One-way infinite 2-walks in planar graphs

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August 27, 2015

Abstract

We prove that every 3-connected 2-indivisible infinite planar graph has a 1-way infinite 2-walk. (A graph is 2-*indivisible* if deleting finitely many vertices leaves at most one infinite component, and a 2-*walk* is a spanning walk using every vertex at most twice.) This improves a result of Timar, which assumed local finiteness. Our proofs use Tutte subgraphs, and allow us to also provide other results when the graph is bipartite or an infinite analog of a triangulation: then the prism over the graph has a spanning 1-way infinite path.

1 Introduction

For terms not defined in this paper, see [20]. All graphs are simple (having no loops or multiple edges) and may be infinite, unless we explicitly state otherwise.

A *cutset* in a graph G is a set $S \subseteq V(G)$ such that $G - S$ is disconnected. A k-*cut* is a cutset S with $|S| = k$. A graph is k-*connected* if it has at least $k+1$ vertices and no cutset S with $|S| < k$. The *connectivity* of a graph is the smallest k for which it is k-connected.

The first major result on the existence of hamilton cycles in graphs embedded in surfaces was by Whitney [21] in 1931, who proved that every 4-connected finite planar triangulation is hamiltonian. Tutte extended this to all 4-connected finite planar graphs in 1956 [18], and gave another proof in 1977 [19]. Tutte actually proved a more general result, using subgraphs which have since been called "Tutte subgraphs" (defined in Section 3).

To extend these results to infinite graphs, one can look for infinite spanning paths. We say that $v_1 v_2 v_3 \cdots$ is a 1-*way infinite path*, and $\cdots v_{-2} v_{-1} v_0 v_1 v_2 \cdots$ is a 2-*way infinite path*, if each v_i is a distinct vertex and consecutive vertices are adjacent.

If deleting finitely many vertices in an infinite graph leaves more than one (two) infinite component(s), then the graph has no 1-way (2-way) infinite spanning path. Nash-Williams

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[12] defined a graph G to be *k-indivisible*, for a positive integer k, if, for any finite $S \subseteq V(G)$, $G - S$ has at most $k-1$ infinite components. He conjectured [12, 13] that every 4-connected 2-indivisible (3-indivisible) infinite planar graph contains a 1-way (2-way) infinite spanning path. The 1-way infinite path conjecture was proved by Dean, Thomas, and Yu [6] in 1996, and Xingxing Yu established the 2-way infinite path conjecture [22, 23, 24, 25, 26].

For connectivity less than 4, we must be more flexible in the types of spanning subgraphs we wish to find, since there are 3-connected finite planar graphs with no hamilton path. Let k be a positive integer. A *k-tree* is a spanning tree with maximum degree at most k. A *k-walk* in a finite graph is a closed spanning walk passing through each vertex at most k times. In a finite graph, a 2-tree is a hamilton path and a 1-walk is a hamilton cycle.

Barnette [1] showed that every 3-connected finite planar graph contains a 3-tree. Gao and Richter [7] later showed that every 3-connected finite planar graph contains a 2-walk. Gao, Richter, and Yu [8, 9] refined this to give information about the locations of vertices included twice in the 2-walk. The 2-walk results strengthen Barnette's result, because Jackson and Wormald [10] showed that in a finite graph a k-walk provides a $(k+1)$-tree.

It is natural to ask if these results generalize to infinite graphs. In 1996, Jung [11] proved that every locally finite 3-connected infinite plane graph with no vertex accumulation points has a 3-tree. A graph is *locally finite* if every vertex has finite degree.

A *1-way (2-way) infinite walk* is a sequence of vertices $v_1 v_2 v_3 \cdots$ $(\cdots v_{-2} v_{-1} v_0 v_1 v_2 \cdots)$ where consecutive vertices are adjacent. A *1-way (2-way) infinite k-walk* is a 1-way (2-way) infinite spanning walk that passes through each vertex at most k times. A graph with a 1-way or 2-way infinite k-walk must be infinite.

In his doctoral dissertation, Timar proved the following two theorems.

Theorem 1.1 (Timar [17, Theorem II.2.5])**.** *Let G be a locally finite 3-connected 2-indivisible infinite planar graph. Then G has a 1-way infinite 2-walk.*

Theorem 1.2 (Timar [17, Theorems II.3.3, II.4.3])**.** *Let G be a locally finite 3-connected 3-indivisible infinite planar graph. Then G has a 2-way infinite 2-walk.*

Our main result extends Theorem 1.1 by not requiring local finiteness and by controlling the location of vertices used twice.

Theorem 1.3. *Let G be a 3-connected 2-indivisible infinite planar graph. Then G has a 1-way infinite 2-walk for which any vertex used twice is in a 3-cut of G.*

Our methods differ from those of Timar. To avoid local finiteness we use structural results, similar to those in [6], for graphs with infinite degree vertices. To build the skeleton of the 2-walk and control the location of vertices that are used twice, we use Tutte subgraphs in a way similar to [8]. We follow a systematic approach to using Tutte subgraphs that we have developed in a survey paper [4], in preparation. Our methods also provide other results when G is bipartite or an infinite analog of a triangulation (Theorems 6.3 and 6.6).

Timar [17, Lemma I.2.14] verified that Jackson and Wormald's proof that a k-walk provides a $(k+1)$-tree applies for infinite graphs, and so Theorem 1.3 allows us to prove a result similar to Jung's. We drop local finiteness, but add 2-indivisibility as both a hypothesis and a conclusion.

Corollary 1.4. *Let G be a 3-connected 2-indivisible infinite planar graph. Then G has a 2-indivisible 3-tree.*

We also make the following natural conjecture.

Conjecture 1.5. *Let G be a 3-connected 3-indivisible infinite planar graph. Then G has a 2-way infinite 2-walk.*

Proving Conjecture 1.5 would likely require significant work along the lines of [22, 23, 24, 25, 26].

Section 2 includes some additional definitions and lemmas, especially for connectivity. Section 3 addresses Tutte subgraph techniques. Section 4 discusses structural results for 3-connected 2-indivisible infinite planar graphs. Section 5 proves the main result, and Section 6 gives stronger theorems for bipartite graphs and analogs of triangulations.

2 Definitions and Connectivity

If G is a connected finite plane graph, X_G denotes the *outer walk* of G, the closed walk bounding the infinite face. If G is 2-connected, then we also call X_G the *outer cycle* of G. We use X_G to denote both a walk and a subgraph; if G is isomorphic to K_2 or K_1, then the subgraph X_G is just G itself.

A *uv-path* P is a path from u to v; P^{-1} denotes the reverse vu-path. If P is a (possibly infinite) path and $x, y \in V(P)$, then $P[x, y]$ denotes the subpath of P from x to y. Given a closed walk W in a plane graph bounding an open disk (such as a cycle or facial boundary walk), the subwalk $W[x, y]$ clockwise from x to y is well-defined provided each of x and y occurs exactly once on W. If $x = y$, then $W[x, y]$ or $P[x, y]$ means the single vertex $x = y$.

A *block* is a 2-connected graph or a graph isomorphic to K_2 or K_1. A *block of a graph* is a maximal subgraph that is a block. Every graph has a unique decomposition into edge-disjoint blocks. A block isomorphic to K_2 is a *trivial* block. A vertex v of a graph is a *cutvertex* if $\{v\}$ is a cutset of the graph.

Let G be a connected graph, and $n \geq 0$ an integer. Suppose that G has finite blocks $B_1, B_2, \ldots, B_n$ and vertices $b_0, b_1, b_2, \ldots, b_{n-1}, b_n$ in G such that $G = b_0$ (if $n = 0$) or $b_0 \in V(B_1) - \{b_1\}$, $b_n \in V(B_n) - \{b_{n-1}\}$, $b_i \in V(B_i) \cap V(B_{i+1})$ for $i = 1, 2, \ldots, n-1$, and $G = \bigcup_{i=1}^{n} B_i$. We say that G is a *chain of blocks* (some sources call this a *linear graph*) and that $(b_0, B_1, b_1, B_2, b_2, \ldots, b_{n-1}, B_n, b_n)$ is a *block-decomposition* of G. In this case, $b_1, b_2, \ldots, b_{n-1}$ are precisely the cutvertices of G. A chain of blocks is a *plane chain of blocks* if it is embedded in the plane so that no block is embedded inside any other block. In a plane chain of blocks G, any internal face in a block will be a face of G.

If G has finite blocks $B_1, B_2, \ldots$ and vertices $b_1, b_2, \ldots$ such that $b_i \in V(B_i) \cap V(B_{i+1})$ for every i, $G = \bigcup_{i=1}^{\infty} B_i$, and G is embedded in the plane so that no block is embedded inside any other block, we say that G is a 1-*way infinite plane chain of finite blocks.* (We will have no need to specify an initial vertex b_0.) In this case, we define X_G to be the 2-way infinite walk traversing the outer face of G in the clockwise direction.

If P is an ab-path in G, then the *chain of blocks along P in G* is the minimal union K of blocks of G that contains P (or $K = a$ if $a = b$). If $K = G$ we say that G is a *chain of* *blocks along P*. We can write $K = (b_0, B_1, b_1, B_2, b_2, \dots, b_{n-1}, B_n, b_n)$ where $n \geq 0$, $b_0 = a$, and $b_n = b$; then $b_0, b_1, \dots, b_n$ are distinct vertices of P and each B_i contains an edge of P.

A *bridge of H*, or *H-bridge*, in G is either (a) an edge of $E(G) - E(H)$ with both ends in H (a *trivial* bridge), or (b) a component C of $G - V(H)$ together with all of the edges with one end in C and the other in H. If J is an H-bridge in G, then $E(J) \cap E(H) = \emptyset$, $V(H) \cap V(J)$ is the set of *attachments* of J on H, and $V(J) - V(H)$ is the set of *internal vertices* of J as an H-bridge. Let $A_G(H)$ be the set of attachments of all H-bridges in G, or in other words, the vertices of H incident with an edge of $E(G) - E(H)$.

We often use a property slightly weaker than being k-connected. Let G be a graph, k a positive integer, and $\emptyset \neq S \subseteq V(G)$. We say G is *k-connected relative to S*, or *(k, S)-connected*, if for every $T \subseteq V(G)$ with $|T| < k$, every component of $G - T$ contains at least one vertex of S. For a subgraph H of G we say G is *(k, H)-connected* if it is $(k, V(H))$-connected. Similar definitions have been used in earlier papers such as [15, 22], but ours differs in that we do not require G to be connected or T to be a cutset.

Our definition has a number of consequences that we use later. We omit the straightforward proofs. Part (c) may be regarded as an alternative (perhaps more intuitive) definition; its proof uses (b) and Menger's Theorem, and it helps to prove later parts.

Lemma 2.1. *Let G be a graph, k a positive integer, and $\emptyset \neq S \subseteq V(G)$.*

(a) If $S = V(G)$ then G is always (k, S)-connected. If $S \neq V(G)$ and G is (k, S)-connected then $|S| \geq k$.

(b) G is (k, S)-connected if and only if the graph obtained from G by adding a new vertex r adjacent to all vertices of S has no cutset T with $|T| < k$ and $r \notin T$.

(c) G is (k, S)-connected if and only if for every $v \in V(G) - S$ there are k paths, disjoint except at v, from v to S in G.

(d) If G is (k, S)-connected, G is a spanning subgraph of G', $1 \leq k' \leq k$, and $S \subseteq S' \subseteq V(G)$, then G' is (k', S')-connected.

(e) Adding or deleting edges with both ends in S does not affect whether or not G is (k, S)-connected.

(f) Suppose G is k-connected and $S \subseteq V(G)$ with $|S| \geq k$. Let H be the union of S, zero or more S-bridges in G, and an arbitrary set of edges with both ends in S. Then H is (k, S)-connected. As a special case, G is (k, S)-connected.

(g) Suppose G is (k, S)-connected, and H is a subgraph of G. Let $S_H = A_G(H) \cup (S \cap V(H))$. Then H is (k, S_H)-connected.

(h) Suppose G is (k, S)-connected and H is a subgraph of G with $S \subseteq V(H)$. If $0 \leq k' \leq k$ and H is k'-connected, then G is also k'-connected. Moreover, if $H \cong K_k$ and $V(H) \neq V(G)$ then G is k-connected.

(i) Construct G' by adding to G a set R of new vertices, each adjacent only to vertices in $R \cup S$. Then G is (k, S)-connected if and only if G' is $(k, R \cup S)$-connected.

To prove a statement for 3-connected finite planar graphs, one often proves it for the following larger class of graphs. A *circuit graph* is an ordered pair (G, C) where G is a finite graph, C is a cycle in G that bounds a face in some plane embedding of G, and G is

$(3, C)$-connected. By Lemma 2.1(h), G is automatically 2-connected. Frequently C is the outer cycle of G. Barnette [1] originally defined a circuit graph as the subgraph inside a cycle in a 3-connected plane graph — this definition can be shown to be equivalent to ours using Lemma 2.1(f), (h) and (i). Also, by Lemma 2.1(f), if G is a 3-connected finite plane graph and C is any facial cycle of G, then (G, C) is a circuit graph.

It is convenient to define a finite plane graph G to be a *circuit block* if either G is an edge, or (G, X_G) is a circuit graph. A (possibly 1-way infinite) *plane chain of circuit blocks* has the obvious meaning.

Lemmas 2.2 and 2.3 below give some useful inductive properties of circuit graphs, or more general $(3, S)$-connected plane graphs. Lemma 2.2 follows from Lemma 2.1(d) and (g), and generalizes [7, Lemma 2]. We use it frequently, often without explicit mention. Lemma 2.3(b) generalizes [7, Lemma 3].

Lemma 2.2. *Suppose G is a $(3, S)$-connected plane graph, and C is a cycle in G with no vertex of S strictly inside C. If the subgraph H of G consisting of C and everything inside C is finite, then (H, C) is a circuit graph.*

Lemma 2.3. *Suppose P is a path in a finite connected plane graph G and $P \subseteq X_G$.*

(a) If G is $(3, P)$-connected then G is a plane chain of circuit blocks along P.

(b) Suppose that $c \in V(X_G) - V(P)$. If G is $(3, P \cup \{c\})$-connected then $G - c$ is a plane chain of circuit blocks along P.

Proof. Let K be the chain of blocks in G (for (a)) or $G - c$ (for (b)) along P. The connectivity requirement means that in (a) there are no K-bridges in G, and in (b) the only $(K \cup \{c\})$-bridges in G are edges incident with c. By Lemma 2.2 all nontrivial blocks of K are circuit graphs. The results follow. □

3 Standard Pieces and Systems of Distinct Representatives

To prove that every 4-connected finite planar graph is hamiltonian, Tutte used what are now known as "Tutte subgraphs." In this section, we describe some Tutte subgraph results, including what we call "standard pieces," that we will use frequently.

Let X be a subgraph (usually given in advance) of a graph G, and let T be another subgraph (often a path or a cycle). Then T is an *X-Tutte subgraph* (or *X-Tutte path* or *X-Tutte cycle*, if appropriate) of G if:

(i) every bridge of T in G has at most three attachments, and

(ii) every bridge of T in G that contains an edge of X has at most two attachments.

Sometimes no X is given and only (i) holds; then we simply say that T is a *Tutte subgraph.*

Our overall strategy for constructing a 1-way infinite spanning 2-walk is to build a 1-way infinite Tutte path P, and then detour into the P-bridges to pick up all remaining vertices. To build P we use a similar strategy to Dean, Thomas and Yu [6]. We build Tutte paths in finite parts of the graph, and then use the argument of Kőnig's Lemma to find finite paths converging to the infinite path P. To avoid using a vertex more than twice when we detour into the P-bridges, we use an idea from Gao, Richter and Yu [8]. We designate an entry point for each nontrivial bridge so that a vertex is used as the entry point of at most one bridge.

The entry points thus form a *system of distinct representatives*, or *SDR*, for the nontrivial P-bridges. Formally an SDR is an injective mapping from a set of subgraphs of a graph G to a set of vertices of G so that each representative vertex belongs to its subgraph. We frequently refer to an SDR simply by its set of representatives. We never need to enter trivial bridges, so for a subgraph P, an *SDR of the P-bridges* means an SDR of the nontrivial P-bridges

Combining the ideas from [6] and [8] is not straightforward; making these work together is one of the main new contributions of this paper. First, finding the finite Tutte paths so that we also have an SDR of their bridges can be complicated — the most technical parts of the proofs in Theorems 5.1 and 5.2 are when we need to join Tutte subgraphs together but also maintain an SDR of the bridges of their union. The general idea of $(3, S)$-connectedness helps here, allowing us to use arguments that would be awkward to formulate just in terms of circuit graphs. Second, when we use a Kőnig's Lemma argument to get finite Tutte paths converging to an infinite Tutte path P, in Theorem 5.5, we also need the SDRs for the finite paths to converge to an SDR for P. This requires a careful technical argument. Moreover, our methods allow us to obtain the stronger results in Section 6.

Throughout this paper, we use a general framework for arguments involving Tutte subgraphs that we have developed in [4]; an early version appeared in [2]. Tutte subgraph arguments are often very technical and hard to follow; our framework attempts to clarify them by emphasizing certain fundamental ideas. Two key concepts are that Tutte subgraphs are constructed by piecing together smaller Tutte subgraphs, and that many of these smaller Tutte subgraphs are obtained using arguments that occur repeatedly.

First we state a simple consequence of the definitions of a bridge, Tutte subgraph, and SDR. For a similar result (but without SDRs), see [14, (2.3)].

Lemma 3.1 (Jigsaw Principle)**.** *Suppose G is the edge-disjoint union of G_1, G_2, $\ldots$, G_k. Suppose each G_i has a subgraph X_i and an X_i-Tutte subgraph T_i with an SDR S_i of the T_i-bridges in G_i. Suppose that $V(T_i) \cap V(T_j) = V(G_i) \cap V(G_j)$ and $S_i \cap S_j = \emptyset$ for $i \neq j$. If $T = \bigcup_{i=1}^{k} T_i$, $X = \bigcup_{i=1}^{k} X_i$, and $S = \bigcup_{i=1}^{k} S_i$ then T is an X-Tutte subgraph of G with SDR S of the T-bridges. Moreover, each T-bridge in G is a T_i-bridge in G_i for some i.*

We can think of a subgraph G_i with its X_i-Tutte subgraph T_i and SDR S_i as a piece of a jigsaw puzzle; we can join pieces if they "fit together" correctly. Usually at least one piece is found by induction. Other pieces are constructed using very standard arguments (here, derived from Theorem 3.2) which form our *standard piece lemmas*, or just *standard pieces*. Each says that a graph with certain properties has a Tutte subgraph of a certain type.

We need three standard pieces involving SDRs, which we call SDR Standard Piece k, or SDR SPk, for $k = 1, 2, 3$. As a mnemonic, k denotes the number of components in the Tutte subgraph. Thomas and Yu gave related results without SDRs, combined into a single theorem [14, (2.4)]. To deal with SDRs it helps to keep the three situations separate; then the reader also knows exactly which is being applied. We postpone the proofs until the end of this section. The figures show an X-Tutte subgraph T having SDR S with X as dashed edges (green, if color is shown), T as solid edges and circled isolated vertices (red), and vertices known *not* to be in S as solid vertices (blue). Solid vertices are used to make SDRs pairwise disjoint when applying the Jigsaw Principle.

- **SDR Standard Piece 1 (SDR SP1)**

  Given: A plane chain of circuit blocks $K = (a = b_0, B_1, b_1, B_2, \ldots, b_{n-1}, B_n, b_n = b)$ with $n \geq 0$, and $u \in V(X_K)$.

  Then there exist: An X_K-Tutte ab-path P through u in K and an SDR S of the P-bridges with $a \notin S$.

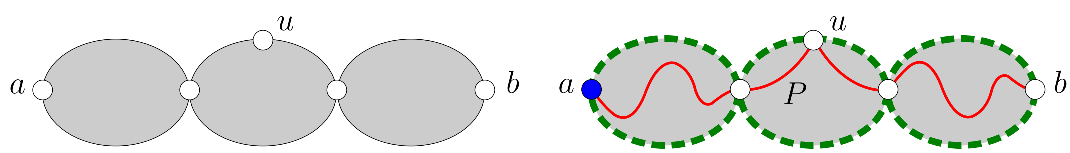

Figure 1: SDR Standard Piece 1

- **SDR Standard Piece 2 (SDR SP2)**

  Given: A connected finite plane graph K and $a, b, c \in V(X_K)$ such that (i) $X_K[a, b]$ is a path avoiding c, and (ii) $K - c$ is a plane chain of circuit blocks $(a = b_0, B_1, b_1, B_2, \ldots, b_{n-1}, B_n, b_n = b)$ with $n \geq 0$.

  Then there exist: An ab-path P avoiding c such that $P \cup \{c\}$ is an $X_K[a, b]$-Tutte subgraph of K, and an SDR S of the $(P \cup \{c\})$-bridges with $a, c \notin S$.

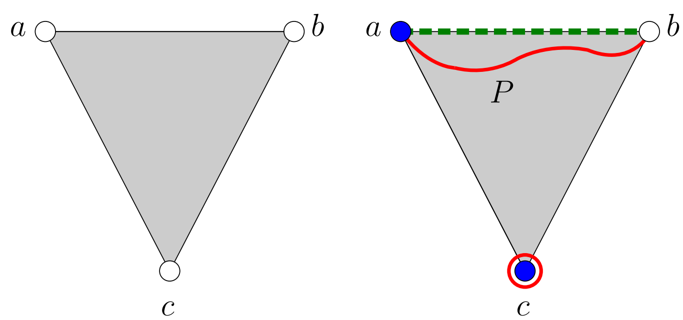

Figure 2: SDR Standard Piece 2

- **SDR Standard Piece 3 (SDR SP3)**

  Given: A connected finite plane graph K and $a, b, c, d \in V(X_K)$ such that (i) $c \neq d$, (ii) $X_K[a, b]$ is a path avoiding c and d, and (iii) K is $(3, X_K[a, b] \cup \{c, d\})$-connected.

  Then there exist: An ab-path P avoiding c and d such that $P \cup \{c, d\}$ is an $X_K[a, b]$-Tutte subgraph of K, and for each $x \in \{c, d\}$ an SDR S of the $(P \cup \{c, d\})$-bridges with $a, x \notin S$.

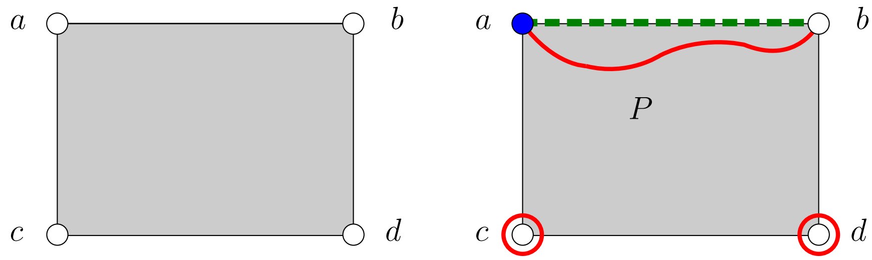

Figure 3: SDR Standard Piece 3 (one of c or d can be solid)

To prove the SDR SPs we use Theorem 3.2 below. It is [8, 9, Theorem 4] modified to include the case where G is just an edge xy (then we take $P = xy$ and $S = \emptyset$). Note that a "Tutte path" in [8, 9] is what we call an "X_G-Tutte path." Corollary 3.3 is used later.

Theorem 3.2. *Let G be a circuit block, $x, u \in V(X_G)$, $y \in V(G)$ with $x \neq y$, and $v \in \{x, u\}$. Then there is an X_G-Tutte xy-path P through u and an SDR S of the P-bridges with $v \notin S$.*

Corollary 3.3. *Let (G, X_G) be a circuit graph, $x \in V(X_G)$, $y \in V(G) - \{x\}$, and $e \in E(X_G)$. Then there is an X_G-Tutte xy-path P through e and an SDR S of the P-bridges with $x \notin S$.*

Proof. Form G' by subdividing the edge e with a new vertex u. Then $(G', X_{G'})$ is a circuit graph by Lemma 2.1(e) and (i). Apply Theorem 3.2 to $(G', X_{G'})$, choosing $v = x$. □

Proof of SDR Standard Pieces 1, 2, and 3. For SDR SP1, if $a = b$, set $P = a = b$ and $S = \emptyset$. Otherwise, for each B_i, by Theorem 3.2, we find an X_{B_i}-Tutte $b_{i-1}b_i$-path P_i and an SDR S_i of the P_i-bridges such that $b_{i-1} \notin S_i$; if $u \in V(B_i) - \{b_{i-1}, b_i\}$, we choose P_i to go through u. By the Jigsaw Principle, $P = \bigcup_{i=1}^n P_i$ and $S = \bigcup_{i=1}^n S_i$ are as desired.

For SDR SP2, apply SDR SP1 to $H = K - c$ to obtain an ab-path X_H-Tutte path P in H and SDR S of the P-bridges in H with $a \notin S$. Every nontrivial $P \cup \{c\}$-bridge J in K is a P-bridge in H unless it contains edges incident with c; in that case J must consist of a P-bridge J' that uses an edge of $X_H[b, a]$, and edges incident with c. Then J has three attachments (two from J', and c) and we may reassign the representative of J' to J. Hence $P \cup \{c\}$ and S (with some reassignment) are as required.

Finally, for SDR SP3, let $H = (b_0 = a, B_1, b_1, B_2, \dots, b_{n-1}, B_n, b_n = b)$ be the plane chain of blocks along $X_K[a, b]$ in $K - \{c, d\}$. Since K is $(3, X_K[a, b] \cup \{c, d\})$-connected, by applying Lemma 2.2 to the nontrivial blocks we see that each B_i is a circuit block.

Every nontrivial $(H \cup \{c, d\})$-bridge has at most one attachment in H, because H is a chain of blocks, but at least three attachments because K is $(3, X_K[a, b] \cup \{c, d\})$-connected, so it must have c, d and exactly one vertex of H as attachments. By planarity there can be at most one such bridge; if it exists, call it J and let u be its attachment in H.

By SDR SP1 there is an X_H-Tutte ab-path P in H, through u if it exists, with an SDR S' of the P-bridges such that $a \notin S'$. Consider the nontrivial $(P \cup \{c, d\})$-bridges in K. The only such bridge that can contain both c and d is J. If J exists, we choose y as its representative, where $\{x, y\} = \{c, d\}$, and set $S = S' \cup \{y\}$; otherwise set $S = S'$. The argument from the proof of SDR SP2 for nontrivial $P \cup \{c\}$ bridges applies here to nontrivial bridges with exactly one of c or d as an attachment. Hence P and S are as desired. □

Remark 3.4. One idea from the proofs of SDR SP2 and SP3 is used often. If we delete a vertex x from a graph, find a Tutte subgraph and an SDR in what remains, and then add x back, x may become a new attachment for some bridges. We ensure that each such bridge previously had only two attachments, so with x there are still only three. Each such bridge already has a representative, so we do not need to use x as its representative.

4 Structural Results

In this section, we give some results about the structure of 3-connected 2-indivisible infinite planar graphs. In these graphs, we find either an infinite plane chain of blocks or a structure called a *net*. Then, in Section 5, we use these structural results to build our 1-way infinite 2-walks.

If G is a 2-indivisible infinite plane graph and C is a cycle in G, then C divides the plane into two closed sets, exactly one of which contains finitely many vertices. Let $I(C)$ (or $I_G(C)$) denote the subgraph of G consisting of all vertices and edges of G inside that closed set containing finitely many vertices. Note that $C \subseteq I(C)$. Dean, Thomas, and Yu [6] defined a *net* in G to be a sequence of cycles $N = (C_1, C_2, C_3, \dots)$ such that

1. $I(C_i)$ is a subgraph of $I(C_{i+1})$ for all $i = 1, 2, 3, \dots$,
2. $\bigcup_{i=1}^{\infty} I(C_i) = G$, and either
3. $C_1, C_2, C_3, \dots$ are pairwise disjoint, or

3$'$. for every $i = 1, 2, 3, \dots$, the graph $C_i \cap C_{i+1}$ is a non-empty path, it is a subgraph of $C_{i+1} \cap C_{i+2}$, and no endpoint of $C_i \cap C_{i+1}$ is an endpoint of $C_{i+1} \cap C_{i+2}$.

If 3 holds we say that N is a *radial net*, and if 3$'$ holds we say that N is a *ladder net*. A graph with a net is locally finite, because for every vertex v there is some i such that v and all its neighbors belong to $I(C_i)$.

In [23], Yu said that an infinite plane graph G is *nicely embedded* or is a *nice embedding* if, for any cycle C in G for which $I(C)$, the finite side of C, is defined, $I(C)$ is contained in the closed disk bounded by C. In a nice embedding, the intuitive idea of the "inside" of a cycle C coincides with $I(C)$, which is why the notation $I(C)$ is used. The following lemma is [23, (2.1)].

Lemma 4.1. *Let G be an infinite plane graph, and suppose G has a sequence of cycles $(C_1, C_2, C_3, \dots)$ such that $I(C_i)$ is a subgraph of $I(C_{i+1})$ for all $i = 1, 2, 3, \dots$, and $\bigcup_{i=1}^{\infty} I(C_i) = G$. Then for any facial cycle C of G, G has a nice embedding in which C is also a facial cycle.*

By Lemma 4.1, if a infinite plane graph has a net, then the graph has a nice embedding. In this paper, we will always assume that such a graph is nicely embedded in the plane.

Let $N = (C_1, C_2, C_3, \dots)$ be a net in a 2-indivisible plane graph G. The *boundary* of N, denoted by ∂N, is the graph $\bigcup_{i=1}^{\infty}(C_i \cap C_{i+1})$. If N is a radial net, $\partial N = \emptyset$, and if N is a ladder net, then ∂N is a 2-way infinite path. If N is a ladder net, we will assign an orientation $\overrightarrow{\partial N}$ to ∂N such that $G - V(\partial N)$ is to the right of every edge in $\overrightarrow{\partial N}$. For $i = 1, 2, 3, \dots$, let D_i be the graph obtained from C_i by deleting $C_i \cap C_{i+1}$ except its endpoints. If N is a radial net, $D_i = C_i$, and if N is a ladder net, then D_i is a path with both ends in ∂N and otherwise disjoint from ∂N. If N is a ladder net, and $C_1 - V(D_1)$ contains at least one vertex, let $D_0 = C_0$ be a subpath of $C_1 - V(D_1)$. Otherwise, set $D_0 = C_0 = \emptyset$ (but we will never see this case in this paper). We say that N is *tight* if

1. $I(C_1) = C_1$ if N is a radial net,
2. $C_1 \cap C_2$ is either empty or contains at least one edge, and
3. for every $i = 1, 2, 3, \ldots$, every D_{i+1}-bridge in $I(C_{i+1}) - V(I(C_i))$ has at most one attachment.

If N is a tight ladder net and every D_1-bridge in $I(C_1) - V(D_0)$ has at most one attachment, we say that N is *tight with respect to* D_0. Note that the definition of a tight ladder net in Dean, Thomas, and Yu [6] also requires that $I(C_1) = C_1$, which would imply that it is tight with respect to D_0; we will not need this.

The following three lemmas are from [6]. The first immediately precedes their (1.2), and the others are their (2.1) and (2.2), respectively.

Lemma 4.2. *If G is a 3-connected planar graph and X is any finite subset of $V(G)$, then $G - X$ has a finite number of components.*

Lemma 4.3. *Let G be a 2-indivisible infinite plane graph such that the deletion of any finite set of vertices in G results in a finite number of components. Then G has at most two vertices of infinite degree.*

Lemma 4.4. *Let G be a locally finite 2-connected 2-indivisible infinite plane graph. Then G has a net.*

How we find our 1-way infinite Tutte path and SDR of its bridges in every 3-connected 2-indivisible infinite plane graph will depend on whether or not the graph contains a net, and, if so, what kind of net the graph contains. Let G be a 3-connected 2-indivisible infinite plane graph, and let F be the set of vertices of infinite degree in G. By Lemmas 4.2 and 4.3, $|F| \leq 2$. If $F = \emptyset$, then Lemma 4.4 guarantees that G has a net. If $|F| \geq 1$, then G contains a spanning subgraph H such that either

1. H contains a ladder net (Lemma 4.5), or
2. H is a 1-way infinite plane chain of circuit blocks (Lemma 4.6).

Thus there are essentially three types of subgraphs to consider: radial nets, ladder nets, and 1-way infinite plane chains of circuit blocks. We will deal with radial nets and ladder nets in Theorem 5.1 and Theorem 5.2, respectively, and then combine the three cases in Theorem 5.5.

We first need two structural lemmas about 2-indivisible infinite plane graphs containing at least one vertex of infinite degree. The first is similar to [6, (2.3)].

Lemma 4.5. *Let G be a 3-connected 2-indivisible infinite plane graph, let F be the set of vertices of infinite degree in G, and assume that $|F| \geq 1$ and that $G - F$ has an infinite block. Then there exists a 2-connected 2-indivisible infinite subgraph H of G such that*

1. H contains a ladder net N,

2. *$F \subseteq V(\partial N)$,*
3. *H is $(3, \partial N)$-connected, and*
4. *the only H-bridges in G are edges incident with at least one vertex in F (so H is a spanning subgraph of G).*

Proof. By Lemmas 4.2 and 4.3, $|F| \leq 2$. By [6, (2.3)], G contains a 2-connected 2-indivisible infinite subgraph H' such that H' contains a net N' and $F \subseteq V(\partial N')$. Since $F \neq \emptyset$, N' must be a ladder net. Also by [6, (2.3)], every H'-bridge of G is finite and has at most three attachments. Since G is 3-connected, any nontrivial H'-bridge in G must have exactly three attachments. By the proof of [6, (2.3)], the attachments of any H'-bridge must be contained in $V(\partial N')$ and at most one of these attachments is in $V(\partial N') - F$. Thus any nontrivial H'-bridge must have exactly two attachments in F and exactly one attachment in $V(\partial N') - F$.

If there are no nontrivial H'-bridges in G (which must happen if $|F| = 1$ and may happen if $|F| = 2$), we set $H = H'$ and $N = N'$. Note that H is $(3, \partial N)$-connected by Lemma 2.1(f). Any H'-bridge is trivial and must have an attachment in F. Therefore H and N are as desired.

Otherwise, we may assume that there is at least one nontrivial H'-bridge. Then $|F| = 2$, so let $F = \{f_1, f_2\}$ and assume that f_1 comes before f_2 in $\overrightarrow{\partial N'}$. If B is a nontrivial H'-bridge with v as its attachment in $V(\partial N') - F$, we show that $v \in \partial N'[f_1, f_2]$. If not, we may assume that v comes before f_1 in $\overrightarrow{\partial N'}$. Then $L = B \cup \partial N'[v, f_2]$ is finite and contains a cycle which separates f_1 from $G - V(L)$, which contradicts the fact that f_1 has infinite degree.

By planarity, there is exactly one nontrivial H'-bridge B with attachments f_1, f_2, and v, where $v \in \partial N'[f_1, f_2]$. Then $H = H' \cup B$ is a 2-connected 2-indivisible spanning subgraph of G. All H-bridges are trivial and have at least one attachment in F. Let D denote the portion of X_B from f_1 to f_2 that does not contain v. Then we modify the net $N' = (C'_1, C'_2, C'_3, \dots)$ to form the net $N = (C_1, C_2, C_3, \dots)$ by setting $C_i = (C'_i - V(\partial N'[f_1, f_2])) \cup D$ for every $i \geq 1$. N is a ladder net, $F \subseteq V(\partial N)$, and H is $(3, \partial N)$-connected by Lemma 2.1(f). □

The second structural lemma is proved in a similar way to that of Lemma 4.5, much as [6, (2.4)] is similar to [6, (2.3)].

Lemma 4.6. *Let G be a 3-connected 2-indivisible infinite plane graph, let F be the set of vertices of infinite degree in G, and assume that $|F| \geq 1$ and that $G - F$ has no infinite block. Then $|F| = 2$, and there exists a connected subgraph H of G such that*

1. *H is a 1-way infinite plane chain of circuit blocks $(B_1, b_1, B_2, b_2, \dots)$,*
2. *$F \subseteq V(X_{B_1}) - \{b_1\}$,*
3. *the only H-bridges in G are edges incident with at least one vertex in F (so H is a spanning subgraph of G).*

In each of the previous two lemmas, the proofs construct the desired subgraph H in such a way that we may assume that H and G are nicely embedded in the plane. The subgraph H is locally finite in both cases. Also, if $|F| = 2$ and $F = \{f_1, f_2\}$, assuming that f_1 comes before f_2 in $\overrightarrow{\partial N}$ (X_H), the construction rules out any H-bridges in G that are edges joining one vertex in F to a vertex in $V(\partial N[f_1, f_2]) - F$ (in $V(X_H[f_1, f_2]) - F$, respectively). We also remark that it follows from these lemmas that every 3-connected 2-indivisible infinite planar graph is countable.

The next two lemmas allow us to consider only tight nets instead of more general nets. The first lemma follows from the proof of [6, (2.6)] (in the first sentence of their proof, they choose C_1 to be any facial cycle of G with $u \in V(C_1)$, which is why we are able to specify such a cycle C in our Lemma 4.7). The second lemma is similar to [6, (2.5)] and Claim 1 in the proof of [23, (3.6)], and can be proven in a similar way as those results — a difference is that, with our slightly modified definition of a tight ladder net, we can treat the first cycle of the net in the same way as all of the other cycles.

Lemma 4.7. *Let G be a 2-connected 2-indivisible plane graph with a radial net, let $u \in V(G)$, and let C be any facial cycle of G with $u \in V(C)$. Then G has a tight radial net $N = (C_1, C_2, C_3, \dots)$ with $C_1 = C$ (and hence u is a vertex of C_1).*

Lemma 4.8. *Let G be a 2-connected 2-indivisible plane graph with a ladder net N'. Let u and v be vertices of $\partial N'$, and suppose that u comes before v in $\overrightarrow{\partial N'}$ if u and v are distinct. Then G has a ladder net $N = (C_1, C_2, C_3, \dots)$ such that $u, v \in V(C_1) - V(D_1)$, $\partial N = \partial N'$, and N is a tight ladder net with respect to $\partial N[u, v]$.*

5 Paths and 2-walks

We will now find a 1-way infinite Tutte path P and an SDR of the nontrivial P-bridges in every 3-connected 2-indivisible infinite planar graph. Using the SDR of the P-bridges, we then detour into each bridge to build our 1-way infinite 2-walk.

Let G be a 2-indivisible plane graph with a net $N = (C_1, C_2, C_3, \dots)$. Then a uv-path P in G is a *forward uv-path* if whenever vertices u, x, y, v occur on P in this order, there is no $i \in \{1, 2, 3, \dots\}$ such that $x \in V(C_{i+2}) - V(C_{i+1})$ and $y \in V(C_i)$. A forward path may move "backward" a little, but only to a limited extent. We will find forward Tutte paths in finite portions of nets, and then use the fact that these paths are forward to show (using a variation of Kőnig's Lemma) that they converge to a 1-way infinite Tutte path.

We first focus on radial nets. The following theorem is similar to [6, (3.4)], but we assume that the graph is $(3, C_1)$-connected, and we find an SDR of the P-bridges.

Theorem 5.1. *Let G be a 2-indivisible plane graph with a tight radial net $N = (C_1, C_2, C_3, \dots)$ such that G is $(3, C_1)$-connected, and let $u \in V(C_1)$. Then for every $k \in \{1, 2, 3, \dots\}$, there exist a vertex $v \in V(C_k)$, a forward C_1-Tutte uv-path P in $I(C_k)$, and an SDR S of the P-bridges.*

Proof. The proof is by induction on k. For $k = 1$, the one-vertex path $v = u$ suffices, with $S = \{u\}$.

So we can assume that $k > 1$ and that the statement holds for positive integers less than k. Let H be the block of $I(C_k) - V(C_1)$ containing C_k, with the embedding inherited from G. Since N is tight and G is nicely embedded, C_2 bounds a face of H containing C_1, and every $(H \cup C_1)$-bridge has at most one vertex of attachment, called a *tip*, in C_2.

Let $t_1, t_2, \ldots, t_n \in V(C_2)$ be all tips of $(H \cup C_1)$-bridges of $I(C_k)$, listed in clockwise cyclic order on C_2. Since G is $(3, C_1)$-connected, $n \geq 3$. For $i = 1, 2, \ldots, n$, let L_i be the union of all $(H \cup C_1)$-bridges that attach at t_i. The cycle C_1 has a collection of pairwise edge-disjoint segments $\{P(t_i)\}_{i=1}^n$ in clockwise cyclic order such that, for every $i = 1, 2, \ldots, n$, $P(t_i)$ contains $V(L_i) \cap V(C_1)$ and the ends of $P(t_i)$ are in $V(L_i)$. Since $n \geq 3$ the collection $\{P(t_i)\}_{i=1}^n$ is well-defined. For each $i = 1, 2, \ldots, n$, let p_i and q_i be the endpoints of $P(t_i)$ so that $p_1, q_1, p_2, q_2, \ldots, p_n, q_n$ occur in C_1 in clockwise order. Let u_1 be the clockwise neighbor of u on C_1. We may assume that $uu_1 \in C_1[q_n, q_1]$; hence $u \neq q_1$.

Now $(C_2, C_3, C_4, \ldots)$ is a tight radial net in H. Since G is $(3, C_1)$-connected, H is $(3, C_2)$-connected by Lemma 2.1(g). Therefore, by induction, there is a vertex $v \in V(C_k)$, a forward C_2-Tutte t_1v-path P' in H, and an SDR S' of the P'-bridges. Since P' is C_2-Tutte and G is 2-connected, every P'-bridge D containing an edge of C_2 has exactly two attachments, which are on C_2; hence $V(D) \cap V(C_j) = \emptyset$ for $j \geq 3$.

We divide the $(H \cup C_1)$-bridges into three types: (i) those with no tip, (ii) those with a tip on P', and (iii) those with a tip not on P'. We will form collections of these bridges, along with portions of C_1 that they span, to form subgraphs where we can apply our standard pieces. If $t_i \in V(P')$, the subgraph K_{t_i} collects together (is the union of) L_i, $P(t_i)$, and every type (i) bridge with all attachments in $P(t_i)$. If $t_i \in V(H) - V(P')$, then any bridge J with tip t_i is of type (iii). The P'-bridge D in H containing t_i has exactly two attachments, c_D and d_D, which are in C_2. One of c_D or d_D is the representative of D in S'. Suppose that $t_\ell, t_{\ell+1}, \ldots, t_m$ are the tips in $V(D) - \{c_D, d_D\}$. The subgraph K_D collects together $D, L_\ell, L_{\ell+1}, \ldots, L_m, C_1[p_\ell, q_m]$, and every type (i) bridge with all attachments in $C_1[p_\ell, q_m]$.

For each K_{t_i} with $i \neq 1$, K_{t_i} is $(3, P(t_i) \cup \{t_i\})$-connected by Lemma 2.1(g), so by Lemma 2.3(b) we may apply SDR SP2 to obtain a p_iq_i-path P_{t_i} in K_{t_i} avoiding t_i such that $P_{t_i} \cup \{t_i\}$ is a $P(t_i)$-Tutte subgraph and an SDR S_{t_i} of the $(P_{t_i} \cup \{t_i\})$-bridges with $q_i, t_i \notin S_{t_i}$ (this is valid even if $p_i = q_i$).

For each K_D, label c_D and d_D so that $c_D \in S'$, $d_D \notin S'$. By Lemma 2.1(g), K_D is $(3, C_1[p_\ell, q_m] \cup \{c_D, d_D\})$-connected. So SDR SP3 gives a $p_\ell q_m$-path P_D in K_D avoiding c_D and d_D such that $P_D \cup \{c_D, d_D\}$ is a $C_1[p_\ell, q_m]$-Tutte subgraph, and an SDR S_D of the $(P_D \cup \{c_D, d_D\})$-bridges with $q_m, d_D \notin S_D$. Then replacing D and its representative c_D by the P_D-bridges in K_D and S_D does not use a representative more than once.

For every segment $C_1[q_i, p_{i+1}]$, $i \neq n$ and $q_i \neq p_{i+1}$, not already included in some K_D, we also collect $C_1[q_i, p_{i+1}]$ and every type (i) bridge with all attachments in $C_1[q_i, p_{i+1}]$ to form a subgraph K_i. By Lemmas 2.1(g) and 2.3(a) we may apply SDR SP1 to K_i to obtain a $C_1[q_i, p_{i+1}]$-Tutte q_ip_{i+1}-path P_i in K_i and an SDR S_i of the P_i-bridges with $p_{i+1} \notin S_i$.

Let P'' be the union of P_{t_i} for every K_{t_i} considered above, P_D for every K_D considered above, and P_i for every K_i considered above. Let S'' be the union of every S_{t_i}, every S_D, and every S_i. Let S'_0 be the set of vertices c_D for every K_D considered above. Then P'' is a q_1q_n-path, and by the Jigsaw Principle $P' \cup P''$ is a $C_1[q_1, q_n]$-Tutte subgraph and $(S' - S'_0) \cup S''$ is an SDR of the $P' \cup P''$-bridges with $q_n \notin (S' - S'_0) \cup S''$.

Finally, we use the bridges with tip t_1 to find vertex-disjoint q_nu- and t_1q_1-paths, and a corresponding SDR. Let F be the union of $C_1[q_n, q_1]$ and every $(H \cup C_1)$-bridge all of whose attachments on C_1 belong to $C_1[q_n, q_1]$. Then F contains K_{t_1}, and possibly some additional type (i) bridges; F also contains u. By Lemma 2.1(g), F is $(3, C_1[q_1, q_n] \cup \{t_1\})$-connected. So, by

Lemma 2.3(b), $F-t_1$ is a plane chain of circuit blocks ($b_0 = q_n, B_1, b_1, B_2, \dots, b_{m-1}, B_m, b_m = q_1$) with $m \geq 1$. Let Y be the q_nq_1-path $X_{F-t_1}[q_n, q_1]$.

Set z, $1 \leq z \leq m$, so that $u \in V(B_z) - \{b_z\}$ (recall that $u \neq q_1$). Let w be the first vertex in $Y[b_{z-1}, q_1]$, other than b_{z-1}, that is a neighbor of t_1 in F. Since $B_m - b_{m-1}$ has such a vertex, w is defined. Let B_r be the block such that $w \in V(B_r) - \{b_{r-1}\}$. Then $z \leq r \leq m$.

Let $H_1 = B_1 \cup B_2 \cup \cdots \cup B_{z-1}$, $H_2 = B_z \cup B_{z+1} \cup \cdots \cup B_r$, and $H_3 = B_{r+1} \cup B_{r+2} \cup \cdots \cup B_m$. Apply SDR SP1 to H_1 to find an X_{H_1}-Tutte q_nb_{z-1}-path Q_1 in H_1 and an SDR T_1 of the Q_1-bridges with $b_{z-1} \notin T_1$. Similarly, apply SDR SP1 to H_3 to find an X_{H_3}-Tutte b_rq_1-path Q_3 in H_3 and an SDR T_3 of the Q_3-bridges with $q_1 \notin T_3$.

Let $H_2' = H_2 \cup b_{z-1}w$, and $Z_2' = X_{H_2'} = b_{z-1}w \cup Y[w, b_r] \cup C_1[b_{z-1}, b_r]$. By choice of r, Z_2' is a cycle. Now $A_G(H_2) \subseteq V(Z_2')$, so by Lemma 2.1(d), (e), and (g), H_2 and hence H_2' are $(3, V(Z_2'))$-connected. Thus, (H_2', Z_2') is a circuit graph. Apply Corollary 3.3 to H_2' to find a Z_2'-Tutte ub_r-path Q_2' through $b_{z-1}w$ and an SDR T_2 of the Q_2'-bridges with $b_r \notin T_2$. Let $Q_2 = Q_2' - b_{z-1}w$ and $Z_2 = Z_2' - b_{z-1}w$, then Q_2 is Z_2-Tutte, consisting of vertex-disjoint $b_{z-1}u$- and wb_r-paths, and T_2 is an SDR of the Q_2-bridges in H_2.

If any Q_2-bridge B in H_2 has three attachments and contains any edges of X_{H_2}, then these edges are in $Y[b_{z-1}, w]$. By choice of w, no internal vertex of B is a neighbor of t_1 in F. Thus B is a Q_2-bridge in F, still with only three attachments. Applying this and Remark 3.4, $Q = Q_1 \cup Q_2 \cup t_1w \cup Q_3$ is $C_1[q_n, q_1]$-Tutte in F, consisting of vertex-disjoint q_nu- and t_1q_1-paths, with an SDR $T = T_1 \cup T_2 \cup T_3$ of the Q-bridges such that $t_1, q_1 \notin T$.

Let $P = P' \cup P'' \cup Q$ and $S = (S' - S_0') \cup S'' \cup T$. Then P is a C_1-Tutte uv-path in $I(C_k)$, and S is an SDR of the P-bridges. We must show that P is a forward uv-path. Suppose that u, x, y, v are vertices in that order along P, with $x \in V(C_{i+2})$ for some $i \in \{1, 2, \dots, k-2\}$. Now $H \cap (P - V(P'))$ is contained in the union of P'-bridges of H that contain an edge of C_2, and these bridges are vertex-disjoint from C_{i+2}, so $x \in V(P')$. But then $y \in V(P')$, so $y \notin V(C_i)$ because P' is forward and $V(P') \cap V(C_1) = \emptyset$. Thus P is a forward uv-path. □

Now let G be a 2-indivisible plane graph with a ladder net $N = (C_1, C_2, C_3, \dots)$. Suppose that $V(C_1) \neq V(D_1)$, so that we may choose a subpath $C_0 = D_0$ of $C_1 - V(D_1)$. For each $i \geq 0$, suppose the path D_i has ends x_i and y_i, where $\dots, x_2, x_1, x_0, y_0, y_1, y_2, \dots$ occur along ∂N in this order. By construction, all of these vertices are distinct, except we might have $x_0 = y_0$. Set $I(C_0) = C_0 = D_0$. See Figure 4. Let r and s be integers such that $0 \leq r \leq s$. The *(r, s)-truncation of G relative to N* is the graph $G_{r,s}$ obtained from $I(C_s)$ by deleting the vertices of $I(C_r) - V(D_r)$. The following lemma is similar to Claim 2 in the proof of [23, (3.6)], but we assume that G is $(3, \partial N)$-connected, and find an SDR of the P-bridges.

Theorem 5.2. *Let G be a 2-indivisible plane graph with a ladder net $N = (C_1, C_2, C_3, \dots)$ such that G is $(3, \partial N)$-connected. Let $\dots, x_2, x_1, x_0, y_0, y_1, y_2, \dots$ be as above, and assume that N is tight with respect to $D_0 = \partial N[x_0, y_0]$. Suppose that $0 \leq r \leq s$. Then there exist an $X_{G_{r,s}}[x_s, y_s]$-Tutte path P in $G_{r,s}$ from x_r to y_s if $s - r$ is even (to x_s if $s - r$ is odd) such that $\{x_r, x_{r+1}, \dots, x_s, y_r, y_{r+1}, \dots, y_s\} \subseteq V(P)$ and P is a forward path in G, and an SDR S of the P-bridges in $G_{r,s}$ such that $y_r \notin S$. Symmetrically, there is also such a path from y_r to x_s (y_s) and a corresponding SDR of its bridges.*

Proof. The proof is by induction on $s - r$. If $s = r$, then $P = D_s$ and $S = \emptyset$ suffice.

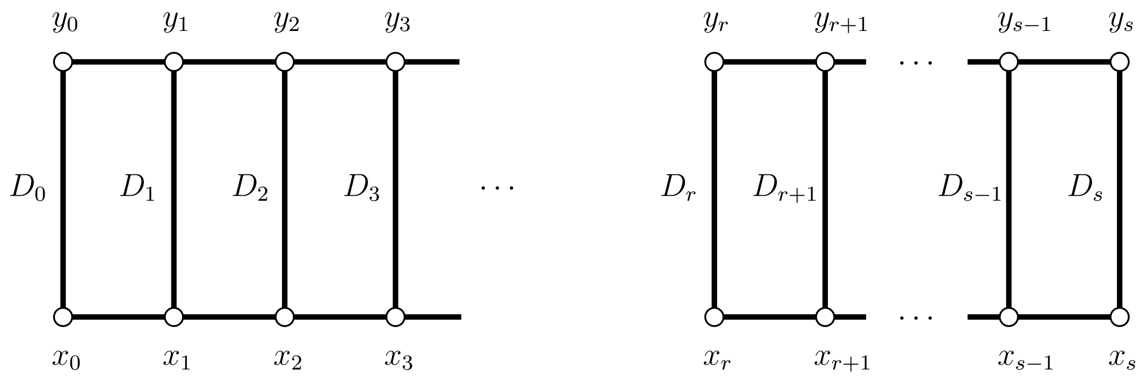

Figure 4: Ladder net notation — the second graph is $G_{r,s}$

So assume that $r < s$. Let $H = G_{r,s}$ and $H' = G_{r+1,s}$. By induction, there is an $X_{H'}[x_s, y_s]$-Tutte path P' in H' from y_{r+1} to y_s if $s - r$ is even (x_s if $s - r$ is odd) such that $\{x_{r+1}, x_{r+2}, \ldots, x_s, y_{r+1}, y_{r+2}, \ldots, y_s\} \subseteq V(P')$ and P' is a forward path in G from y_{r+1} to y_s (x_s), and an SDR S' of the P'-bridges such that $x_{r+1} \notin S'$. By Lemma 2.1(g), since G is $(3, \partial N)$-connected, H is $(3, X_H)$-connected. Since X_H is a cycle, (H, X_H) is a circuit graph.

Since N is tight, every $(H' \cup D_r)$-bridge in $G_{r,s}$ has at most one attachment in D_{r+1}. Let $t_1, t_2, \ldots, t_n \in V(D_{r+1})$ be all of the tips of $(H' \cup D_r)$-bridges in $G_{r,s}$, listed in order in $X_{H'}[x_{r+1}, y_{r+1}]$. Then $n \geq 2$, $t_1 = x_{r+1}$, and $t_n = y_{r+1}$. So we may proceed similarly to the proof of Theorem 5.1. Define subpaths $P(t_i)$ of D_r with ends p_i and q_i, so that $p_1 = x_r, q_1, p_2, q_2, \ldots, p_n, q_n = y_r$ occur in this order along D_r. Collect together subgraphs K_{t_i} ($i \neq 1, n$), K_D, and K_i, and use the SDR standard pieces to to find a $q_1 p_n$-path P'' such that $P' \cup P''$ is a $D_r[q_1, p_n]$-Tutte subgraph and an SDR S'' of the $(P' \cup P'')$-bridges such that $p_n \notin S''$ (for each piece, we ensure that the vertex in D_r closest to q_n is not in its SDR). We must still deal with K_{t_1} and K_{t_n}.

Consider K_{t_1}. Let v be the clockwise neighbor of $t_1 = x_{r+1}$ in X_H. By Lemmas 2.1(g) and 2.3(b), $K_{t_1} - t_1$ is a plane chain of circuit blocks $A' = (a_0 = v, A_1, a_1, A_2, \ldots, a_{m-1}, A_m, a_m = q_1)$ with $m \geq 0$ along $X_H[v, q_1]$. Let $A = (a_\alpha, A_{\alpha+1}, a_{\alpha+1}, \ldots, a_m)$ be the subchain of A' along $X_H[x_r, q_1]$ (with a_α as in A'). By SDR SP1 there is an X_A-Tutte $x_r q_1$-path P_1 through a_α in A and an SDR S_1' of the P_1-bridges with $q_1 \notin S_1'$. If $\alpha > 0$ there is a nontrivial $(P_1 \cup \{t_1\})$-bridge containing $t_1 v$ with attachments t_1 and a_α, so take $t_1 = x_{r+1}$ as its representative and set $S_1 = S_1' \cup \{x_{r+1}\}$; otherwise set $S_1 = S_1'$. By Remark 3.4, $P_1 \cup \{t_1\}$ is an $X_H[x_{r+1}, q_1]$-Tutte subgraph of K_{t_1}; S_1 is an SDR of the $(P_1 \cup \{t_1\})$-bridges with $q_1 \notin S_1$.

Finally, consider K_{t_n}. Let w be the counterclockwise neighbor of $t_n = y_{r+1}$ in X_H. By Lemmas 2.1(g) and 2.3(b), $K_{t_n} - t_n$ is a plane chain of circuit blocks $(b_0 = p_n, B_1, b_1, B_2, \ldots, b_{z-1}, B_z, b_z = w)$ with $z \geq 0$. If $z = 0$, then $p_n = y_r = w$; let $P_n = y_r t_n$ and $S_n = \emptyset$. If $z > 0$ let B_β be any block with $y_r \in V(B_\beta)$. For each j, apply Theorem 3.2 to find an X_{B_j}-Tutte $b_{j-1}b_j$-path R_j in B_j, through y_r if $j = \beta$, and an SDR U_j of the R_j-bridges such that $b_j \notin U_j$ if $j < \beta$, $y_r \notin U_j$ if $j = \beta$, and $b_{j-1} \notin U_j$ if $j > \beta$. Set $P_n = \left(\bigcup_{j=1}^{z} R_j\right) \cup w t_n$ and $S_n = \bigcup_{j=1}^{z} U_j$. In either case, P_n is an $X_H[p_n, y_{r+1}]$-Tutte $p_n y_{r+1}$-path in K_{t_n} (using Remark 3.4 if $z > 0$) and S_n is an SDR of the P_n-bridges with $y_r, y_{r+1} \notin S_n$ (but y_{r+1} may already be in S').

Let $P = P' \cup P'' \cup P_1 \cup P_n$, and $S = S' \cup S'' \cup S_1 \cup S_n$. Then P is an $X_H[x_s, y_s]$-Tutte path in $H = G_{r,s}$ from x_r to y_s (x_s) such that $\{x_r, x_{r+1}, \dots, x_s, y_r, y_{r+1}, \dots, y_s\} \subseteq V(P)$, and S is an SDR of the P-bridges such that $y_r \notin S$. By an argument similar to the one in Theorem 5.1, P is a forward path from x_r to y_s (x_s). $\square$

We will build our 1-way infinite Tutte path from a sequence of finite forward Tutte paths. The following lemma will help to compare bridges of different subgraphs in different graphs.

Lemma 5.3. *Let G be a 2-indivisible plane graph with a net $N = (C_1, C_2, C_3, \dots)$. Let i be a positive integer and $w \in V(D_{i+2})$. For $k = 1, 2$ suppose that $I(C_{i+1}) \subseteq G_k \subseteq G$, and that $R_k \subseteq G_k$ is a forward path in G from u through w. Suppose that $R_1[u, w] = R_2[u, w]$ and $J \subseteq I(C_i)$. Then J is an R_1-bridge in G_1 if and only if it is an R_2-bridge in G_2, and J has the same attachments in both cases.*

Proof. For J to be a bridge, it must be connected and have at least two vertices. Given this, J is an R_k-bridge in G_k with attachment set S if and only if (i) $E(J) \subseteq E(G_k) - E(R_k)$, (ii) $A_{G_k}(J) \subseteq V(R_k)$, (iii) $J - V(R_k)$ is connected or empty, and (iv) $V(J) \cap V(R_k) = S$. Since R_k is forward, after w it contains no vertex of $I(C_i)$ and hence no vertex or edge of J, so we may replace R_k in (i)–(iv) with $R_k[u, w]$. Since every edge of G incident with a vertex of J belongs to $I(C_{i+1})$ and $I(C_{i+1}) \subseteq G_k \subseteq G$, we may replace G_k in (i)–(iv) with $I(C_{i+1})$. But then the conditions are identical for $k = 1$ or 2. $\square$

We are now ready to find a 1-way infinite Tutte path and an SDR of its bridges in every 3-connected 2-indivisible infinite plane graph. We actually prove another similar result first, for graphs that have a net but satisfy a somewhat weaker connectivity condition. The proof of this theorem is a variation of Kőnig's Lemma and is similar to [6, (3.7) and (3.8)], [23, (3.5)], and [24, Lemma 5.2].

Theorem 5.4. *Let G be a 2-indivisible infinite plane graph with a net $N = (C_1, C_2, C_3, \dots)$. If N is a radial net, let $\Delta = C_1$ and $u = v \in V(C_1)$. If N is a ladder net, let $\Delta = \partial N$ and $u, v \in \partial N$. Assume that G is $(3, \Delta)$-connected. Then there exist a 1-way infinite Δ-Tutte path P in G from u through v and an SDR S of the P-bridges.*

Proof. In either case, G is locally finite since the neighbors of any vertex belong to a finite graph $I(C_n)$ for large enough n. Also, G is 2-connected by Lemma 2.1(h), taking H to be C_1 when N is a radial net, and the 2-connected subgraph $\bigcup_{i=1}^{\infty} C_i$ when N is a ladder net. Therefore, by Lemma 4.7 or 4.8, we may assume that N is a tight net, and that if N is a ladder net then $u, v \in V(C_1) - V(D_1)$ and N is a tight ladder net with respect to $D_0 = \partial N[u, v]$.

Throughout this paper, a system of distinct representatives has been represented by a set of vertices. For our infinite limiting argument in this proof, we must also keep track of the specific assignment of vertices as representatives of different bridges. If P is a subgraph of G, $\mathcal{B}$ is the set of nontrivial P-bridges, and $\sigma : \mathcal{B} \to V(G)$ is an injection such that, for every $B \in \mathcal{B}$, $\sigma(B) \in V(B) \cap V(P)$, then we say that σ is an *assignment function* for $\mathcal{B}$.

The range of σ in $V(G)$ is an SDR S of the P-bridges in $\mathcal{B}$, and the existence of an SDR S guarantees the existence of σ. If we restrict the codomain of σ to S, then σ is a bijection.

If N is a radial net, then by Theorem 5.1 for $n = 1, 2, 3, \ldots$ we can find a forward C_1-Tutte path Q_n in $I(C_n)$ from u to some vertex in D_n (in this case, $D_n = C_n$) and an SDR of the Q_n-bridges in $I(C_n)$. If N is a ladder net, then by Theorem 5.2 for $n = 1, 2, 3, \ldots$ we can find a forward $(\partial N \cap C_n)$-Tutte path Q_n in $G_{0,n} = I(C_n)$ from u through v to some vertex in D_n and an SDR of the Q_n-bridges in $I(C_n)$. In either case, each Q_n is a forward $(\Delta \cap I(C_n))$-Tutte path in $I(C_n)$ from u through v. Let $\mathcal{T}_n$ be the collection of nontrivial Q_n-bridges in $I(C_n)$, and let $\alpha_n : \mathcal{T}_n \to V(G)$ be an assignment function for $\mathcal{T}_n$.

We now construct an infinite sequence of paths and assignment functions in G, which will converge to the path P and to an assignment function giving an SDR of the P-bridges, respectively. For subgraphs H, K of a graph F, let $\mathrm{NB}(H, F; K)$ denote the set of nontrivial H-bridges in F that are subgraphs of K.

Let $A_0 = \{1, 2, 3, \ldots\}$. Suppose $i \geq 1$ and we have an infinite set A_{i-1} of positive integers. Let P_i be a path in $I(C_{i+2})$ from u to a vertex of D_{i+2} such that, for an infinite set $A'_i \subseteq A_{i-1}$ of values of n, Q_n has P_i as an initial segment. Such a P_i exists because A_{i-1} is infinite, $I(C_{i+2})$ is finite and each Q_n, $n \in A_{i-1}$, uses a vertex of D_{i+2}. (There may be more than one such path, but we fix just one as P_i.) Suppose $n \in A'_i$; necessarily $n \geq i + 2$. Since Q_n is a forward path from u through v, P_i is also a forward path from u through v. By Lemma 5.3, if we define $\mathcal{B}_i = \mathrm{NB}(P_i, I(C_{i+2}); I(C_i))$, then also $\mathcal{B}_i = \mathrm{NB}(Q_n, I(C_n); I(C_i)) \subseteq \mathcal{T}_n$. Let $\sigma_i : \mathcal{B}_i \to V(G)$ be an assignment function for $\mathcal{B}_i$ such that, for an infinite set $A_i \subseteq A'_i$ of values of n, $\alpha_n|_{\mathcal{B}_i} = \sigma_i$. Such a σ_i exists because A'_i is infinite and $\mathcal{B}_i$ is finite.

Constructing A_i from A_{i-1} in this way gives an infinite sequence of sets $A_0 = \{1, 2, 3, \ldots\}$ $\supseteq A_1 \supseteq A_2 \ldots$. Suppose $i < j$ and choose $n \in A_j \subseteq A_i$. Since P_i is an initial segment of Q_n from u to D_{i+2} in $I(C_{i+2})$, and P_j is an initial segment of Q_n from u to D_{j+2} in $I(C_{j+2})$, we have $P_i \subseteq P_j$. Hence $P = \bigcup_{i=1}^{\infty} P_i$ is a 1-way infinite path; since each P_i is a forward path from u through v, P is also a forward path from u through v. Also, $\mathcal{B}_i = \mathrm{NB}(Q_n, I(C_n); I(C_i)) \subseteq \mathrm{NB}(Q_n, I(C_n); I(C_j)) = \mathcal{B}_j$. Hence $\mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \mathcal{B}_3 \subseteq \ldots \subseteq \mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{B}_i$. Furthermore, since $\sigma_i = \alpha_n|_{\mathcal{B}_i}$ and $\sigma_j = \alpha_n|_{\mathcal{B}_j}$, σ_j is an extension of σ_i. Hence we can define a function $\sigma : \mathcal{B} \to V(G)$ where $\sigma|_{\mathcal{B}_i} = \sigma_i$ for all i. Since each σ_i is injective, σ is injective, and its range S is an SDR of $\mathcal{B}$.

Now we show that each P-bridge J in G is finite. Suppose not. Then $J - V(P)$ is infinite, and by planarity it intersects infinitely many D_i. In particular, for some $i \geq 4$, $J - V(P)$ contains a path M from D_{i-3} to D_i; by terminating M at its first vertex in D_i, we may assume that $M \subseteq I(C_i)$. Choose $n \in A_i$. Then $M \subseteq I(C_i) - V(P) = I(C_i) - V(P_i) = I(C_i) - V(Q_n) \subseteq I(C_n) - V(Q_n)$, so M is contained in some Q_n-bridge J' in $I(C_n)$. Since Q_n is a Tutte path in $I(C_n)$, J' has at most three attachments on Q_n. But since $M \subseteq J'$, J' must have at least one attachment on Q_n in each of D_{i-3}, D_{i-2}, D_{i-1}, and D_i, a contradiction.

By Lemma 5.3, for each $i \geq 1$ we have $\mathcal{B}_i = \mathrm{NB}(P_i, I(C_{i+2}); I(C_i)) = \mathrm{NB}(P, G; I(C_i))$. Therefore, if $J \in \mathcal{B}$ then $J \in \mathcal{B}_i$ for some i and hence J is a P-bridge in G. Conversely, if J is a nontrivial P-bridge in G then, because J is finite, $J \subseteq I(C_i)$ for some i, and hence $J \in \mathcal{B}_i \subseteq \mathcal{B}$. So $\mathcal{B}$ is precisely the set of nontrivial P-bridges in G.

For each P-bridge J in G, with $J \subseteq I(C_i)$ and $n \in A_i$, Lemma 5.3 gives that the number of attachments of J on P in G is the same as the number of attachments of J on Q_n in

$I(C_n)$, namely at most three, and at most two if J contains an edge of Δ.

We conclude that P is a Δ-Tutte path in G, and S is an SDR of the P-bridges. $\square$

Theorem 5.5. *Let G be a 3-connected 2-indivisible infinite plane graph, and F the set of vertices of infinite degree in G. If $F = \emptyset$, then G has a net N; let $u = v \in V(G)$ be arbitrary if N is a radial net, and let $u, v \in V(\partial N)$ if N is a ladder net. If $F = \{f_1\}$, let $u = v = f_1$. If $F = \{f_1, f_2\}$, let $u = f_1$ and $v = f_2$. Then there exist a 1-way infinite Tutte path P in G from u through v and an SDR S of the P-bridges.*

Proof. If $F = \emptyset$ then we can use Lemma 4.4 to find a net in G, Lemma 4.7 to adjust a radial net so that $u = v \in V(C_1)$, Lemma 2.1(f) to show that G is $(3, C_1)$- or $(3, \partial N)$-connected, and Theorem 5.4 to find P and S.

So suppose that $|F| \geq 1$. Assume first that $G - F$ has no infinite block. By Lemma 4.6, $|F| = 2$ and G has a spanning 1-way infinite plane chain of finite circuit blocks $H = (B_1, b_1, B_2, b_2, \dots)$ such that $F \subseteq V(X_{B_1}) - \{b_1\}$. If $i > 1$ then by Theorem 3.2 we find an X_{B_i}-Tutte $b_{i-1}b_i$-path P_i in B_i and an SDR S_i of the P_i-bridges in B_i such that $b_{i-1} \notin S_i$. In B_1, by Theorem 3.2, we find an X_{B_1}-Tutte ub_1-path P_1 in B_1 through v and an SDR S_1 of the P_1-bridges in B_1 such that $u \notin S_1$. Let $P = \bigcup_{i=1}^{\infty} P_i$ and $S = \bigcup_{i=1}^{\infty} S_i$, then P is an X_H-Tutte path from u through v in H with an SDR S of the P-bridges in H. Now assume that $G - F$ has an infinite block. By Lemma 4.5, $G - F$ has a spanning subgraph H with a ladder net N so that H is $(3, \partial N)$-connected. Apply Theorem 5.4 to H to obtain a ∂N-Tutte path P from u through v in H with an SDR S of the P-bridges in H. In both cases, all edges of G not in H are incident with at least one vertex of F, and applying Remark 3.4 shows that P is a Tutte path in G and S is still an SDR of the P-bridges in G. $\square$

We now find 1-way infinite 2-walks, using a 1-way infinite Tutte path P provided by Theorem 5.4 or 5.5 as the skeleton of each 2-walk. To detour into each nontrivial P-bridge to visit the remaining vertices, we use some results of Gao, Richter and Yu from the proof of [8, Theorem 6]. Lemma 5.6 is the more technical result from which their Theorem 6 follows, modified to include the case of a trivial block. Gao, Richter and Yu also examine the structure of a bridge L of a Tutte subgraph in a circuit graph. Parts (a) and (b) of Lemma 5.7 correspond to when L has three or two vertices of attachment, respectively; part (b) is just a special case of our Lemma 2.3(b).

For a plane graph G, an *internal* 3*-cut* is a 3-cut A of G such that $G - A$ has a component vertex-disjoint from X_G. Let $N_G(v)$ denote the set of vertices adjacent to v in G.

Lemma 5.6 ([8, proof of Theorem 6])**.** *Let (G, X_G) be a circuit block, and let $x, y \in V(X_G)$ with $x \neq y$. Then G contains a closed 2-walk visiting x and y exactly once, such that every vertex visited twice is either in an internal 3-cut A of G, or in a 2-cut A of G with $A \subseteq V(X_G[x, y])$ or $A \subseteq V(X_G[y, x])$.*

Lemma 5.7 ([8, proof of Theorem 6])**.** *Let L be a plane graph.*

(a) Suppose L is $(3, \{a, b, c\})$-connected, where a, b, c are distinct vertices each appearing once on X_L, in that clockwise order. Then $L - \{b, c\}$ is a plane chain of circuit blocks $K = (a = b_0, B_1, b_1, \dots, b_{k-1}, B_k, b_k = d)$ where $N_L(b) \subseteq V(X_K[a, d]) \cup \{c\}$ and $N_L(c) \subseteq V(X_K[d, a]) \cup \{b\}$.

(b) Suppose L is $(3, X_L[a,b])$-connected, where $a \neq b$. Then $L - b$ is a plane chain of circuit blocks $K = (a = b_0, B_1, b_1, \ldots, b_{k-1}, B_k, b_k = d)$ where d is the neighbor of b in $X_L[a,b]$, and $N_L(b) \subseteq V(X_K[d,a])$.

Theorem 5.8. *Let G be a 2-indivisible infinite plane graph with a net $N = (C_1, C_2, C_3, \ldots)$. If N is a radial net, let $\Delta = C_1$. If N is a ladder net, let $\Delta = \partial N$. Assume that $u \in V(\Delta)$ and that G is $(3, \Delta)$-connected. Then G contains a 1-way infinite 2-walk beginning at u such that every vertex used more than once belongs to a 2- or 3-cut of G.*

Proof. By Theorem 5.4, G contains a 1-way infinite Δ-Tutte path P beginning at u and an SDR S of the P-bridges. Our walk W will traverse P, beginning at u, until we reach a vertex $a \in S$ that is a representative of a nontrivial P-bridge L.

If L has three attachments a, b, c, then L cannot contain an edge of Δ, so L is $(3, \{a,b,c\})$-connected by Lemma 2.1(g). Then $L - \{b,c\}$ is a plane chain of circuit blocks $K = (a = b_0, B_1, b_1, \ldots, b_{k-1}, B_k, b_k = d)$ as in Lemma 5.7(a), and we can apply Lemma 5.6 to each block B_i to get a 2-walk using b_{i-1} and b_i only once. Combining these 2-walks yields a 2-walk W_a in K. Each vertex used twice by W_a is (i) some b_i, $i \geq 1$, which is in a 3-cut $\{b_i, b, c\}$ of G, (ii) in an internal 3-cut of K, which is also a 3-cut of G, or (iii) in a 2-cut A of K contained in either $X_K[a,d]$ or $X_K[d,a]$, so that one of $A \cup \{b\}$ or $A \cup \{c\}$ is a 3-cut of G.

If L has two attachments a, b then necessarily $a, b \in V(\Delta)$, and one of $X_L[a,b]$ or $X_L[b,a]$, call it R, is a subpath of Δ. By Lemma 2.1(g), L is $(3, R)$-connected. Then $L - b$ is a plane chain of circuit blocks K as in Lemma 5.7(b) (or its mirror image), and we can apply Lemma 5.6 to each block, combining the resulting 2-walks to obtain a 2-walk W_a. Each vertex used twice by W_a is (i) some b_i, $i \geq 1$, which is in a 2-cut $\{b_i, b\}$ of G, (ii) in an internal 3-cut of K, which is also a 3-cut of G, or (iii) in a 2-cut A of K, which is either a 2-cut of G or such that $A \cup \{b\}$ is a 3-cut of G.

Splicing W_a into P for every representative $a \in S$ gives the required 2-walk W. The vertices used twice by W are the vertices used twice by each W_a and the vertices $a \in S$ themselves, each of which lies in a 2- or 3-cut of G. $\square$

The following result can be proved in the same way as Theorem 5.8, using Theorem 5.5 instead of Theorem 5.4. In fact, the proof is simpler, because now P has no bridges with two attachments. Our main result, Theorem 1.3, follows immediately from this.

Theorem 5.9. *Let G be a 3-connected 2-indivisible infinite plane graph, and let F be the set of vertices of infinite degree in G. If $F = \emptyset$, then G has a net N. Let $u \in V(G)$ be arbitrary if N is a radial net, and let $u \in V(\partial N)$ if N is a ladder net. Otherwise, let $u \in F$. Then G contains a 1-way infinite 2-walk beginning at u such that every vertex used more than once belongs to a 3-cut of G.*

6 Infinite Prisms

In this final section, we discuss spanning paths in prisms over infinite planar graphs. The *prism* over a graph G is the Cartesian product $G \square K_2$ of G with the complete graph K_2. In [3], we showed that prisms over bipartite circuit graphs and near-triangulations are hamiltonian. A *near-triangulation* is a finite plane graph where every face is a triangle, except for possibly the outer face, which is bounded by a cycle. Here we extend these results to infinite graphs by showing that if G is a 2-indivisible infinite analog of a bipartite circuit graph or near-triangulation then $G\square K_2$ has a 1-way infinite spanning path. Tracing this path in the original graph G gives a 1-way infinite 2-walk. So for these classes of graphs we can strengthen the existence of a 1-way infinite 2-walk in a somewhat different way from what we did in Theorems 5.8 and 5.9, where we controlled the location of the vertices used twice.

In the prism $G\square K_2$, we may identify G with one of its two copies in the prism. Let v be a vertex in G. In the prism, we let v denote the copy of the vertex in the graph that is identified with G, and we let v^* denote the other copy. We use the same notation for edges. An edge of the form vv^* is called a *vertical* edge. Below we often take $u \in V(G)$ and find a path in $G\square K_2$ beginning at u; then there is also a symmetric path beginning at u^*.

Our results for infinite bipartite graphs depend on the following result for finite graphs.

Lemma 6.1 ([3, Theorem 2.4])**.** *Let (G, X_G) be a bipartite circuit graph and let u, v be two distinct vertices in X_G. Then there is a hamilton cycle in $G\square K_2$ that uses the vertical edges at u and v.*

Theorem 6.2. *Let G be a 2-indivisible infinite bipartite plane graph with a net $N = (C_1, C_2, C_3, \dots)$. If N is a radial net, let $\Delta = C_1$. If N is a ladder net, let $\Delta = \partial N$. Assume that $u \in V(\Delta)$ and that G is $(3, \Delta)$-connected. Then $G\square K_2$ contains a 1-way infinite spanning path beginning at u.*

Proof. By Theorem 5.4, G contains a 1-way infinite Tutte path $P = v_1v_2v_3v_4\dots$ with $v_1 = u$. Let $P' = v_1v_1^*v_2^*v_2v_3v_3^*v_4^*v_4\dots$ be the 1-way infinite spanning path of $P\square K_2$ starting at u and using every vertical edge.

We modify P' to detour into $L\square K_2$ for each nontrivial P-bridge L, as follows. As in the proof of Theorem 5.8, L is decomposed using Lemma 5.7. Instead of using Lemma 5.6 to obtain 2-walks in circuit blocks that we splice together, we use Lemma 6.1 (extended to allow blocks that are edges) to find hamilton cycles in the prisms over circuit blocks, which we splice together at vertical edges (deleting those vertical edges) and then splice into P'. $\square$

The following result can be proved similarly, using Theorem 5.5 instead of Theorem 5.4.

Theorem 6.3. *Let G be a 3-connected 2-indivisible infinite bipartite plane graph, and let F be the set of vertices of infinite degree in G. If $F = \emptyset$, then G has a net N. Let $u \in V(G)$ be arbitrary if N is a radial net, and let $u \in V(\partial N)$ if N is a ladder net. Otherwise, let $u \in F$. Then $G\square K_2$ contains a 1-way infinite spanning path beginning at u.*

We also want to give results for infinite versions of triangulations or near-triangulations. These are based on the following result for finite graphs.

Lemma 6.4 ([3, Theorem 2.7])**.** *Let G be a finite near-triangulation and let u, v be two distinct vertices in X_G. Then there is a hamilton cycle in $G\square K_2$ that uses the vertical edges at u and v.*

Theorem 6.5. *Let G be a connected 2-indivisible infinite nicely embedded plane graph with a net $N = (C_1, C_2, C_3, \dots)$. If N is a radial net, let $\Delta = C_1$ and suppose that every face is bounded by a triangle except perhaps one face bounded by C_1. If N is a ladder net, let $\Delta = \partial N$ and suppose that every finite walk bounding a face is a triangle. Then for every $u \in V(\Delta)$, $G \square K_2$ contains a 1-way infinite spanning path beginning at u.*

Proof. We show that G is $(3, \Delta)$-connected. First notice that if N is a radial net and v is any vertex, or if N is a ladder net and $v \notin \partial N$, then $v \in I(C_i) - V(C_i)$ for some i. This follows from the definition of a net. Consequently, every edge incident with v is incident with two faces bounded by finite walks, since they are also faces of the finite graph $I(C_i)$.

Now suppose there is $T \subseteq V(G)$ with $|T| \leq 2$ so that $G - T$ has a component K with no vertex of Δ. Since G is connected, $|T| \geq 1$.

Suppose first that $|T| = 1$, with $T = \{t_1\}$. Then t_1 has neighbors v_0, v_1 consecutive in its rotation with $v_0 \notin V(K)$ and $v_1 \in V(K)$. Let f_0 be the face containing $v_0 t_1 v_1$. If N is a radial net, then f_0 is not bounded by C_1, since $v_1 \notin V(C_1)$. If N is a ladder net, then f_0 is a face bounded by a finite walk since $v_1 \notin \partial N$. In either case, f_0 is a triangle. Since t_1 is the only place where we can change between vertices in K and not in K, $f_0 = t_1 \dots v_0 t_1 v_1 \dots t_1$. Hence, f_0 has length at least 4, and has a repeated vertex, either of which is a contradication.

Now suppose that $|T| = 2$, with $T = \{t_1, t_2\}$. Then there is a sequence $v_0, v_1, \dots, v_k, v_{k+1}$ of neighbors of t_1 in clockwise order with $k \geq 1$, $v_1, v_2, \dots, v_k \in K$, and $v_0, v_{k+1} \notin K$; these vertices are distinct except possibly $v_0 = v_{k+1}$. Let f_0 be the face containing $v_0 t_1 v_1$ and f_k the face containing $v_k t_1 v_{k+1}$. By the same reasoning as in the case $|T| = 1$, f_0 is a triangle. Now since t_1 and t_2 are the only places we can change between vertices in K and not in K, $f_0 = t_i \dots v_0 t_1 v_1 \dots t_i$ where possibly $v_0 = t_i$, but $v_1 \neq t_i$. Since f_0 is a triangle we must have $v_0 = t_2$ and $f_0 = t_2 t_1 v_1 t_2$. By similar reasoning, $v_{k+1} = t_2$ and $f_k = t_2 t_1 v_k t_2$. Since we cannot have two edges $t_1 t_2$, the neighbors of t_1 must be exactly $t_2, v_1, v_2, \dots v_k$. Since, from above, $\{t_2\}$ cannot isolate a component with no vertex of Δ, $t_1 \in V(\Delta)$. But Δ is 2-regular, which means one of $v_1, v_2, \dots, v_k$ is in Δ, a contradiction.

Hence no such T exists, and G is $(3, \Delta)$-connected.

We now proceed as in the proof of Theorem 6.2, using Theorem 5.4, with Lemma 6.4 instead of Lemma 6.1. This works because our conditions on faces guarantee that every circuit graph that we consider as a subgraph of a bridge L of the 1-way infinite walk P is actually a near-triangulation. □

We can also prove the following, using Theorem 5.5 and Lemma 6.4. We say a face has *bounded extent* if it is bounded in the metric space sense.

Theorem 6.6. *Let G be a 3-connected 2-indivisible infinite plane graph with a nice embedding in which every finite walk bounding a face is a triangle. Let F be the set of vertices of infinite degree in G. If $F = \emptyset$, then G has a net N. Let $u \in V(G)$ be arbitrary if N is a radial net, and let $u \in V(\partial N)$ if N is a ladder net. Otherwise, let $u \in F$. Then $G \square K_2$ contains a 1-way infinite spanning path beginning at u.*

As a special case, suppose G is a locally finite 2-indivisible infinite plane graph in which every edge is incident with two faces of bounded extent, each bounded by a triangle. Then for every $u \in V(G)$, $G \square K_2$ contains a 1-way infinite spanning path beginning at u.

Proof. In the special case we can prove that the graph is 3-connected by the same reasoning we used in the previous theorem. Therefore there is a net, which must be a radial net, because for a ladder net N the edges of ∂N fail the condition. The embedding must be nice because the side of any cycle with finitely many vertices partitions into a finite number of triangles, each of which is bounded in the metric space sense. □

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