a rotation $i$ places clockwise. Note that $C_i = C_{k+i}$ for each $i$. Thus, although $\{C_i | i \in Z_{2k}\}$ covers every edge twice, it does not represent an embedding because the rotation graphs will all be unions of 2-cycles. However, let us take the distinct cycles here as a starting point and form $\mathcal{F}_1 = \{C_i | 0 \leq i \leq k-1\}$. $\mathcal{F}_1$ covers every edge of $K_n$ exactly once using $k = \frac{n-1}{2}$ hamilton cycles. Consider the rotation graphs $R_v(\mathcal{F}_1)$.

For $R_v(\mathcal{F}_1)$ with $v \neq \infty$, we may think of $\mathcal{F}_1$ as containing cycles $C_v, C_{v+1}, \ldots, C_{v+k-1}$. Now $C_v$ contains $[\infty, v, v-1]$, and for $1 \leq i \leq k-1$, $C_{v-i}$ contains $[(v+i)+(i-1) = v+2i-1, (v+i)-i = v, (v+i)+i = v+2i]$. So, $R_v(\mathcal{F}_1)$ contains $[\infty, v-1 = v+2k-1]$, and $[v+2i-1, v+2i]$ for $1 \leq i \leq k-1$. In other words, $R_v(\mathcal{F}_1)$ is a matching $\{[v+1, v+2], [v+3, v+4], \ldots, [v+2k-3, v-2k+2], [v+2k-1, \infty]\}$.

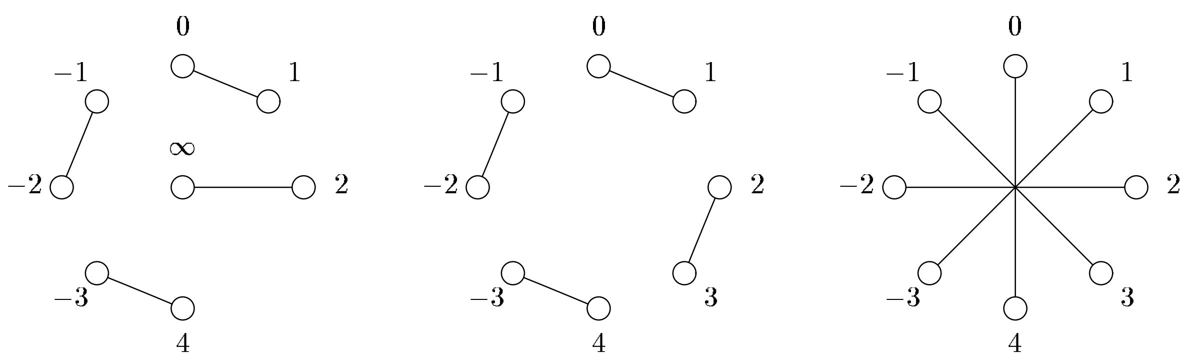
For $R_\infty(\mathcal{F}_1)$, each $C_i$ contains $[i-k = i+k, \infty, i]$ so $R_\infty(\mathcal{F}_1)$ is a matching $M_\infty = \{[0, k], [1, k+1], \ldots, [k-1, 2k-1]\}$.

Now all the rotation graphs $R_v(\mathcal{F}_1)$ are distinct, because they have different vertex sets. However, suppose we rename the vertex $\infty$ to $v$ in $R_v(\mathcal{F}_1)$, and call the resulting graph $R'_v(\mathcal{F}_1)$ (if $v = \infty$ there is no change). Then $R'_v(\mathcal{F}_1)$ is one of only three graphs: it is the matching $M_0 = \{[1, 2], [3, 4], \ldots, [2k-3, 2k-2], [2k-1, 0]\}$ when $v$ is even, the matching $M_1 = \{[0, 1], [2, 3], \ldots, [2k-2, 2k-1]\}$ when $v$ is odd, and the matching $M_\infty$ when $v = \infty$. Therefore, working with the graphs $R'_v(\mathcal{F}_1)$ rather than the graphs $R_v(\mathcal{F}_1)$ greatly reduces the number of cases we must consider.

For example, suppose $n = 9$. Then $k = 4$ and $\mathcal{F}_1$ consists of

$$\begin{array}{ll} C_0 = (\infty, 0, 7, 1, 6, 2, 5, 3, 4), & C_2 = (\infty, 2, 1, 3, 0, 4, 7, 5, 6), \\ C_1 = (\infty, 1, 0, 2, 7, 3, 6, 4, 5), & C_3 = (\infty, 3, 2, 4, 1, 5, 0, 6, 7). \end{array}$$

The graphs $R_3(\mathcal{F}_1)$, $R'_3(\mathcal{F}_1) = M_1$, and $R_\infty(\mathcal{F}_1) = R'_\infty(\mathcal{F}_1) = M_\infty$ are shown in Figure 2.2.




$R_3(\mathcal{F}_1)$ $\quad$ $R'_3(\mathcal{F}_1) = M_1$ $\quad$ $R_\infty(\mathcal{F}_1) = R'_\infty(\mathcal{F}_1) = M_\infty$

Figure 2.2: Some rotation graphs for $n = 9$

In general, for a multiset $\mathcal{F}$ of nontrivial closed walks in $K_{2k+1}$ we define $R'_v(\mathcal{F})$ to be the graph obtained from $R_v(\mathcal{F})$ by renaming $\infty$ as $v$ (when $v = \infty$, $R'_v(\mathcal{F})$ is the same as $R_v(\mathcal{F})$). $R'_v(\mathcal{F})$ is always a graph on vertex set $\mathbf{Z}_{2k} = \{0, 1, 2, \ldots, 2k-1\}$. $R'_v(\mathcal{F})$ is isomorphic to $R_v(\mathcal{F})$, so $\mathcal{F}$ represents an embedding if and only if each $R'_v(\mathcal{F})$ is a cycle.

Now $\mathcal{F}_1$ will provide half of the hamilton cycles for our embedding. We obtain the remaining cycles by applying a permutation to $\mathcal{F}_1$, as follows. Let $\sigma$ be a permutation of $V(K_{2k+1})$. If we think of $\sigma$ as just a renaming of the vertices, then there is a natural action of $\sigma$ on any structure (e.g., subgraph, walk, or multiset of walks) associated with $K_{2k+1}$. For example, for a subgraph $H$ we have $\sigma(H)$ with $V(\sigma(H)) = \{\sigma(v) | v \in V(H)\}$ and $E(\sigma(H)) = \{\sigma(u)\sigma(v) | uv \in E(H)\}$. If $\mathcal{F}$ is a multiset of nontrivial closed walks in $K_{2k+1}$, then since $\sigma$ is just a renaming of the vertices, we see that $R_{\sigma(v)}(\sigma(\mathcal{F})) = \sigma(R_v(\mathcal{F}))$ for all $v$. Moreover, if $\sigma(\infty) = \infty$, then the renaming of the vertices in forming the graphs $R'_v$ commutes with the renaming given by $\sigma$ in the sense that $R'_{\sigma(v)}(\sigma(\mathcal{F})) = \sigma(R'_v(\mathcal{F}))$ for all $v$. This may be restated as $R'_u(\sigma(\mathcal{F})) = \sigma(R'_{\sigma^{-1}(u)}(\mathcal{F}))$ for all $u$.

Therefore, if we apply a permutation $\sigma$ with $\sigma(\infty) = \infty$ to $\mathcal{F}_1$, we obtain $\sigma(\mathcal{F}_1)$ with the property that for any $u$, $R'_u(\sigma(\mathcal{F}_1))$ is $\sigma(M_0)$ when $\sigma^{-1}(u)$ is even, $\sigma(M_1)$ when $\sigma^{-1}(u)$ is odd, or $\sigma(M_\infty)$ when $u = \infty$. For each $v$, $R'_v(\mathcal{F}_1 \cup \sigma(\mathcal{F}_1)) = R'_v(\mathcal{F}_1) \cup R'_v(\sigma(\mathcal{F}_1))$. Therefore, $R'_v(\mathcal{F}_1 \cup \sigma(\mathcal{F}_1))$ is a union of two matchings: $M_0 \cup \sigma(M_0)$, $M_0 \cup \sigma(M_1)$, $M_1 \cup \sigma(M_0)$, $M_1 \cup \sigma(M_1)$, or $M_\infty \cup \sigma(M_\infty)$. To ensure that $\mathcal{F}_1 \cup \sigma(\mathcal{F}_1)$ represents an embedding we need only choose $\sigma$ such that each of these five subgraphs of $K_{2k}$ is a cycle.

An embedding found in this way will always be 2-face-colorable because each edge is covered once by $\mathcal{F}_1$ and once by $\sigma(\mathcal{F}_1)$. We may color the faces in $\mathcal{F}_1$ white and those in $\sigma(\mathcal{F}_1)$ black to obtain a 2-face-coloring of the embedding.

For example, suppose $n = 9$ and $\sigma = [\sigma(0), \sigma(1), \ldots, \sigma(7), \sigma(\infty)] = [0, 2, 6, 3, 5, 1, 7, 4, \infty]$. Then $\sigma(\mathcal{F}_1)$ consists of

$$\sigma(C_0) = (\infty, 0, 4, 2, 7, 6, 1, 3, 5), \qquad \sigma(C_2) = (\infty, 6, 2, 3, 0, 5, 4, 1, 7),$$
$$\sigma(C_1) = (\infty, 2, 0, 6, 4, 3, 7, 5, 1), \qquad \sigma(C_3) = (\infty, 3, 6, 5, 2, 1, 0, 7, 4).$$

From Figure 2.3 we can see that all of $M_0 \cup \sigma(M_0)$, $M_0 \cup \sigma(M_1)$, $M_1 \cup \sigma(M_0)$, $M_1 \cup \sigma(M_1)$ and $M_\infty \cup \sigma(M_\infty)$ are hamilton cycles. The modified rotation graphs are $R'_0 = M_0 \cup \sigma(M_0)$, $R'_1 = M_1 \cup \sigma(M_1)$, $R'_2 = M_0 \cup \sigma(M_1)$, $R'_3 = M_1 \cup \sigma(M_1)$, $R'_4 = M_0 \cup \sigma(M_1)$, $R'_5 = M_1 \cup \sigma(M_0)$, $R'_6 = M_0 \cup \sigma(M_0)$, $R'_7 = M_1 \cup \sigma(M_0)$, and $R'_\infty = R_\infty = M_\infty \cup \sigma(M_\infty)$. As these are all cycles, the actual rotation graphs $R_v$ are all cycles, and $\mathcal{F}_1 \cup \sigma(\mathcal{F}_1)$ represents an embedding.

In fact, instead of finding a permutation and checking that certain associated matchings have desirable properties, we look for the matchings first. Note that $M_0 \cup M_1$ is a hamilton cycle in $K_{2k}$. For any matchings $L_0$ and $L_1$ such that $L_0 \cup L_1$ is a hamilton cycle in $K_{2k}$, there exist permutations (in fact, $2k$ of them) $\sigma$ of $\mathbf{Z}_{2k} = V(K_{2k})$ with $\sigma(M_0) = L_0$ and $\sigma(M_1) = L_1$. $M_\infty$ is the matching joining antipodal vertices of the cycle $M_0 \cup M_1$, so for every such $\sigma$ the matching $L_\infty$ joining antipodal vertices of $L_0 \cup L_1 = \sigma(M_0 \cup M_1)$ will be $\sigma(M_\infty)$. Therefore, to find an embedding it suffices to find matchings $L_0$ and $L_1$ such that all of $L_0 \cup L_1$, $M_0 \cup L_0$, $M_0 \cup L_1$, $M_1 \cup L_0$, $M_1 \cup L_1$ and $M_\infty \cup L_\infty$ are hamilton cycles.

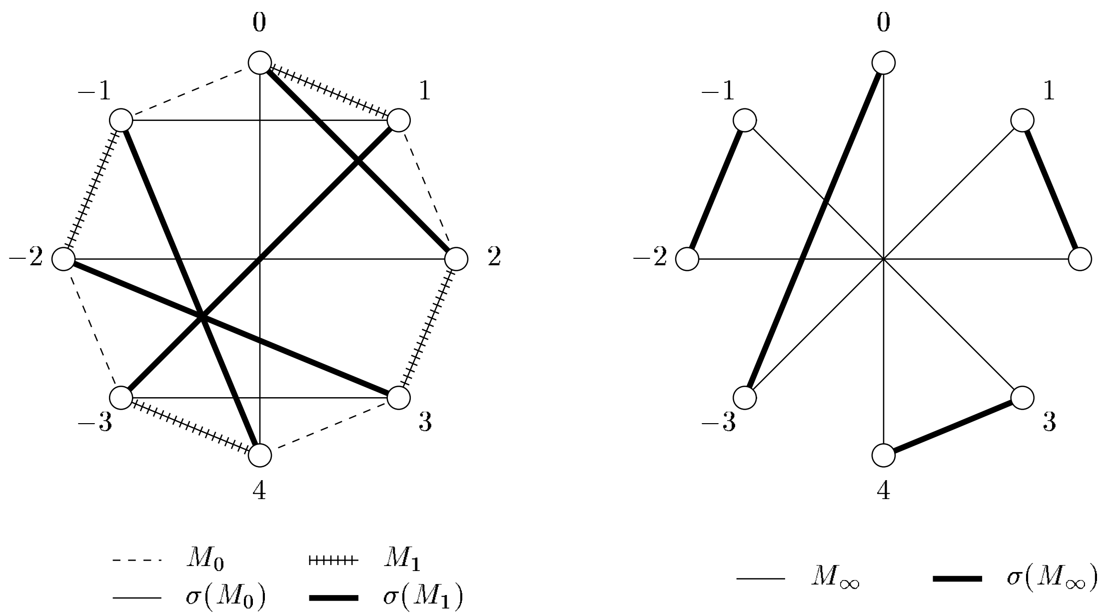



Figure 2.3: Matchings and their images under $\sigma$ for $n = 9$

Given $L_0$ and $L_1$, for definiteness we restrict ourselves to one possible permutation $\sigma$, as follows. Write $L_0 \cup L_1 = (v_0, v_1, \ldots, v_{2k-1})$ where $v_0 = 0$ and $[v_0, v_1] \in L_1$. We take $\sigma = [\sigma(0), \sigma(1), \ldots, \sigma(2k-1), \sigma(\infty)] = [v_0, v_1, \ldots, v_{2k-1}, \infty]$.

In our example for $n = 9$, it suffices to find the matchings $L_0 = \{[0,4], [1,7], [2,6], [3,5]\}$ and $L_1 = \{[1,5], [0,2], [3,6], [4,7]\}$. Then $L_\infty$ is the antipodal matching of the cycle $L_0 \cup L_1 = (0, 2, 6, 3, 5, 1, 7, 4)$, i.e., $L_\infty = \{[0,5], [1,2], [6,7], [3,4]\}$. The pictures in Figure 2.3 verify that we have the required properties. There are eight permutations with $\sigma(\infty) = \infty$, $\sigma(M_0) = L_0$ and $\sigma(M_1) = L_1$. For example, we could take $\sigma = [\sigma(0), \sigma(1), \ldots, \sigma(7), \sigma(\infty)] = [3, 6, 2, 0, 4, 7, 1, 5, \infty]$. However, for definiteness we take $\sigma = [0, 2, 6, 3, 5, 1, 7, 4, \infty]$.

An Euler's formula calculation shows that embedding we obtain for $K_9$ has Euler characteristic $8 + 9 - 36 = -19$, which is odd. Therefore, the embedding must be nonorientable.

Below we find suitable matchings $L_0$ and $L_1$ for all even $n \geq 8$. We deal with $n \equiv 1 \bmod 4$ and $3 \bmod 4$ separately. Once we have found an embedding, we show it is nonorientable.

2.1.4.1. $n \equiv 1 \bmod 4$, $n \geq 9$

Then $k$ is even and $2k \geq 8$. We generalize our previous example when $n = 9$. Take $L_0 = \{[0,k]\} \cup \{[i,-i] | 1 \leq i \leq k-1\}$, and $L_1 = \{[1,k+1], [k-1,k+2], [k,k+3]\} \cup \{[1+i, 1-i] | 1 \leq i \leq k-3\}$. For example, $L_0$ and $L_1$ for $n = 21$ are shown in Figure 2.4. It is easy to verify that $M_0 \cup L_0$, $M_0 \cup L_1$, $M_1 \cup L_0$ and $M_1 \cup L_1$ are all hamilton cycles in $K_{2k}$. $L_0 \cup L_1$ is the cycle $(0, 2, -2, 4, -4, \ldots, 2-k, k-2, k-1, 1-k, 1, -1, 3, -3, \ldots, k-3, 3-k, k)$ giving $L_\infty = \{[0, 1-k], [k-1, k]\} \cup \{[2i-1, 2i], [-(2i-1), -2i] | 1 \leq i \leq \frac{k-2}{2}\}$. For example, $L_\infty$ for $n = 21$ is shown in Figure 2.4. It is also not difficult to verify that $M_\infty \cup L_\infty$ is a hamilton

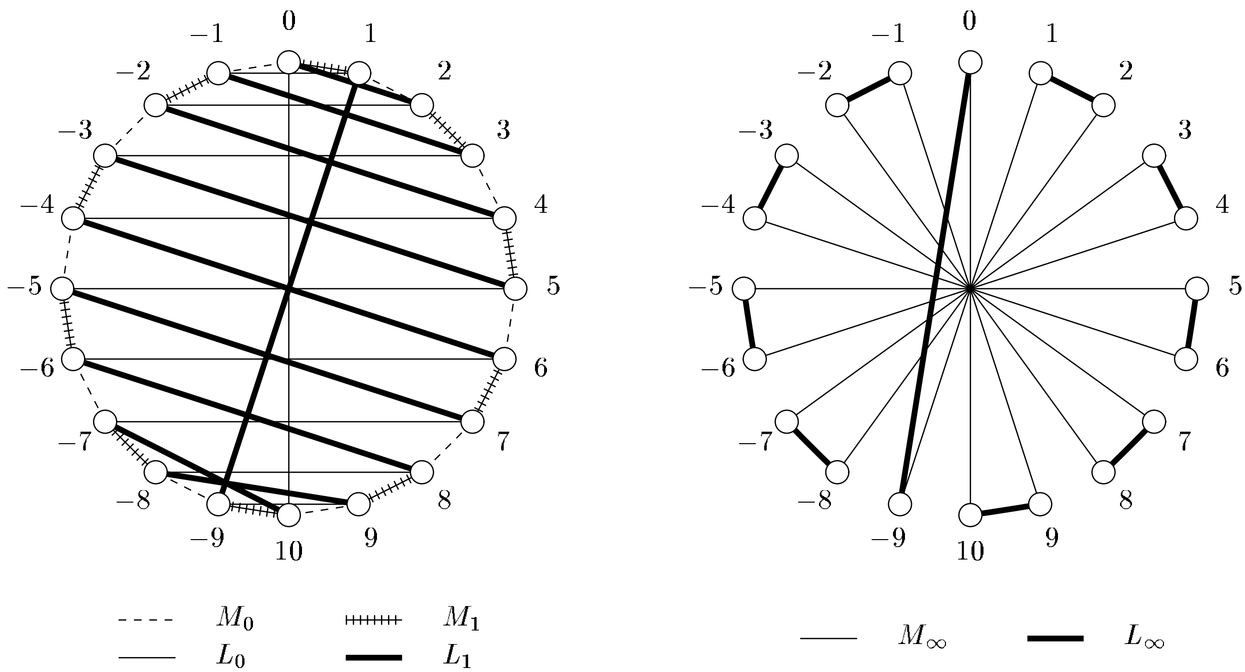



Figure 2.4: Matchings for $n = 21$

cycle. Therefore, $L_0$ and $L_1$ determine a permutation $\sigma$ for which $\mathcal{F}_1 \cup \sigma(\mathcal{F}_1)$ represents an embedding.

Euler's formula shows that we have an embedding on a surface whose Euler characteristic is $2 - \frac{(n-2)(n-3)}{2}$. Since $n \equiv 1 \bmod 4$, this is odd. Therefore, the embedding must be nonorientable.

2.1.4.2. $n \equiv 3 \bmod 4$, $n \geq 11$

Then $k$ is odd and $2k \geq 10$. Write $k = 2p + 1$, so $n = 4p + 3$. As when $n \equiv 1 \bmod 4$, we take $L_0 = \{[0, k]\} \cup \{[i, -i] | 1 \leq i \leq k - 1\}$ which is now $\{[0, 2p + 1]\} \cup \{[i, -1] | 1 \leq i \leq 2p\}$. But now we take $L_1$ constructed in a different way: $L_1 = \{[2, 1-2p], [p+3, -p], [p+4, 1-p]\} \cup \{[2+i, 2-i] | 1 \leq i \leq p$ and $p+3 \leq i \leq k - 1 = 2p\}$. For example, $L_0$ and $L_1$ for $n = 35$ are shown in Figure 2.5. It is again easy to verify that $M_0 \cup L_0$, $M_0 \cup L_1$, $M_1 \cup L_0$ and $M_1 \cup L_1$ are all hamilton cycles.

We must still show that $L_0 \cup L_1$ is a cycle, and then that $L_\infty \cup M_\infty$ is a cycle. It helps to keep in mind that all edges of $L_0$ join a vertex $i$ to $-i$, except for $[0, 2p + 1]$, and that all edges of $L_1$ join a vertex $i$ to $4 - i$, except for $[2, -(2p - 1)]$, $[p + 3, -p]$, and $[p + 4, 1 - p]$. The details depend on the value of $p$ mod 4, i.e., $n$ mod 16, although they are similar in all four cases. We describe one case in full.

Suppose that $p \equiv 0 \bmod 4$, i.e., $n \equiv 3 \bmod 16$. Then $L_0 \cup L_1$ is a cycle $(v_0, v_1, \ldots, v_{4p+1} = v_{-1})$, where we index its vertices by elements of $\mathbf{Z}_{2k} = \mathbf{Z}_{4p+2}$. Because of page width limitations, we display it as the union of six consecutive paths $P_0$, $P_1$, …, $P_5$ (of unequal length), where $P_i$ is antipodal to $P_{3+i}$. To assist in determining $L_\infty$, we list these in the order $P_0, P_3, P_1, P_4, P_2, P_5$. For each path, most of the edges are the 'regular' edges of $L_0$ or $L_1$; the 'irregular' edges occur only as the first or last edge. To show the resulting $\sigma$, each vertex $v_i$ is also labelled above or below with its position $i$; recall that $\sigma$ maps $i$ to $v_i$.

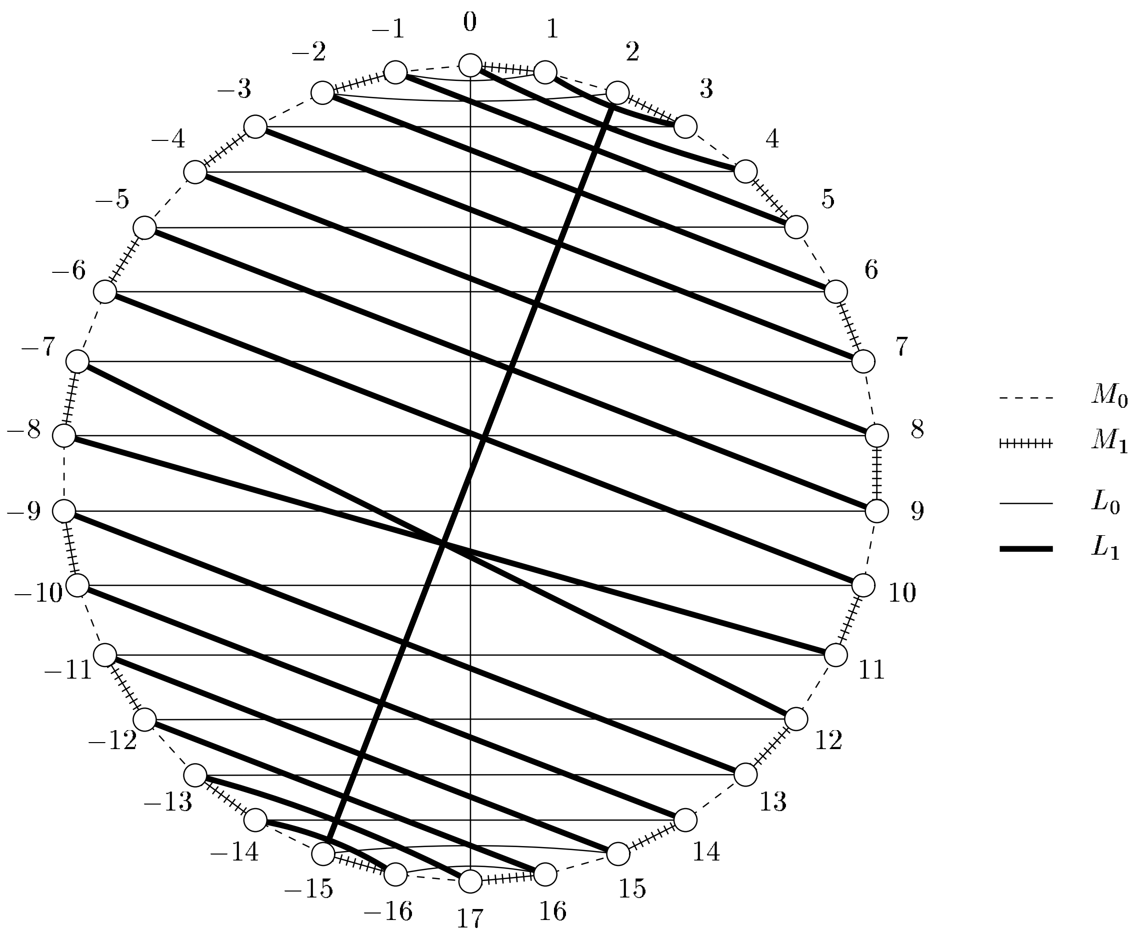



Figure 2.5: Matchings for $n = 35$

$$
\begin{array}{rlccccccc}
 & & 0 & 1 & 2 & 3 & \ldots & \frac{p}{2}-1 & \frac{p}{2} \\
P_0 & = [ & 0, & 4, & -4, & 8, & \ldots, & p, & -p\ ] \\
P_3 & = [ & -2p, & 2p, & 4-2p, & 2p-4, & \ldots, & p+4, & 1-p\ ] \\
 & & 2p+1 & -2p & 1-2p & 2-2p & \ldots & -(\frac{3p}{2}+2) & -(\frac{3p}{2}+1)
\end{array}
$$

$$
\begin{array}{rlccccccc}
 & & \frac{p}{2} & \frac{p}{2}+1 & \frac{p}{2}+2 & \frac{p}{2}+3 & \ldots & p-1 & p \\
P_1 & = [ & -p, & p+3, & -(p+3), & p+7, & \ldots, & 2p-1, & 1-2p\ ] \\
P_4 & = [ & 1-p, & p-1, & 5-p, & p-5, & \ldots, & 3, & 1\ ] \\
 & & -(\frac{3p}{2}+1) & -\frac{3p}{2} & 1-\frac{3p}{2} & 2-\frac{3p}{2} & \ldots & -(p+2) & -(p+1)
\end{array}
$$

$$
\begin{array}{rlcccccccccccc}
 & & p & p+1 & p+2 & p+3 & p+4 & \ldots & \frac{3p}{2} & \frac{3p}{2}+1 & \ldots & 2p & 2p+1 \\
P_2 & = [ & -(2p-1), & 2, & -2, & 6, & -6, & \ldots, & 2-p, & p+2, & \ldots, & 2-2p, & -2p\ ] \\
P_5 & = [ & 1, & -1, & 5, & -5, & 9, & \ldots, & p+1, & -(p+1), & \ldots, & 2p+1, & 0\ ] \\
 & & -(p+1) & -p & 1-p & 2-p & 3-p & \ldots & -(\frac{p}{2}+1) & -\frac{p}{2} & \ldots & -1 & 0
\end{array}
$$

The matching $L_\infty$ can now be read off by matching corresponding elements of $P_0$ and $P_3$, of $P_1$ and $P_4$, and of $P_2$ and $P_5$. Thus, $L_\infty = \{[0, -2p], [4, 2p], \ldots\}$.

We wish to show that $M_\infty \cup L_\infty$ is a hamilton cycle in $\mathbf{Z}_{2k}$. We proceed indirectly. We know that $M_\infty \cup L_\infty$ is a 2-regular graph with vertex set $\mathbf{Z}_{2k}$. Suppose we contract every edge of $M_\infty$ in $M_\infty \cup L_\infty$, to produce $L_\infty^*$. Then $L_\infty^*$ is still a 2-regular graph, and it is a single cycle if and only if $M_\infty \cup L_\infty$ is a single cycle. Contracting $M_\infty = \{[i, i+k] | 0 \le i \le k-1\}$ identifies each vertex $i$ with its antipodal vertex $i+k$. This corresponds to the natural projection of $\mathbf{Z}_{2k}$ onto $\mathbf{Z}_k = \mathbf{Z}_{2p+1}$, where we take each vertex modulo $2p+1$. Therefore, it suffices to show that when we project $L_\infty$ in this way, the result $L_\infty^*$ is a cycle spanning all vertices of $\mathbf{Z}_{2p+1}$.

From the endvertices of $P_0, \ldots, P_5$ we obtain three edges of $L_\infty^*$:

$$\{[0, -2p], [-p, 1-p], [-(2p-1), 1]\} = \{[0, 1], [-p, 1-p], [1, 2]\}.$$

From the second, fourth, etc. vertices of $P_0$ and $P_3$ we get edges

$$\begin{aligned}&\{[4, 2p], [8, 2p-4], \ldots, [p, p+4]\} = \{[4, -1], [8, -5], \ldots, [p, -(p+1)]\}\\ &\quad = \{[i, 3-i] \mid i \equiv 0 \bmod 4,\ 4 \le i \le p\},\end{aligned}$$

and from the third, fifth, etc. vertices of $P_0$ and $P_3$ we get edges

$$\begin{aligned}&\{[-4, -(2p-4)], [-8, -(2p-8)], \ldots, [-(p-4), -(p+4)]\}\\ &\quad = \{[-4, 5], [-8, 9], \ldots, [-(p-4), p-3]\}\\ &\quad = \{[i, 1-i] \mid i \equiv 1 \bmod 4,\ 5 \le i \le p-3\}.\end{aligned}$$

From the second, fourth, etc. vertices of $P_1$ and $P_4$ we get edges

$$\begin{aligned}&\{[p+3, p-1], [p+7, p-5], \ldots, [2p-1, 3]\} = \{[2-p, p-1], [6-p, p-5], \ldots, [-2, 3]\}\\ &\quad = \{[i, 1-i] \mid i \equiv 3 \bmod 4,\ 3 \le i \le p-1\},\end{aligned}$$

and from the third, fifth, etc. vertices of $P_1$ and $P_4$ we get edges

$$\begin{aligned}&\{[-(p+3), 5-p], [-(p+7), 9-p], \ldots, [-(2p-5), -3]\}\\ &\quad = \{[p-2, 5-p], [p-6, 9-p], \ldots, [6, -3]\}\\ &\quad = \{[i, 3-i] \mid i \equiv 2 \bmod 4,\ 6 \le i \le p-2\}.\end{aligned}$$

From the second, fourth, etc. vertices of $P_2$ and $P_5$ we get edges

$$\begin{aligned}&\{[2, -1], [6, -5], \ldots, [p-2, 3-p]\} \cup \{[p+2, -(p+1)], [p+6, -(p+5)], \ldots, [2p-2, -(2p-3)]\}\\ &\quad = \{[2, -1], [6, -5], \ldots, [p-2, 3-p]\} \cup \{[1-p, p], [5-p, p-4], \ldots, [-3, 4]\}\\ &\quad = \{[i, 1-i] \mid i \equiv 2 \text{ or } 4 \bmod 4,\ 2 \le i \le p\},\end{aligned}$$

and from the third, fifth, etc. vertices of $P_2$ and $P_5$ we get edges

$$\begin{aligned}&\{[-2, 5], [-6, 9], \ldots, [2-p, p+1]\} \cup \{[-(p+2), p+5], [-(p+6), p+9], \ldots, [3, 0]\}\\ &\quad = \{[-2, 5], [-6, 9], \ldots, [2-p, p+1]\} \cup \{[p-1, 4-p], [p-5, 8-p], \ldots, [3, 0]\}\\ &\quad = \{[i, 3-i] \mid i \equiv 1 \text{ or } 3 \bmod 4,\ 3 \le i \le p+1\}.\end{aligned}$$

Altogether, we see that the edge set of $L_\infty^*$ is $\{[0, 1], [1, 2], [-p, 1-p]\} \cup \{[i, 3-i] \mid 3 \le i \le p+1\} \cup \{[i, 1-i] \mid 2 \le i \le p\}$. This clearly forms a single cycle, as we illustrate in Figure 2.6 for the case $n = 35$. Therefore $M_\infty \cup L_\infty$ is a hamilton cycle on $\mathbf{Z}_{2k} = \mathbf{Z}_{4p+2}$, as required.

Now we demonstrate that the embedding represented by $\mathcal{F}_1 \cup \sigma(\mathcal{F}_1)$ is nonorientable. Initially, orient all $C_i$ in the direction in which they were originally described, and orient each $\sigma(C_i)$ in the direction it inherits

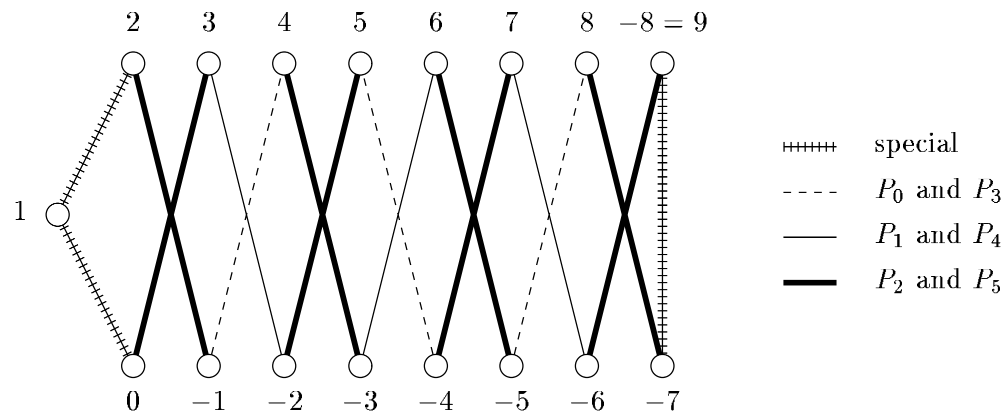



Figure 2.6: Cycle $L^*_\infty$ for $n = 35$

from $C_i$. Notice that $\sigma$ maps $\frac{p}{2} \mapsto -p$, $2p+1 \mapsto -2p$, and $-(\frac{p}{2}+1) \mapsto p+1$. Therefore, the oriented edges $[\frac{p}{2}, -(\frac{p}{2}+1)]$ and $[2p+1, \infty]$ in the oriented $C_0$ map to oriented edges $[-p, p+1]$ and $[-2p, \infty]$ in the oriented $\sigma(C_0)$. Now $C_1$ contains the oriented edges $[p+1, -p]$ and $[-2p, \infty]$. These edges are incompatibly oriented in $\sigma(C_0)$ and $C_1$. If we have an orientation where every edge is used once in each direction, it must reverse the initial orientation of exactly one of $\sigma(C_0)$ or $C_1$ to use $[-2p, \infty]$ once in each direction. However, then $[-p, p+1]$ is used twice in the same direction. Therefore, the embedding is nonorientable.

The details of verifying that $L_0 \cup L_1$ and $L_\infty \cup M_\infty$ are cycles are similar for the other three cases, $n \equiv 7$, 11 or 15 mod 16. We have to break $L_0 \cup L_1$ up in slightly different ways, but in all cases $L^*_\infty$ is the cycle with edge set $\{[0,1], [1,2], [-p, 1-p]\} \cup \{[i, 3-i] \mid 3 \le i \le p+1\} \cup \{[i, 1-i] \mid 2 \le i \le p\}$. If the reader wishes to get an idea of the details, we suggest checking the special cases $n = 39$, 43 and 47, which are large enough to illustrate the general principles.

For the proofs of nonorientability, we can in each case examine just two faces. When $n \equiv 7$ mod 16, the edges $[p-1, -(p+2)]$ and $[2p-6, 4-2p]$ are incompatibly oriented in $\sigma(C_0)$ and $C_{-1}$. When $n \equiv 11$ mod 16, the edges $[\infty, 0]$ and $[2p-2, 1-2p]$ are incompatibly oriented in $\sigma(C_0)$ and $C_0$. Finally, when $n \equiv 15$ mod 16, the edges $[-1, 3]$ and $[1-2p, 2p]$ are incompatibly oriented in $\sigma(C_0)$ and $C_1$. We omit the details.

This concludes the proof of Theorem 2.1.

## 3. The nonorientable genus of $\overline{K_m} + K_n$

In this section we determine the nonorientable genus of all graphs $\overline{K_m} + K_n$ with $m \ge n-1$. First we state a lower bound on the Euler genus of embeddings of $\overline{K_m} + K_n$ in general, whether orientable or nonorientable, and whether or not $l \ge m-1$. If $n = 0$, 1 or 2, the graphs $\overline{K_m} + K_n$ are all planar, so we take $n \ge 3$.

**Theorem 3.1.** *Suppose $m \geq 0$ and $n \geq 3$. Then any embedding of $\overline{K_m} + K_n$ has Euler genus $\gamma$ such that*

$$\gamma \geq \begin{cases} \lceil \frac{(m-2)(n-2)}{2} \rceil & \text{if } m \geq n-1, \\ \lceil \frac{(2m+n-4)(n-3)}{6} \rceil & \text{if } m \leq n-1. \end{cases}$$

**Proof.** Suppose we have an embedding of $\overline{K_m} + K_n$ with minimum Euler genus $\gamma$. By a result of Youngs [**26**], it is a cellular embedding, so we may apply Euler's formula: if $v = m+n$, $e = mn + \frac{1}{2}n(n-1)$ and $f$ denote the number of vertices, faces, and edges respectively, then $\gamma = 2 + e - v - f$.

Let $f_i$ denote the number of facial walks of length $i$. Since $\overline{K_m} + K_n$ is simple, connected, and not $K_2$, $f_i = 0$ for $i \leq 2$. By simple counting $f = f_3 + f_4 + f_5 + \ldots$ and $2e = 3f_3 + 4f_4 + 5f_5 + \ldots$. Therefore, $4f = 4f_3 + 4f_4 + 4f_5 + \ldots \leq f_3 + (3f_3 + 4f_4 + 5f_5 + \ldots) = f_3 + 2e$, or $f \leq \frac{1}{4}(f_3 + 2e)$. Hence, $\gamma = 2 + e - v - f \geq 2 + e - v - \frac{1}{4}(f_3 + 2e) = 2 + \frac{1}{2}e - v - \frac{1}{4}f_3$. So, an upper bound on $f_3$ will give a lower bound on $\gamma$.

Any facial walk of length 3 must be a triangle (3-cycle). For $0 \leq i \leq 3$ let $t_i$ be the number of triangles in the embedding that use $i$ edges of the $K_n$. Then $t_0 = t_2 = 0$. Since every edge of $K_n$ is used by at most two triangles, $t_1 \leq n(n-1)$ and $t_3 \leq \frac{1}{3}[n(n-1) - t_1]$. Thus, $f_3 = t_1 + t_3 \leq \frac{1}{3}[n(n-1) + 2t_1]$. Every triangle that uses exactly one edge of the $K_n$ must also use two edges of the $K_{m,n}$ that joins $\overline{K_m}$ to $K_n$. Since every edge of the $K_{m,n}$ may be used by at most two triangles, we have $t_1 \leq mn$.

If $m \geq n-1$, $t_1 \leq n(n-1)$ is the stronger bound on $t_1$, giving an upper bound on $f_3$ that gives the first bound on $\gamma$. If $m \leq n-1$, $t_1 \leq mn$ is the stronger bound on $t_1$, giving an upper bound on $f_3$ that gives the second bound on $\gamma$. ∎

A natural conjecture then follows.

**Conjecture 3.2.** *Suppose $m \geq 0$ and $n \geq 3$. Then, perhaps with a finite number of small exceptions,*

$$g(\overline{K_m} + K_n) = \begin{cases} \lceil \frac{(m-2)(n-2)}{4} \rceil & \text{if } m \geq n-1, \\ \lceil \frac{(2m+n-4)(n-3)}{12} \rceil & \text{if } m \leq n-1, \end{cases}$$

*and*

$$\widetilde{g}(\overline{K_m} + K_n) = \begin{cases} \lceil \frac{(m-2)(n-2)}{2} \rceil & \text{if } m \geq n-1, \\ \lceil \frac{(2m+n-4)(n-3)}{6} \rceil & \text{if } m \leq n-1. \end{cases}$$

Theorem 3.1 and Conjecture 3.2 are not new. They have been used implicitly for many years by many researchers. However, we feel it is useful to provide a statement in complete generality, covering both the case of 'small' $m$ ($m \leq n-1$) and 'large' $m$ ($m \geq n-1$). The conjectured embeddings for small and large $m$ have different qualities. The embeddings for small $m$ would be triangulations, or very close to triangulations. The embeddings for large $m$ would be minimum genus embeddings of complete bipartite graphs with extra edges added.

In the case of very small $m$ the results on Conjecture 3.2 are intimately related to the results on the Map Color Theorem due to Ringel, Youngs and others – see [**22**] for an overview. When $m = 0$ or 1, $\overline{K_m} + K_n$ is the complete graph $K_n$ or $K_{n+1}$, respectively, whose genus conforms to the conjectured values except for $\widetilde{g}(K_7)$. For $m = 2$, $\overline{K_2} + K_n = K_{n+2} - K_2$ is obtained from a complete graph by deleting one edge. In many cases in the Map Color Theorem, either the genus of $K_{n+2} - K_2$ must be the same as for $K_{n+2}$, or else embeddings of $K_{n+2}$ are found by first embedding $K_{n+2} - K_2$ and then performing some kind of manipulation. Therefore, much was known about the genus of $K_{n+2} - K_2 = \overline{K_2} + K_n$ as a result of proving the Map Color Theorem. The orientable genus of $K_{n+2} - K_2$ was determined [**22**, p. 180], except for the case $n + 2 \equiv 2 \bmod 24$, which was settled by Jungerman [**11**]. The nonorientable genus of $K_{n+2} - K_2$ was also determined [**16**, **20**, **22**] (see especially Satz 11 and the remark on p. 200 of [**20**]), except for the case $n+2 \equiv 8 \bmod 12$, $n + 2 \geq 20$, which was settled by Korzhik [**14**]. Conjecture 3.2 holds for $\overline{K_2} + K_n = K_{n+2} - K_2$ in all cases except for $\widetilde{g}(\overline{K_2} + K_6 = K_8 - K_2)$.

The orientable genus of certain graphs $\overline{K_3} + K_n$ and $\overline{K_5} + K_n$ also played a role in the proof of the Map Color Theorem. In addition, the orientable genus of various graphs $\overline{K_m} + K_n$ with $m = 3$, 4, 5 and 6 were investigated during the 1970's and early 1980's by authors including Guy, Jungerman, Ringel and Youngs. In general they investigated cases where a triangular embedding is to be expected, using current graph techniques. We omit a detailed summary of the results supporting Conjecture 3.2, but refer the reader to [**8**, **9**, **10**, **11**, **12**, **22**]. Some orientable counterexamples to Conjecture 3.2 were found by Jungerman [**10**] using a computer search, namely $\overline{K_3} + K_6 = K_9 - K_3$, $\overline{K_5} + K_6 = K_{11} - K_5$, and $\overline{K_6} + K_7 = K_{13} - K_6$. Perhaps it is overly optimistic in Conjecture 3.2 to expect there to be only a finite number of exceptions altogether. However, we believe strongly that there will be only a finite number of exceptions for each given $m$.

For values of $m$ that are small in the sense that $m \leq n-1$, but not bounded by a fixed number, Korzhik has conducted extensive investigations which are summarized in [**15**]. He obtains triangular embeddings for $\overline{K_m} + K_n$ in many situations, both orientable and nonorientable, again using current graph techniques. For all of these results, $m < n/2$.

Much less has been done for the case of large $m$, when $m \geq n - 1$. Craft [**3**, Theorem 5.3; **4**, Theorem 1] has verified the conjectured orientable genus of $\overline{K_m} + K_n$ when $n$ is even and $m \geq 2n - 4$. His method involves a doubling construction starting from a 'graphical surface' constructed of 'tubes' and 'spheres', and a surgical construction called 'crowning' which accomplishes some of the same ends as our diamond sum, below.

In the remainder of this section we show that the conjectured formula for nonorientable genus of $\overline{K_m} + K_n$ is correct for $m \geq n - 1$, except when $(m, n) = (4, 5)$. Our proof is by reduction to a basis case. The basis uses the embeddings of $K_n$ with hamilton cycle faces that we found in Section 2. The reduction uses a construction we call the 'diamond sum', which was also the reduction technique used to determine the nonorientable genus of all complete tripartite graphs [**5**, **6**, **13**].

**The Diamond Sum.** Our reduction procedure was introduced in a different form by Bouchet [**2**], who used it to obtain a new inductive proof for the genus of complete bipartite graphs. Magajna, Mohar and Pisanski reinterpreted Bouchet's construction in the context of quadrangular embeddings [**17**], and more details were given by Mohar, Parsons, and Pisanski [**18**]. The general version here is due to Kawarabayashi, Stephens and Zha [**13**].

Suppose $\Psi_1$ is an embedding of a simple graph $G_1$ on surface $\Sigma_1$ and $\Psi_2$ is an embedding of a simple graph $G_2$ on $\Sigma_2$. Let $u$ be a vertex of degree $k \geq 1$ in $G_1$, with neighbors $u_0, u_1, \ldots, u_{k-1}$ in cyclic order around $u$. Suppose there is a vertex $v$ of degree $k$ in $G_2$, with neighbors $v_0, v_1, \ldots, v_{k-1}$ in cyclic order around $v$. We can find a closed disk $D_1$ that intersects $G_1$ in $u$ and the edges $uu_0$, $uu_1$, …, $uu_{k-1}$, with $\partial D_1 \cap G_1 = \{u_0, u_1, \ldots, u_{k-1}\}$. Similarly, we can find a closed disk $D_2$ that intersects $G_2$ in $v$ and the edges $vv_0$, $vv_1$, …, $vv_{k-1}$, with $\partial D_2 \cap G_2 = \{v_0, v_1, \ldots, v_{k-1}\}$. Remove the interiors of $D_1$ and $D_2$, and identify $\partial D_1$ with $\partial D_2$ so that $u_i$ is identified with $v_i$ for $0 \leq i \leq k-1$. The result is called a *diamond sum* $\Psi_1 \diamondsuit \Psi_2$, and it embeds a graph we denote by $G_1 \diamondsuit G_2$ on the surface $\Sigma_1 \# \Sigma_2$, where $\#$ denotes the connected sum of two surfaces. Note that $\Psi_1 \diamondsuit \Psi_2$ will be nonorientable if either (or both) of $\Psi_1$ or $\Psi_2$ is nonorientable.

The diamond sum is not unique; it depends on $u$ and $v$, and how we match the neighbors of $u$ to the neighbors of $v$. When $k = 1$ or 2, this does not determine the direction in which we identify $\partial D_1$ with $\partial D_2$, and we also need to choose that direction. However, when we apply the diamond sum we will have $k \geq 3$, and every permutation of the neighbors of $u$ will be an automorphism of $G_1$, so given $u$ and $v$ the graph $G_1 \diamondsuit G_2$ will be unique up to isomorphism.

In particular, we shall let $n \geq 3$, and take $G_1$ to be $\overline{K_p} + K_n$ with $u$ one of the vertices of the $\overline{K_p}$, and $G_2$ to be $K_{q,n} = \overline{K_q} + \overline{K_n}$ with $v$ one of the vertices of the $\overline{K_q}$. Then the graph $(\overline{K_p} + K_n) \diamondsuit K_{q,n}$ is $\overline{K_{p+q-2}} + K_n$.

**Theorem 3.3.** *Suppose $m \geq 0$, $n \geq 0$, and $m \geq n-1$. The nonorientable genus of $\overline{K_m} + K_n$ is given by*

$$\widetilde{g}(\overline{K_m} + K_n) = \begin{cases} 0 & \text{if } n \leq 2, \\ \lceil \frac{(m-2)(n-2)}{2} \rceil & \text{if } n \geq 3 \text{ and } (m,n) \neq (4,5), \\ 4 & \text{if } (m,n) = (4,5). \end{cases}$$

**Proof.** If $n \leq 2$, $\overline{K_m} + K_n$ is planar and the formula holds. In the other cases, $\lceil \frac{(m-2)(n-2)}{2} \rceil$ is a lower bound on the genus, by Theorem 3.1, so except when $(m,n) = (4,5)$ we need only show that an embedding with that genus exists.

Suppose that $n \geq 3$ and $n \neq 5$. If $n \neq 3$ then by Theorem 2.1 there is an embedding of $K_n$ on $N_{(n-2)(n-3)/2}$ with all faces hamilton cycles. This is also true for $n = 3$, recalling that $N_0$ is the sphere. By adding a vertex in the interior of the $n-1$ hamilton cycle faces, and joining the new vertices to all original vertices, we obtain an embedding of $\overline{K_{n-1}} + K_n$ on $N_{(n-2)(n-3)/2}$. By Ringel [**21**] there is an embedding of $K_{m-n+3,n}$ on $N_{\lceil (m-n+1)(n-2)/2 \rceil}$. Applying the diamond sum operation as described above, we obtain an embedding of $(\overline{K_{n-1}} + K_n) \diamondsuit K_{m-n+3,n} = \overline{K_m} + K_n$ on $N_{(n-2)(n-3)/2} \# N_{\lceil (m-n+1)(n-2)/2 \rceil} = N_{\lceil (m-2)(n-2)/2 \rceil}$, as required.

Suppose that $(m,n) = (4,5)$. By Theorem 3.1, if $\overline{K_4} + K_5$ embeds on $N_k$ then $k \geq 3$. Suppose that $k = 3$. Then this is a minimum Euler genus embedding, so by Youngs [**26**] it is a cellular embedding and

Euler's formula applies. Thus, $f = 2+e-v-\gamma = 2+30+9-3 = 20$. However, $2e = 60 = 3f_3+4f_4+5f_5+\ldots = 3f+(f_4+2f_5+\ldots) = 60+(f_4+2f_5+\ldots)$, so $0 = f_4 = f_5 = \ldots$ and all faces are triangles. Thus, removing each vertex of the $\overline{K_4}$ leaves a face that is a hamilton cycle of the $K_5$. Therefore, we obtain an embedding of $K_5$ with 4 hamilton cycle faces, which does not exist by Theorem 2. Hence, $k \neq 3$. We observe that $\overline{K_4}+K_5 \subseteq \overline{K_4}+K_6$ and from above $\overline{K_4}+K_6$ embeds on $N_4$, so $\widetilde{g}(\overline{K_4}+K_5) = 4$.

Suppose that $(m,n) = (5,5)$. We must show that $\overline{K_5}+K_5$ embeds on $N_5$. Such an embedding is shown in Figure 3.1, where we represent $N_5$ as a torus with three added crosscaps, shown as dotted circles with an 'X' in the center.

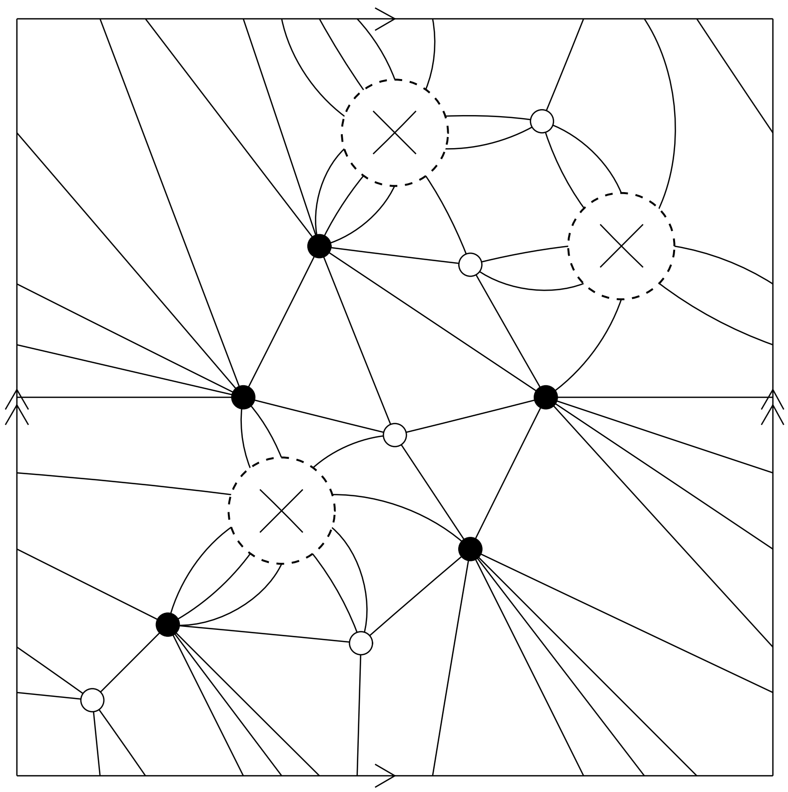


Figure 3.1: $\overline{K_5}+K_5$ on $N_5$

Suppose that $n = 5$ and $m \geq 6$. Observe first that $\overline{K_6}+K_5 \subseteq \overline{K_5}+K_6$ and from above $\overline{K_5}+K_6$ embeds on $N_6$. Thus, $\overline{K_6}+K_5$ embeds on $N_6$. Now by Ringel [**21**] there is an embedding of $K_{m-4,5}$ on $N_{\lceil 3(m-6)/2\rceil}$. Applying the diamond sum operation we obtain an embedding of $(\overline{K_6}+K_5)\diamondsuit K_{m-4,5} = \overline{K_m}+K_5$ on $N_6 \# N_{\lceil 3(m-6)/2\rceil} = N_{\lceil 3(m-2)/2\rceil}$, as required. ∎

## 4. The nonorientable genus of $\overline{K_m}+G$

As a consequence of the results in the previous section, we can also determine the nonorientable genus of a very general family of graphs, where we join an edgeless graph $\overline{K_m}$ to a graph $G$ on at most $m+1$

vertices.

**Theorem 4.1.** *Suppose that $m \geq 0$, and $G$ is a graph on at most $m+1$ vertices. Then the nonorientable genus of $\overline{K_m} + G$ is*

$$\widetilde{g}(\overline{K_m} + G) = \begin{cases} 0 & \text{if } |V(G)| \leq 2, \\ 4 & \text{if } m = 4,\ |V(G)| = 5, \text{ and } K_{1,4} \subseteq G \text{ or } K_4 \subseteq G, \\ \lceil \frac{(m-2)(|V(G)|-2)}{2} \rceil & \text{otherwise.} \end{cases}$$

**Proof.** Let $n = |V(G)|$. If $n \leq 2$, $\overline{K_m} + G$ is planar and the result holds, so suppose that $n \geq 3$. Now $K_{m,n} \subseteq \overline{K_m} + G \subseteq \overline{K_m} + K_n$. If $(m,n) \neq (4,5)$, then by Ringel [**21**] and by Theorem 3.3, both $K_{m,n}$ and $\overline{K_m} + K_n$ have nonorientable genus $\lceil \frac{(m-2)(n-2)}{2} \rceil$, and therefore that is also the nonorientable genus of $\overline{K_m} + G$.

Suppose therefore that $(m,n) = (4,5)$. We have $K_{4,5} \subseteq \overline{K_4} + G \subseteq \overline{K_4} + K_5$, where $\widetilde{g}(K_{4,5}) = 3$ by Ringel [**21**] and $\widetilde{g}(\overline{K_4} + K_5) = 4$ by Theorem 3.3. Therefore $\widetilde{g}(\overline{K_4} + G)$ is either 3 or 4.

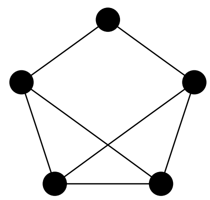
From [**5**] $\overline{K_4} + K_{1,4} = K_{4,4,1}$ does not embed on $N_3$, so if $K_{1,4} \subseteq G$ then $\widetilde{g}(\overline{K_4} + G) = 4$, as required.

Suppose $K_4 \subseteq G$, and $\widetilde{g}(\overline{K_4} + G) = 3$. Then $\overline{K_4} + (K_4 \cup K_1)$, which is a subgraph of $\overline{K_4} + G$, has an embedding on $N_3$. This is a minimum Euler genus embedding, and hence Euler's formula applies. We have $f = 2 + e - v - \gamma = 2 + 26 - 9 - 3 = 16$. We also have $2e = 52 = 3f_3 + 4f_4 + 5f_5 + \ldots = 3f + (f_4 + 2f_5 + \ldots) = 48 + (f_4 + 2f_5 + \ldots)$. Therefore, $f_4 + 2f_5 + \ldots = 4$. However, $f_3 \leq 12$ because every triangular face must use one of the 6 edges of the $K_4$. Hence $f_3 = 12$, $f_4 = 4$, every triangular face uses exactly one edge of the $K_4$ and one vertex of the $\overline{K_4}$, and every edge of the $K_4$ appears in two triangular faces. Since the graph is simple with no vertices of degree 1, all the faces of length 4 are 4-cycles. Each vertex of the $K_4$ has degree 7 and is incident with 6 triangular faces; therefore it is incident with one 4-cycle face. The $K_1$ vertex has degree 4 and does not belong to any 3-cycles; therefore it belongs to all four 4-cycle faces. By adding one edge in each 4-cycle face joining the $K_1$ vertex to the $K_4$ vertex, we obtain an embedding of $\overline{K_4} + K_5$ on $N_3$, which contradicts Theorem 3.3. Thus, if $K_4 \subseteq G$ then $\widetilde{g}(\overline{K_4} + G) = 4$, as required.

Finally, suppose that $G$ contains neither $K_{1,4}$ nor $K_4$. Since $K_{1,4} \not\subseteq G$, $\overline{G}$ contains at least one edge incident with every vertex. Moreover, since $G \neq K_4 \cup K_1$, $\overline{G}$ contains two independent edges $e_1$ and $e_2$. The vertex that is not a vertex of $e_1$ or $e_2$ must be a vertex of $e_3 \in E(\overline{G})$, and $\{e_1, e_2, e_3\}$ forms a subgraph of $\overline{G}$ isomorphic to $P_3 \cup K_2$. Thus, we may assume that $G \subseteq \overline{P_3 \cup K_2}$. Figure 4.1 shows an embedding of $\overline{K_4} + \overline{P_3 \cup K_2}$ on $N_3$, represented as a torus with an added crosscap. Thus, $\widetilde{g}(\overline{K_4} + G) = 3$, as the formula requires. ∎

We can also use the diamond sum operation to obtain the nonorientable genus of some graphs that are not covered by Theorem 4.1. We employ the following upper bound.

**Lemma 4.2.** *Suppose that $H$ is an $m$-vertex graph, where $m \geq 0$, and suppose that $G_1, G_2, \ldots, G_k$ are*



$\overline{P_3 \cup K_2}$

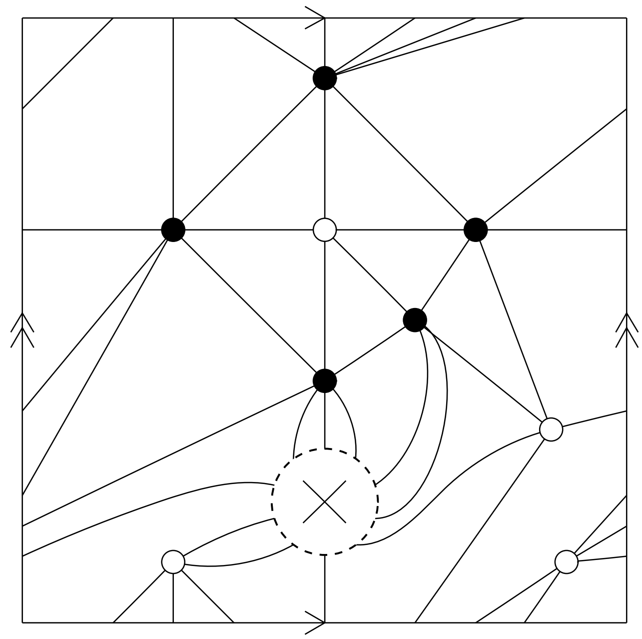


Figure 4.1: $\overline{K_4} + \overline{P_3 \cup K_2}$ on $N_3$

*disjoint graphs, where $k \geq 1$. Then*

$$g(H + (G_1 \cup G_2 \cup \ldots \cup G_k)) \leq g(H + \overline{K_k}) + \sum_{i=1}^{k} g(\overline{K_m} + (K_1 \cup G)), \qquad \textit{and}$$

$$\widetilde{g}(H + (G_1 \cup G_2 \cup \ldots \cup G_k)) \leq \widetilde{g}(H + \overline{K_k}) + \sum_{i=1}^{k} \widetilde{g}(\overline{K_m} + (K_1 \cup G)).$$

**Proof.** Take minimum genus embeddings (orientable or nonorientable, as appropriate) of $H + \overline{K_k}$ and of each $\overline{K_m} + (K_1 \cup G_i)$. Let the vertices of the $\overline{K_k}$ be $v_1, v_2, \ldots, v_k$. Do $k$ simultaneous diamond sums, identifying the neighbors of $v_i$ with the neighbors of the $K_1$ in $\overline{K_m} + (K_1 \cup G_i)$, for each $i$. The resulting graph is $H + (G_1 \cup G_2 \cup \ldots \cup G_k)$, and it has an embedding of the given genus. ∎

The following corollary provides an example of when this can be used to calculate the nonorientable genus exactly.

**Corollary 4.3.** *Suppose that $m \geq 2$, and $G$ is a graph with $k \geq 1$ components. Suppose further that*

(i) *each component of $G$ has at most $m$ vertices;*

(ii) *if $m = 4$, no component of $G$ is isomorphic to $K_4$;*

(iii) *either $m$ is even, or else $k$ is even and each component has an odd number of vertices.*

*Then*

$$\widetilde{g}(\overline{K_m} + G) = \frac{(m-2)(|V(G)| - 2)}{2}.$$

**Proof.** Apply Lemma 4.2 with $H = \overline{K_m}$. Since $\overline{K_m} + \overline{K_k} = K_{m,k}$, by Ringel [**21**] and (iii) we have $\widetilde{g}(\overline{K_m} + \overline{K_k}) = \frac{1}{2}(m-2)(k-2)$. Suppose the components of $G$ are $G_1, G_2, \ldots, G_k$, where each $G_i$ has $n_i$ vertices. By (i), Theorem 4.1 applies to $\overline{K_m} + (K_1 \cup G_i)$, and by (ii) and (iii) this gives $\widetilde{g}(\overline{K_m} + (K_1 \cup G_i)) = \frac{1}{2}(m-2)(n_i-1)$. Thus, Lemma 4.2 yields $\widetilde{g}(\overline{K_m}+G) \leq \frac{1}{2}(m-2)(n_1+n_2+\ldots+n_k-2) = \frac{1}{2}(m-2)(|V(G)|-2)$. However, since $\overline{K_m} + G$ contains $K_{m,|V(G)|}$, this is also a lower bound on $\widetilde{g}(\overline{K_m} + G)$, and the result follows. ∎

Corollary 4.3 may apply when Theorem 4.1 does not. We illustrate with a concrete example. By Corollary 4.3 we have $\widetilde{g}(\overline{K_4} + 5K_3) = 13$. Theorem 4.1 does not apply here, because the edgeless graph $\overline{K_4}$ is too small relative to the other side of the join, $5K_3$.

We note that Craft [**3**, Theorem 5.6] showed that the orientable genus of $\overline{K_m}+G$ is $\lceil (m-2)(|V(G)|-2)/4 \rceil$ provided every component of $G$ has even order and order at most $m/2$. In fact, although Craft's techniques are different from ours, his results suggested to us that it would be profitable to look at the case where $G$ is disconnected. Craft also has several other interesting results on the orientable genus of joins.

## 5. Problems with Wei and Liu's paper [24]

Wei and Liu [**24**] claim to prove that $K_n$ has an embedding on the surface $N_{(n-2)(n-3)/2}$ with all faces hamilton cycles of $K_n$. However, as we have stated earlier, there are problems with Wei and Liu's paper, and we describe them in this section.

When $n$ is even, Wei and Liu give a correct construction, essentially the same as we give in 2.1.1. They correctly show that the embedding is nonorientable. However, they fail to show that their collection of facial cycles actually represents an embedding. If $n = 2k + 2$, Wei and Liu construct $n - 1$ paths by letting

$$P_1 = [0, 2k, 1, 2k-1, 2, 2k-2, \ldots, k+1, k-1, k]$$

and for $i$, $2 \leq i \leq n-1$, they obtain $P_i$ by adding $i - 1$ modulo $n - 1 = 2k + 1$ to each vertex of $P_1$. Then for each $i$ a cycle $C_i$ is formed by joining both ends of $P_i$ to the vertex $n - 1$. In their proof that this construction gives an embedding (their Lemma 2.4, Case 1), they state 'By symmetry, we are allowed to consider only the edges incident with the vertex $n - 1$.' But their vertex $n - 1$ corresponds to our vertex $\infty$, which is not obviously similar to the other vertices in the construction. Wei and Liu therefore fail to prove that the rotation graph around vertices $0, 1, 2, \ldots, n-2$ is a cycle.

For the case where $n$ is odd, Wei and Liu's construction does not give an embedding. We quote their construction, with explanations added in brackets [ ]:

> 'Let us start from the $n - 1$ circuits of $K_{n-1}$ [i.e., $C_1, C_2, \ldots, C_{n-2}$ as constructed above] to get $n - 1$ circuits which form an SCDC [small cycle double cover] of $K_n$. In $C_1$ (the first row), put the vertex $n-1$ between the vertices $n-2$ and 0. The new circuit is denoted by $C_1'$. In $C_i$ ($2 \leq i \leq n-2$), put the vertex

$n-1$ between the vertices $L(i,1)$ and $L(i,2)$ [the first two vertices in $P_i$] where $L(i,2)-L(i,1)=-1$. The new circuit is denoted by $C'_i$. The $(n-1)$-th circuit $C'_{n-1}$ is $(n-1, n-2, 0, 1, \ldots, n-3)$.'

For example, for $n = 9$ this gives the following 8 cycles:

$$\begin{array}{ll} C'_1 = (0,6,1,5,2,4,3,7,8), & C'_5 = (4,8,3,5,2,6,1,0,7), \\ C'_2 = (1,8,0,2,6,3,5,4,7), & C'_6 = (5,8,4,6,3,0,2,1,7), \\ C'_3 = (2,8,1,3,0,4,6,5,7), & C'_7 = (6,8,5,0,4,1,3,2,7), \\ C'_4 = (3,8,2,4,1,5,0,6,7), & C'_8 = (8,7,0,1,2,3,4,5,6). \end{array}$$

The rotation graph at vertex 0 then consists of two cycles, $(8,2,3,4,5,6)$ and $(1,7)$, not a single cycle. Therefore, this does not describe an embedding. In general, the rotation graph at vertex 0 consists of cycles $(n-1, 2, 3, \ldots, n-3)$ and $(1, n-2)$, and we do not have an embedding.

Wei and Liu have another construction for odd $n$, but they claim only that it is a cycle double cover of $K_n$, not that it is an embedding, and in fact it also does not give an embedding.

## 6. Conclusion

We observe that arguments from this paper may be used to replace some of the more technical arguments in our paper with Zha on the nonorientable genus of complete tripartite graphs. In particular, embeddings of $K_{m,m,1}$ on $N_{(m-1)(m-2)/2}$ when $m$ is even are treated as special cases in [**6**], and the argument for $m \equiv 0$ mod 4 is particularly complicated. By regarding $K_{m,m,1}$ as $\overline{K_m} + K_{1,m}$ we obtain the required embeddings from Theorem 4.1. In more generality, Theorem 4.1 gives an alternate proof of the value of $\widetilde{g}(K_{l,m,n})$ when $l \geq m+n-1$.

In future work we would naturally like to determine the orientable genus of $\overline{K_m} + K_n$ with $m \geq n-1$, which is approachable using the diamond sum technique. We have some partial results. In particular, Conjecture 3.2 holds for $g(\overline{K_m} + K_n)$ if $n$ is even, $n = 4$, $n = 8$ or $n \geq 12$, and $m \geq n$.

We would also like to determine the orientable genus of $K_{l,m,n}$, with $l \geq m \geq n$, which White [**25**] conjectured to be $\lceil (l-2)(m+n-2)/4 \rceil$. Again we can use the diamond sum technique. In collaboration with Zha, we have some partial results confirming White's conjecture. The proof depends on the values of $m$ and $n$ modulo 4. Of the sixteen cases, we can determine the genus completely for seven: where $(m,n)$ modulo 4 is $(0,0)$, $(0,2)$, $(2,0)$, $(2,1)$, $(2,2)$, $(2,3)$ and $(4,4)$. We have results for all but very small $n$ in five other cases: $(1,1)$ for $n \geq 5$, $(0,3)$ and $(1,3)$ for $n \geq 7$, and $(0,1)$ and $(3,1)$ for $n \geq 9$.

The genus problems for both complete tripartite graphs and for $\overline{K_m} + K_n$ with $m \geq n-1$ can be thought of as special cases of a more general problem. Consider a genus embedding (orientable or nonorientable) of $K_{m,n}$. This will be a quadrangulation (all faces 4-cycles), or close to a quadrangulation. For simplicity, consider only the cases that are quadrangulations (when $(m-2)(n-2)$ is even for nonorientable embeddings, or divisible by 4 for orientable embeddings). A minimum genus embedding of $K_{m,n} = \overline{K_m} + \overline{K_n}$ then has $mn/2$ quadrangular faces, exactly $m$ of which are incident with each vertex of the $\overline{K_n}$. We may ask the following question.

**Question 6.1.** *Given an $n$-vertex graph $G$ with maximum degree at most $m$, when is the genus (orientable or nonorientable) of $\overline{K_m} + G$ the same as for $K_{m,n}$? In other words, when can we minimally embed $K_{m,n} = \overline{K_m} + \overline{K_n}$ so that the edges of $G$ can be added between the vertices of the $\overline{K_n}$, without increasing the genus?*

Even more generally, we may ask the following.

**Question 6.2.** *When it is possible to take an $m$-vertex graph $F$ of maximum degree at most $n$, and an $n$-vertex graph $G$ of maximum degree at most $m$, such that $|E(F)| + |E(G)| \leq mn/2$, and find a minimum genus embedding of $K_{m,n}$ that can be extended to an embedding of $F + G$ without increasing the genus?*

Another area for future research is the genus (orientable or nonorientable) of $\overline{K_m} + K_n$ with $m < n-1$. As previously mentioned, Korzhik [**15**] has some very general results here, but much still remains to be done. In particular, we know of no results for $n/2 \leq m < n-1$.


### Acknowledgement

The authors would like to thank Xiaoya Zha for many useful discussions.


### References


[**1**] C. P. Bonnington and C. H. C. Little, "The Foundations of Topological Graph Theory", Springer, New York, 1995.

[**2**] A. Bouchet, Orientable and nonorientable genus of the complete bipartite graph, *J. Combin. Theory Ser. B* **24** (1978) 24-33.

[**3**] David L. Craft, "Surgical techniques for constructing minimal orientable imbeddings of joins and compositions of graphs", Ph.D. thesis, Western Michigan University, 1991.

[**4**] David L. Craft, On the genus of joins and compositions of graphs, *Discrete Math.* **178** (1998) 25-50.

[**5**] M. N. Ellingham, Chris Stephens and Xiaoya Zha, Counterexamples to the nonorientable genus conjecture for complete tripartite graphs, *Eur. J. Combin.*, to appear.

[**6**] M. N. Ellingham, Chris Stephens and Xiaoya Zha, The nonorientable genus of complete tripartite graphs, submitted.

[**7**] J. L. Gross and T. W. Tucker, "Topological Graph Theory", Wiley Interscience, New York, 1987)

[**8**] Richard K. Guy and Gerhard Ringel, Triangular imbedding of $K_n - K_6$, *J. Combin. Theory Ser. B* **21** (1976) 140-145.

[**9**] Richard K. Guy and J. W. T. Youngs, A smooth and unified proof of cases 6, 5 and 3 of the Ringel-Youngs Theorem, *J. Combin. Theory Ser. B* **15** (1973) 1-11.

[**10**] M. Jungerman, Orientable triangular embeddings of $K_{18} - K_3$ and $K_{13} - K_3$, *J. Combin. Theory Ser. B* **16** (1974) 293-294.

[**11**] M. Jungerman, The genus of $K_n - K_2$, *J. Combin. Theory Ser. B* **18** (1975) 53-58.

[**12**] M. Jungerman and G. Ringel, Minimal triangulations on orientable surfaces, *Acta Math.* **145** (1980) 121-154.

[**13**] Ken-ichi Kawarabayashi, Chris Stephens and Xiaoya Zha, Orientable and nonorientable genera of some complete tripartite graphs, *SIAM J. Discrete Math.*, to appear.

[**14**] Vladimir P. Korzhik, A nonorientable triangular embedding of $K_n - K_2$, $n \equiv 8 \pmod{12}$, *Discrete Math.* **141** (1995) 195-211.

[**15**] Vladimir P. Korzhik, Triangular embeddings of $K_n - K_m$ with unboundedly large $m$, *Discrete Math.* **190** (1998) 149-162.

[**16**] Vladimir P. Korzhik, Another proof of the Map Color Theorem for nonorientable surfaces, *J. Combin. Theory Ser. B* **86** (2002) 221-253.

[**17**] Z. Magajna, B. Mohar and T. Pisanski, Minimal ordered triangulations of surfaces, *J. Graph Theory* **10** (1986) 451-460.

[**18**] B. Mohar, T. D. Parsons and T. Pisanski, The genus of nearly complete bipartite graphs, *Ars Combin.* **20B** (1985) 173-183.

[**19**] Bojan Mohar and Carsten Thomassen, "Graphs on Surfaces", Johns Hopkins University Press, Baltimore, 2001.

[**20**] G. Ringel, Bestimmung der Maximalzahl der Nachbargebiete auf nichtorientierbaren Flächen, *Math. Ann.* **127** (1954) 181-214.

[**21**] G. Ringel, Der vollständige paare Graph auf nichtorientierbaren Flächen, *J. Reine Angew. Math.* **220** (1965) 88-93.

[**22**] G. Ringel, "Map Color Theorem", Springer-Verlag, Berlin, 1974.

[**23**] Martin Škoviera and Jozef Širáň, Relative embeddings of graphs, in "Graph theory", (Dubrovnik, 1985), edited by R. Tošić, D. Acketa and V. Petrović, Univ. Novi Sad, Novi Sad (1986) 211-217.

[**24**] Erling Wei and Yanpei Liu, Determination of regular embeddings of graphs on surfaces, *Util. Math.* **59** (2001) 237-251.

[**25**] A. T. White, "The genus of cartesian products of graphs", Ph.D. thesis, Michigan State University, 1969.

[**26**] J. W. T. Youngs, Minimal embeddings and the genus of a graph, *J. Math. Mech.* **12** (1963) 303-315.