One-way infinite 2-walks in planar graphs

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Abstract

We prove that every 3-connected 2-indivisible infinite planar graph has a 1-way infinite 2-walk. (A graph is 2-indivisible if deleting finitely many vertices leaves at most one infinite component, and a 2-walk is a spanning walk using every vertex at most twice.) This improves a result of Timar, which assumed local finiteness. Our proofs use Tutte subgraphs, and allow us to also provide other results when the graph is bipartite or an infinite analog of a triangulation: then the prism over the graph has a spanning 1-way infinite path.

1 Introduction

For terms not defined in this paper, see [20]. All graphs are simple (having no loops or multiple edges) and may be infinite, unless we explicitly state otherwise.

A cutset in a graph $G$ is a set $S \subseteq V(G)$ such that $G - S$ is disconnected. A $k$-cut is a cutset $S$ with $|S| = k$. A graph is $k$-connected if it has at least $k + 1$ vertices and no cutset $S$ with $|S| < k$. The connectivity of a graph is the smallest $k$ for which it is $k$-connected.

The first major result on the existence of hamilton cycles in graphs embedded in surfaces was by Whitney [21] in 1931, who proved that every 4-connected finite planar triangulation is hamiltonian. Tutte extended this to all 4-connected finite planar graphs in 1956 [18], and gave another proof in 1977 [19]. Tutte actually proved a more general result, using subgraphs which have since been called “Tutte subgraphs” (defined in Section 3).

To extend these results to infinite graphs, one can look for infinite spanning paths. We say that $v_1v_2v_3 \cdots$ is a 1-way infinite path, and $\cdots v_{-2}v_{-1}v_0v_1v_2 \cdots$ is a 2-way infinite path, if each $v_i$ is a distinct vertex and consecutive vertices are adjacent.

If deleting finitely many vertices in an infinite graph leaves more than one (two) infinite component(s), then the graph has no 1-way (2-way) infinite spanning path. Nash-Williams

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defined a graph $G$ to be $k$-indivisible, for a positive integer $k$, if, for any finite $S \subseteq V(G)$, $G - S$ has at most $k - 1$ infinite components. He conjectured [12, 13] that every 4-connected 2-indivisible (3-indivisible) infinite planar graph contains a 1-way (2-way) infinite spanning path. The 1-way infinite path conjecture was proved by Dean, Thomas, and Yu [6] in 1996, and Xingxing Yu established the 2-way infinite path conjecture [22, 23, 24, 25, 26].

For connectivity less than 4, we must be more flexible in the types of spanning subgraphs we wish to find, since there are 3-connected finite planar graphs with no hamilton path. Let $k$ be a positive integer. A $k$-tree is a spanning tree with maximum degree at most $k$. A $k$-walk in a finite graph is a closed spanning walk passing through each vertex at most $k$ times. In a finite graph, a 2-tree is a hamilton path and a 1-walk is a hamilton cycle.

Barnette [1] showed that every 3-connected finite planar graph contains a 3-tree. Gao and Richter [7] later showed that every 3-connected finite planar graph contains a 2-walk. Gao, Richter, and Yu [8, 9] refined this to give information about the locations of vertices included twice in the 2-walk. The 2-walk results strengthen Barnette’s result, because Jackson and Wormald [10] showed that in a finite graph a $k$-walk provides a $(k + 1)$-tree.

It is natural to ask if these results generalize to infinite graphs. In 1996, Jung [11] proved that every locally finite 3-connected infinite plane graph with no vertex accumulation points has a 3-tree. A graph is locally finite if every vertex has finite degree.

A 1-way (2-way) infinite walk is a sequence of vertices $v_1v_2v_3\cdots(v_{-2}v_{-1}v_0v_1v_2\cdots)$ where consecutive vertices are adjacent. A 1-way (2-way) infinite $k$-walk is a 1-way (2-way) infinite spanning walk that passes through each vertex at most $k$ times. A graph with a 1-way or 2-way infinite $k$-walk must be infinite.

In his doctoral dissertation, Timar proved the following two theorems.

**Theorem 1.1** (Timar [17, Theorem II.2.5]). Let $G$ be a locally finite 3-connected 2-indivisible infinite planar graph. Then $G$ has a 1-way infinite 2-walk.

**Theorem 1.2** (Timar [17, Theorems II.3.3, II.4.3]). Let $G$ be a locally finite 3-connected 3-indivisible infinite planar graph. Then $G$ has a 2-way infinite 2-walk.

Our main result extends Theorem 1.1 by not requiring local finiteness and by controlling the location of vertices used twice.

**Theorem 1.3.** Let $G$ be a 3-connected 2-indivisible infinite planar graph. Then $G$ has a 1-way infinite 2-walk for which any vertex used twice is in a 3-cut of $G$.

Our methods differ from those of Timar. To avoid local finiteness we use structural results, similar to those in [6], for graphs with infinite degree vertices. To build the skeleton of the 2-walk and control the location of vertices that are used twice, we use Tutte subgraphs in a way similar to [8]. We follow a systematic approach to using Tutte subgraphs that we have developed in a survey paper [4], in preparation. Our methods also provide other results when $G$ is bipartite or an infinite analog of a triangulation (Theorems 6.3 and 6.6).

Timar [17, Lemma I.2.14] verified that Jackson and Wormald’s proof that a $k$-walk provides a $(k + 1)$-tree applies for infinite graphs, and so Theorem 1.3 allows us to prove a result similar to Jung’s. We drop local finiteness, but add 2-indivisibility as both a hypothesis and a conclusion.
Corollary 1.4. Let $G$ be a 3-connected 2-indivisible infinite planar graph. Then $G$ has a 2-indivisible 3-tree.

We also make the following natural conjecture.

Conjecture 1.5. Let $G$ be a 3-connected 3-indivisible infinite planar graph. Then $G$ has a 2-way infinite 2-walk.

Proving Conjecture 1.5 would likely require significant work along the lines of [22, 23, 24, 25, 26].

Section 2 includes some additional definitions and lemmas, especially for connectivity. Section 3 addresses Tutte subgraph techniques. Section 4 discusses structural results for 3-connected 2-indivisible infinite planar graphs. Section 5 proves the main result, and Section 6 gives stronger theorems for bipartite graphs and analogs of triangulations.

2 Definitions and Connectivity

If $G$ is a connected finite plane graph, $X_G$ denotes the outer walk of $G$, the closed walk bounding the infinite face. If $G$ is 2-connected, then we also call $X_G$ the outer cycle of $G$. We use $X_G$ to denote both a walk and a subgraph; if $G$ is isomorphic to $K_2$ or $K_1$, then the subgraph $X_G$ is just $G$ itself.

A $uv$-path $P$ is a path from $u$ to $v$; $P^{-1}$ denotes the reverse $vu$-path. If $P$ is a (possibly infinite) path and $x, y \in V(P)$, then $P[x, y]$ denotes the subpath of $P$ from $x$ to $y$. Given a closed walk $W$ in a plane graph bounding an open disk (such as a cycle or facial boundary walk), the subwalk $W[x, y]$ clockwise from $x$ to $y$ is well-defined provided each of $x$ and $y$ occurs exactly once on $W$. If $x = y$, then $W[x, y]$ or $P[x, y]$ means the single vertex $x = y$.

A block is a 2-connected graph or a graph isomorphic to $K_2$ or $K_1$. A block of a graph is a maximal subgraph that is a block. Every graph has a unique decomposition into edge-disjoint blocks. A block isomorphic to $K_2$ is a trivial block. A vertex $v$ of a graph is a cutvertex if $\{v\}$ is a cutset of the graph.

Let $G$ be a connected graph, and $n \geq 0$ an integer. Suppose that $G$ has finite blocks $B_1, B_2, \ldots, B_n$ and vertices $b_0, b_1, b_2, \ldots, b_{n-1}, b_n$ in $G$ such that $G = b_0$ (if $n = 0$) or $b_0 \in V(B_1) - \{b_1\}$, $b_n \in V(B_n) - \{b_{n-1}\}$, $b_i \in V(B_i) \cap V(B_{i+1})$ for $i = 1, 2, \ldots, n-1$, and $G = \bigcup_{i=1}^{n} B_i$. We say that $G$ is a chain of blocks (some sources call this a linear graph) and that $(b_0, B_1, b_1, B_2, b_2, \ldots, b_{n-1}, B_n, b_n)$ is a block-decomposition of $G$. In this case, $b_1, b_2, \ldots, b_{n-1}$ are precisely the cutvertices of $G$. A chain of blocks is a plane chain of blocks if it is embedded in the plane so that no block is embedded inside any other block. In a plane chain of blocks $G$, any internal face in a block will be a face of $G$.

If $G$ has finite blocks $B_1, B_2, \ldots$ and vertices $b_1, b_2, \ldots$ such that $b_i \in V(B_i) \cap V(B_{i+1})$ for every $i$, $G = \bigcup_{i=1}^{\infty} B_i$, and $G$ is embedded in the plane so that no block is embedded inside any other block, we say that $G$ is a 1-way infinite plane chain of finite blocks. (We will have no need to specify an initial vertex $b_0$.) In this case, we define $X_G$ to be the 2-way infinite walk traversing the outer face of $G$ in the clockwise direction.

If $P$ is an $ab$-path in $G$, then the chain of blocks along $P$ in $G$ is the minimal union $K$ of blocks of $G$ that contains $P$ (or $K = a$ if $a = b$). If $K = G$ we say that $G$ is a chain of
blocks along \( P \). We can write \( K = (b_0, B_1, b_1, B_2, b_2, \ldots, b_{n-1}, B_n, b_n) \) where \( n \geq 0, b_0 = a, \) and \( b_n = b; \) then \( b_0, b_1, \ldots, b_n \) are distinct vertices of \( P \) and each \( B_i \) contains an edge of \( P \).

A bridge of \( H \), or \( H \)-bridge, in \( G \) is either (a) an edge of \( E(G) - E(H) \) with both ends in \( H \) (a trivial bridge), or (b) a component \( C \) of \( G - V(H) \) together with all of the edges with one end in \( C \) and the other in \( H \). If \( J \) is an \( H \)-bridge in \( G \), then \( E(J) \cap E(H) = \emptyset, \) \( V(H) \cap V(J) \) is the set of attachments of \( J \) on \( H \), and \( V(J) - V(H) \) is the set of internal vertices of \( J \) as an \( H \)-bridge. Let \( A_G(H) \) be the set of attachments of all \( H \)-bridges in \( G \), or in other words, the vertices of \( H \) incident with an edge of \( E(G) - E(H) \).

We often use a property slightly weaker than being \( k \)-connected. Let \( G \) be a graph, \( k \) a positive integer, and \( \emptyset \neq S \subseteq V(G) \). We say \( G \) is \( k \)-connected relative to \( S \), or \((k,S)\)-connected, if for every \( T \subseteq V(G) \) with \( |T| < k \), every component of \( G - T \) contains at least one vertex of \( S \). For a subgraph \( H \) of \( G \) we say \( G \) is \((k,H)\)-connected if it is \((k,V(H))\)-connected. Similar definitions have been used in earlier papers such as [15, 22], but ours differs in that we do not require \( G \) to be connected or \( T \) to be a cutset.

Our definition has a number of consequences that we use later. We omit the straightforward proofs. Part (c) may be regarded as an alternative (perhaps more intuitive) definition; its proof uses (b) and Menger’s Theorem, and it helps to prove later parts.

**Lemma 2.1.** Let \( G \) be a graph, \( k \) a positive integer, and \( \emptyset \neq S \subseteq V(G) \).

(a) If \( S = V(G) \) then \( G \) is always \((k,S)\)-connected. If \( S \neq V(G) \) and \( G \) is \((k,S)\)-connected then \( |S| \geq k \).

(b) \( G \) is \((k,S)\)-connected if and only if the graph obtained from \( G \) by adding a new vertex \( r \) adjacent to all vertices of \( S \) has no cutset \( T \) with \( |T| < k \) and \( r \notin T \).

(c) \( G \) is \((k,S)\)-connected if and only if for every \( v \in V(G) - S \) there are \( k \) paths, disjoint except at \( v \), from \( v \) to \( S \) in \( G \).

(d) If \( G \) is \((k,S)\)-connected, \( G \) is a spanning subgraph of \( G' \), \( 1 \leq k' \leq k \), and \( S \subseteq S' \subseteq V(G) \), then \( G' \) is \((k',S')\)-connected.

(e) Adding or deleting edges with both ends in \( S \) does not affect whether or not \( G \) is \((k,S)\)-connected.

(f) Suppose \( G \) is \( k \)-connected and \( S \subseteq V(G) \) with \( |S| \geq k \). Let \( H \) be the union of \( S \), zero or more \( S \)-bridges in \( G \), and an arbitrary set of edges with both ends in \( S \). Then \( H \) is \((k,S)\)-connected. As a special case, \( G \) is \((k,S)\)-connected.

(g) Suppose \( G \) is \((k,S)\)-connected, and \( H \) is a subgraph of \( G \). Let \( S_H = A_G(H) \cup (S \cap V(H)) \). Then \( H \) is \((k,S_H)\)-connected.

(h) Suppose \( G \) is \((k,S)\)-connected and \( H \) is a subgraph of \( G \) with \( S \subseteq V(H) \). If \( 0 \leq k' \leq k \) and \( H \) is \( k' \)-connected, then \( G \) is also \( k' \)-connected. Moreover, if \( H \cong K_k \) and \( V(H) \neq V(G) \) then \( G \) is \( k \)-connected.

(i) Construct \( G' \) by adding to \( G \) a set \( R \) of new vertices, each adjacent only to vertices in \( R \cup S \). Then \( G \) is \((k,S)\)-connected if and only if \( G' \) is \((k,R \cup S)\)-connected.

To prove a statement for \( 3 \)-connected finite planar graphs, one often proves it for the following larger class of graphs. A circuit graph is an ordered pair \((G,C)\) where \( G \) is a finite graph, \( C \) is a cycle in \( G \) that bounds a face in some plane embedding of \( G \), and \( G \) is
(3, C)-connected. By Lemma 2.1(h), G is automatically 2-connected. Frequently C is the outer cycle of G. Barnette [1] originally defined a circuit graph as the subgraph inside a cycle in a 3-connected plane graph — this definition can be shown to be equivalent to ours using Lemma 2.1(f), (h) and (i). Also, by Lemma 2.1(f), if G is a 3-connected finite plane graph and C is any facial cycle of G, then (G, C) is a circuit graph.

It is convenient to define a finite plane graph G to be a circuit block if either G is an edge, or (G, XG) is a circuit graph. A (possibly 1-way infinite) plane chain of circuit blocks has the obvious meaning.

Lemmas 2.2 and 2.3 below give some useful inductive properties of circuit graphs, or more general (3, S)-connected plane graphs. Lemma 2.2 follows from Lemma 2.1(d) and (g), and generalizes [7, Lemma 2]. We use it frequently, often without explicit mention. Lemma 2.3(b) generalizes [7, Lemma 3].

**Lemma 2.2.** Suppose G is a (3, S)-connected plane graph, and C is a cycle in G with no vertex of S strictly inside C. If the subgraph H of G consisting of C and everything inside C is finite, then (H, C) is a circuit graph.

**Lemma 2.3.** Suppose P is a path in a finite connected plane graph G and P ⫋ XG.

(a) If G is (3, P)-connected then G is a plane chain of circuit blocks along P.

(b) Suppose that c ∈ V(XG) − V(P). If G is (3, P ∪ {c})-connected then G − c is a plane chain of circuit blocks along P.

**Proof.** Let K be the chain of blocks in G (for (a)) or G − c (for (b)) along P. The connectivity requirement means that in (a) there are no K-bridges in G, and in (b) the only (K ∪ {c})-bridges in G are edges incident with c. By Lemma 2.2 all nontrivial blocks of K are circuit graphs. The results follow.

### 3 Standard Pieces and Systems of Distinct Representatives

To prove that every 4-connected finite planar graph is hamiltonian, Tutte used what are now known as “Tutte subgraphs.” In this section, we describe some Tutte subgraph results, including what we call “standard pieces,” that we will use frequently.

Let X be a subgraph (usually given in advance) of a graph G, and let T be another subgraph (often a path or a cycle). Then T is an X-Tutte subgraph (or X-Tutte path or X-Tutte cycle, if appropriate) of G if:

(i) every bridge of T in G has at most three attachments, and

(ii) every bridge of T in G that contains an edge of X has at most two attachments.

Sometimes no X is given and only (i) holds; then we simply say that T is a Tutte subgraph.

Our overall strategy for constructing a 1-way infinite spanning 2-walk is to build a 1-way infinite Tutte path P, and then detour into the P-bridges to pick up all remaining vertices. To build P we use a similar strategy to Dean, Thomas and Yu [6]. We build Tutte paths in finite parts of the graph, and then use the argument of König’s Lemma to find finite paths.
converging to the infinite path \( P \). To avoid using a vertex more than twice when we detour into the \( P \)-bridges, we use an idea from Gao, Richter and Yu [8]. We designate an entry point for each nontrivial bridge so that a vertex is used as the entry point of at most one bridge.

The entry points thus form a system of distinct representatives, or SDR, for the nontrivial \( P \)-bridges. Formally an SDR is an injective mapping from a set of subgraphs of a graph \( G \) to a set of vertices of \( G \) so that each representative vertex belongs to its subgraph. We frequently refer to an SDR simply by its set of representatives. We never need to enter trivial bridges, so for a subgraph \( P \), an SDR of the \( P \)-bridges means an SDR of the nontrivial \( P \)-bridges.

Combining the ideas from [6] and [8] is not straightforward; making these work together is one of the main new contributions of this paper. First, finding the finite Tutte paths so that we also have an SDR of their bridges can be complicated — the most technical parts of the proofs in Theorems 5.1 and 5.2 are when we need to join Tutte subgraphs together but also maintain an SDR of the bridges of their union. The general idea of \((3, S)\)-connectedness helps here, allowing us to use arguments that would be awkward to formulate just in terms of circuit graphs. Second, when we use a König's Lemma argument to get finite Tutte paths converging to an infinite Tutte path \( P \), in Theorem 5.5, we also need the SDRs for the finite paths to converge to an SDR for \( P \). This requires a careful technical argument. Moreover, our methods allow us to obtain the stronger results in Section 6.

Throughout this paper, we use a general framework for arguments involving Tutte subgraphs that we have developed in [4]; an early version appeared in [2]. Tutte subgraph arguments are often very technical and hard to follow; our framework attempts to clarify them by emphasizing certain fundamental ideas. Two key concepts are that Tutte subgraphs are constructed by piecing together smaller Tutte subgraphs, and that many of these smaller Tutte subgraphs are obtained using arguments that occur repeatedly.

First we state a simple consequence of the definitions of a bridge, Tutte subgraph, and SDR. For a similar result (but without SDRs), see [14, (2.3)].

**Lemma 3.1 (Jigsaw Principle).** Suppose \( G \) is the edge-disjoint union of \( G_1, G_2, \ldots, G_k \). Suppose each \( G_i \) has a subgraph \( X_i \) and an \( X_i \)-Tutte subgraph \( T_i \) with an SDR \( S_i \) of the \( T_i \)-bridges in \( G_i \). Suppose that \( V(T_i) \cap V(T_j) = V(G_i) \cap V(G_j) \) and \( S_i \cap S_j = \emptyset \) for \( i \neq j \). If \( T = \bigcup_{i=1}^k T_i, \ X = \bigcup_{i=1}^k X_i, \) and \( S = \bigcup_{i=1}^k S_i \) then \( T \) is an \( X \)-Tutte subgraph of \( G \) with SDR \( S \) of the \( T \)-bridges. Moreover, each \( T \)-bridge in \( G \) is a \( T_i \)-bridge in \( G_i \) for some \( i \).

We can think of a subgraph \( G_i \) with its \( X_i \)-Tutte subgraph \( T_i \) and SDR \( S_i \) as a piece of a jigsaw puzzle; we can join pieces if they “fit together” correctly. Usually at least one piece is found by induction. Other pieces are constructed using very standard arguments (here, derived from Theorem 3.2) which form our standard piece lemmas, or just standard pieces. Each says that a graph with certain properties has a Tutte subgraph of a certain type.

We need three standard pieces involving SDRs, which we call SDR Standard Piece \( k \), or SDR \( SPk \), for \( k = 1, 2, 3 \). As a mnemonic, \( k \) denotes the number of components in the Tutte subgraph. Thomas and Yu gave related results without SDRs, combined into a single theorem [14, (2.4)]. To deal with SDRs it helps to keep the three situations separate; then the reader also knows exactly which is being applied. We postpone the proofs until the end of this section. The figures show an \( X \)-Tutte subgraph \( T \) having SDR \( S \) with \( X \) as dashed
edges (green, if color is shown), $T$ as solid edges and circled isolated vertices (red), and vertices known not to be in $S$ as solid vertices (blue). Solid vertices are used to make SDRs pairwise disjoint when applying the Jigsaw Principle.

• **SDR Standard Piece 1 (SDR SP1)**

  **Given:** A plane chain of circuit blocks $K = (a = b_0, B_1, b_1, B_2, \ldots, b_{n-1}, B_n, b_n = b)$ with $n \geq 0$, and $u \in V(X_K)$.

  **Then there exist:** An $X_K$-Tutte $ab$-path $P$ through $u$ in $K$ and an SDR $S$ of the $P$-bridges with $a \notin S$.

  ![Figure 1: SDR Standard Piece 1](image)

• **SDR Standard Piece 2 (SDR SP2)**

  **Given:** A connected finite plane graph $K$ and $a, b, c \in V(X_K)$ such that (i) $X_K[a, b]$ is a path avoiding $c$, and (ii) $K - c$ is a plane chain of circuit blocks $(a = b_0, B_1, b_1, B_2, \ldots, b_{n-1}, B_n, b_n = b)$ with $n \geq 0$.

  **Then there exist:** An $ab$-path $P$ avoiding $c$ such that $P \cup \{c\}$ is an $X_K[a, b]$-Tutte subgraph of $K$, and an SDR $S$ of the $(P \cup \{c\})$-bridges with $a, c \notin S$.

  ![Figure 2: SDR Standard Piece 2](image)

• **SDR Standard Piece 3 (SDR SP3)**

  **Given:** A connected finite plane graph $K$ and $a, b, c, d \in V(X_K)$ such that (i) $c \neq d$, (ii) $X_K[a, b]$ is a path avoiding $c$ and $d$, and (iii) $K$ is $(3, X_K[a, b] \cup \{c, d\})$-connected.

  **Then there exist:** An $ab$-path $P$ avoiding $c$ and $d$ such that $P \cup \{c, d\}$ is an $X_K[a, b]$-Tutte subgraph of $K$, and for each $x \in \{c, d\}$ an SDR $S$ of the $(P \cup \{c, d\})$-bridges with $a, x \notin S$.
Lemma 2.2 to the nontrivial blocks we see that each \( B \) in \( \mathcal{B} \) at most one such bridge; if it exists, call it \( S \).

Proof of SDR Standard Pieces 1, 2, and 3.

For SDR SP1, if \( a = b \), set \( P = a = b \) and \( S = \emptyset \). Otherwise, for each \( B_i \), by Theorem 2.2, we find an \( X_{B_i} \)-Tutte \( b_{i-1}b_i \)-path \( P_i \) and an SDR \( S_i \) of the \( P_i \)-bridges such that \( b_{i-1} \notin S_i \); if \( u \in V(B_i) - \{b_{i-1}, b_i\} \), we choose \( P_i \) to go through \( u \). By the Jigsaw Principle, \( P = \bigcup_{i=1}^n P_i \) and \( S = \bigcup_{i=1}^n S_i \) are as desired.

For SDR SP2, apply SDR SP1 to \( H = K - c \) to obtain an \( ab \)-path \( X_{H} \)-Tutte path \( P \) in \( H \) and SDR \( S \) of the \( P \)-bridges in \( H \) with \( a \notin S \). Every nontrivial \( P \cup \{c\} \)-bridge \( J \) in \( K \) is a \( P \)-bridge in \( H \) unless it contains edges incident with \( c \); in that case \( J \) must consist of a \( P \)-bridge \( J' \) that uses an edge of \( X_H[b,a] \), and edges incident with \( c \). Then \( J \) has three attachments (two from \( J' \), and \( c \)) and we may reassign the representative of \( J' \) to \( J \). Hence \( P \cup \{c\} \) and \( S \) (with some reassignment) are as required.

Finally, for SDR SP3, let \( H = (b_0 = a, B_1, b_1, B_2, \ldots, b_{n-1}, B_n, b_n = b) \) be the plane chain of blocks along \( X_K[a,b] \) in \( K - \{c,d\} \). Since \( K \) is \( (3, X_K[a,b] \cup \{c,d\}) \)-connected, by applying Lemma 2.2 to the nontrivial blocks we see that each \( B_i \) is a circuit block.

Every nontrivial \( (H \cup \{c,d\}) \)-bridge has at most one attachment in \( H \), because \( H \) is a chain of blocks, but at least three attachments because \( K \) is \( (3, X_K[a,b] \cup \{c,d\}) \)-connected, so it must have \( c, d \) and exactly one vertex of \( H \) as attachments. By planarity there can be at most one such bridge; if it exists, call it \( J \) and let \( u \) be its attachment in \( H \).

By SDR SP1 there is an \( X_H \)-Tutte \( ab \)-path \( P \) in \( H \), through \( u \) if it exists, with an SDR \( S' \) of the \( P \)-bridges such that \( a \notin S' \). Consider the nontrivial \( (P \cup \{c,d\}) \)-bridges in \( K \). The only such bridge that can contain both \( c \) and \( d \) is \( J \). If \( J \) exists, we choose \( y \) as its representative, where \( \{x,y\} = \{c,d\} \), and set \( S = S' \cup \{y\} \); otherwise set \( S = S' \). The argument from the proof of SDR SP2 for nontrivial \( P \cup \{c\} \) bridges applies here to nontrivial bridges with exactly one of \( c \) or \( d \) as an attachment. Hence \( P \) and \( S \) are as desired.

Figure 3: SDR Standard Piece 3 (one of \( c \) or \( d \) can be solid)
Remark 3.4. One idea from the proofs of SDR SP2 and SP3 is used often. If we delete a vertex \( x \) from a graph, find a Tutte subgraph and an SDR in what remains, and then add \( x \) back, \( x \) may become a new attachment for some bridges. We ensure that each such bridge previously had only two attachments, so with \( x \) there are still only three. Each such bridge already has a representative, so we do not need to use \( x \) as its representative.

4 Structural Results

In this section, we give some results about the structure of 3-connected 2-indivisible infinite planar graphs. In these graphs, we find either an infinite plane chain of blocks or a structure called a net. Then, in Section 5, we use these structural results to build our 1-way infinite 2-walks.

If \( G \) is a 2-indivisible infinite plane graph and \( C \) is a cycle in \( G \), then \( C \) divides the plane into two closed sets, exactly one of which contains finitely many vertices. Let \( I(C) \) (or \( I_G(C) \)) denote the subgraph of \( G \) consisting of all vertices and edges of \( G \) inside that closed set containing finitely many vertices. Note that \( C \subseteq I(C) \). Dean, Thomas, and Yu [6] defined a net in \( G \) to be a sequence of cycles \( N = (C_1, C_2, C_3, \ldots) \) such that

1. \( I(C_i) \) is a subgraph of \( I(C_{i+1}) \) for all \( i = 1, 2, 3, \ldots \),
2. \( \bigcup_{i=1}^\infty I(C_i) = G \), and either
3. \( C_1, C_2, C_3, \ldots \) are pairwise disjoint, or
3’. for every \( i = 1, 2, 3, \ldots \), the graph \( C_i \cap C_{i+1} \) is a non-empty path, it is a subgraph of \( C_{i+1} \cap C_{i+2} \), and no endpoint of \( C_i \cap C_{i+1} \) is an endpoint of \( C_{i+1} \cap C_{i+2} \).

If 3 holds we say that \( N \) is a radial net, and if 3’ holds we say that \( N \) is a ladder net. A graph with a net is locally finite, because for every vertex \( v \) there is some \( i \) such that \( v \) and all its neighbors belong to \( I(C_i) \).

In [23], Yu said that an infinite plane graph \( G \) is nicely embedded or is a nice embedding if, for any cycle \( C \) in \( G \) for which \( I(C) \), the finite side of \( C' \), is defined, \( I(C) \) is contained in the closed disk bounded by \( C \). In a nice embedding, the intuitive idea of the “inside” of a cycle \( C \) coincides with \( I(C) \), which is why the notation \( I(C) \) is used. The following lemma is [23, (2.1)].

Lemma 4.1. Let \( G \) be an infinite plane graph, and suppose \( G \) has a sequence of cycles \( (C_1, C_2, C_3, \ldots) \) such that \( I(C_i) \) is a subgraph of \( I(C_{i+1}) \) for all \( i = 1, 2, 3, \ldots \), and \( \bigcup_{i=1}^\infty I(C_i) = G \). Then for any facial cycle \( C \) of \( G \), \( G \) has a nice embedding in which \( C \) is also a facial cycle.

By Lemma 4.1, if a infinite plane graph has a net, then the graph has a nice embedding. In this paper, we will always assume that such a graph is nicely embedded in the plane.

Let \( N = (C_1, C_2, C_3, \ldots) \) be a net in a 2-indivisible plane graph \( G \). The boundary of \( N \), denoted by \( \partial N \), is the graph \( \bigcup_{i=1}^\infty (C_i \cap C_{i+1}) \). If \( N \) is a radial net, \( \partial N = \emptyset \), and if \( N \) is a ladder net, then \( \partial N \) is a 2-way infinite path. If \( N \) is a ladder net, we will assign an orientation \( \overrightarrow{\partial N} \) to \( \partial N \) such that \( G - V(\partial N) \) is to the right of every edge in \( \overrightarrow{\partial N} \). For \( i = 1, 2, 3, \ldots \), let \( D_i \) be the graph obtained from \( C_i \) by deleting \( C_i \cap C_{i+1} \) except its endpoints. If \( N \) is
a radial net, $D_i = C_i$, and if $N$ is a ladder net, then $D_i$ is a path with both ends in $\partial N$ and otherwise disjoint from $\partial N$. If $N$ is a ladder net, and $C_1 - V(D_1)$ contains at least one vertex, let $D_0 = C_0$ be a subpath of $C_1 - V(D_1)$. Otherwise, set $D_0 = C_0 = \emptyset$ (but we will never see this case in this paper). We say that $N$ is tight if
1. $I(C_1) = C_1$ if $N$ is a radial net,
2. $C_1 \cap C_2$ is either empty or contains at least one edge, and
3. for every $i = 1, 2, 3, \ldots$, every $D_{i+1}$-bridge in $I(C_{i+1}) - V(I(C_i))$ has at most one attachment.

If $N$ is a tight ladder net and every $D_1$-bridge in $I(C_1) - V(D_0)$ has at most one attachment, we say that $N$ is tight with respect to $D_0$. Note that the definition of a tight ladder net in Dean, Thomas, and Yu [6] also requires that $I(C_1) = C_1$, which would imply that it is tight with respect to $D_0$; we will not need this.

The following three lemmas are from [6]. The first immediately precedes their (1.2), and the others are their (2.1) and (2.2), respectively.

**Lemma 4.2.** If $G$ is a 3-connected planar graph and $X$ is any finite subset of $V(G)$, then $G - X$ has a finite number of components.

**Lemma 4.3.** Let $G$ be a 2-indivisible infinite plane graph such that the deletion of any finite set of vertices in $G$ results in a finite number of components. Then $G$ has at most two vertices of infinite degree.

**Lemma 4.4.** Let $G$ be a locally finite 2-connected 2-indivisible infinite plane graph. Then $G$ has a net.

How we find our 1-way infinite Tutte path and SDR of its bridges in every 3-connected 2-indivisible infinite plane graph will depend on whether or not the graph contains a net, and, if so, what kind of net the graph contains. Let $G$ be a 3-connected 2-indivisible infinite plane graph, and let $F$ be the set of vertices of infinite degree in $G$. By Lemmas 4.2 and 4.3, $|F| \leq 2$. If $F = \emptyset$, then Lemma 4.4 guarantees that $G$ has a net. If $|F| \geq 1$, then $G$ contains a spanning subgraph $H$ such that either
1. $H$ contains a ladder net (Lemma 4.5), or
2. $H$ is a 1-way infinite plane chain of circuit blocks (Lemma 4.6).

Thus there are essentially three types of subgraphs to consider: radial nets, ladder nets, and 1-way infinite plane chains of circuit blocks. We will deal with radial nets and ladder nets in Theorem 5.1 and Theorem 5.2, respectively, and then combine the three cases in Theorem 5.5.

We first need two structural lemmas about 2-indivisible infinite plane graphs containing at least one vertex of infinite degree. The first is similar to [6, (2.3)].

**Lemma 4.5.** Let $G$ be a 3-connected 2-indivisible infinite plane graph, let $F$ be the set of vertices of infinite degree in $G$, and assume that $|F| \geq 1$ and that $G - F$ has an infinite block. Then there exists a 2-connected 2-indivisible infinite subgraph $H$ of $G$ such that
1. $H$ contains a ladder net $N$,
2. \( F \subseteq V(\partial N) \),
3. \( H \) is \((3, \partial N)\)-connected, and
4. the only \( H \)-bridges in \( G \) are edges incident with at least one vertex in \( F \) (so \( H \) is a spanning subgraph of \( G \)).

Proof. By Lemmas 4.2 and 4.3, \(|F| \leq 2\). By \([6, (2.3)]\), \( G \) contains a 2-connected 2-indivisible infinite subgraph \( H' \) such that \( H' \) contains a net \( N' \) and \( F \subseteq V(\partial N') \). Since \( F \neq \emptyset \), \( N' \) must be a ladder net. Also by \([6, (2.3)]\), every \( H' \)-bridge of \( G \) is finite and has at most three attachments. Since \( G \) is 3-connected, any nontrivial \( H' \)-bridge in \( G \) must have exactly three attachments. By the proof of \([6, (2.3)]\), the attachments of any \( H' \)-bridge must be contained in \( V(\partial N') \) and at most one of these attachments is in \( V(\partial N') - F \). Thus any nontrivial \( H' \)-bridge must have exactly two attachments in \( F \) and exactly one attachment in \( V(\partial N') - F \).

If there are no nontrivial \( H' \)-bridges in \( G \) (which must happen if \(|F| = 1 \) and may happen if \(|F| = 2\)), we set \( H = H' \) and \( N = N' \). Note that \( H \) is \((3, \partial N)\)-connected by Lemma 2.1(f). Any \( H' \)-bridge is trivial and must have an attachment in \( F \). Therefore \( H \) and \( N \) are as desired.

Otherwise, we may assume that there is at least one nontrivial \( H' \)-bridge. Then \(|F| = 2\), so let \( F = \{f_1, f_2\} \) and assume that \( f_1 \) comes before \( f_2 \) in \( \overrightarrow{\partial N} \). If \( B \) is a nontrivial \( H' \)-bridge with \( v \) as its attachment in \( V(\partial N') - F \), we show that \( v \in \partial N'[f_1, f_2] \). If not, we may assume that \( v \) comes before \( f_1 \) in \( \overrightarrow{\partial N} \). Then \( L = B \cup \partial N'[v, f_2] \) is finite and contains a cycle which separates \( f_1 \) from \( G - V(L) \), which contradicts the fact that \( f_1 \) has infinite degree.

By planarity, there is exactly one nontrivial \( H' \)-bridge \( B \) with attachments \( f_1, f_2, \) and \( v \), where \( v \in \partial N'[f_1, f_2] \). Then \( H = H' \cup B \) is a 2-connected 2-indivisible spanning subgraph of \( G \). All \( H \)-bridges are trivial and have at least one attachment in \( F \). Let \( D \) denote the portion of \( X_B \) from \( f_1 \) to \( f_2 \) that does not contain \( v \). Then we modify the net \( N' = (C'_1, C'_2, C'_3, \ldots) \) to form the net \( N = (C_1, C_2, C_3, \ldots) \) by setting \( C_i = (C'_i - V(\partial N'[f_1, f_2])) \cup D \) for every \( i \geq 1 \). \( N \) is a ladder net, \( F \subseteq V(\partial N) \), and \( H \) is \((3, \partial N)\)-connected by Lemma 2.1(f). \( \square \)

The second structural lemma is proved in a similar way to that of Lemma 4.5, much as \([6, (2.4)]\) is similar to \([6, (2.3)]\).

**Lemma 4.6.** Let \( G \) be a 3-connected 2-indivisible infinite plane graph, let \( F \) be the set of vertices of infinite degree in \( G \), and assume that \(|F| \geq 1 \) and that \( G - F \) has no infinite block. Then \(|F| = 2\), and there exists a connected subgraph \( H \) of \( G \) such that

1. \( H \) is a 1-way infinite plane chain of circuit blocks \((B_1, b_1, B_2, b_2, \ldots)\),
2. \( F \subseteq V(X_{B_1}) - \{b_1\} \),
3. the only \( H \)-bridges in \( G \) are edges incident with at least one vertex in \( F \) (so \( H \) is a spanning subgraph of \( G \)).

In each of the previous two lemmas, the proofs construct the desired subgraph \( H \) in such a way that we may assume that \( H \) and \( G \) are nicely embedded in the plane. The subgraph \( H \) is locally finite in both cases. Also, if \(|F| = 2 \) and \( F = \{f_1, f_2\} \), assuming that \( f_1 \) comes before \( f_2 \) in \( \overrightarrow{\partial N} (X_H) \), the construction rules out any \( H \)-bridges in \( G \) that are edges joining
one vertex in $F$ to a vertex in $V(\partial N[f_1, f_2]) - F$ (in $V(X_H[f_1, f_2]) - F$, respectively). We also remark that it follows from these lemmas that every 3-connected 2-indivisible infinite planar graph is countable.

The next two lemmas allow us to consider only tight nets instead of more general nets. The first lemma follows from the proof of [6, (2.6)] (in the first sentence of their proof, they choose $C_1$ to be any facial cycle of $G$ with $u \in V(C_1)$, which is why we are able to specify such a cycle $C$ in our Lemma 4.7). The second lemma is similar to [6, (2.5)] and Claim 1 in the proof of [23, (3.6)], and can be proven in a similar way as those results — a difference is that, with our slightly modified definition of a tight ladder net, we can treat the first cycle of the net in the same way as all of the other cycles.

**Lemma 4.7.** Let $G$ be a 2-connected 2-indivisible plane graph with a radial net, let $u \in V(G)$, and let $C$ be any facial cycle of $G$ with $u \in V(C)$. Then $G$ has a tight radial net $N = (C_1, C_2, C_3, \ldots)$ with $C_1 = C$ (and hence $u$ is a vertex of $C_1$).

**Lemma 4.8.** Let $G$ be a 2-connected 2-indivisible plane graph with a ladder net $N'$. Let $u$ and $v$ be vertices of $\partial N'$, and suppose that $u$ comes before $v$ in $\overrightarrow{\partial N'}$ if $u$ and $v$ are distinct. Then $G$ has a ladder net $N = (C_1, C_2, C_3, \ldots)$ such that $u, v \in V(C_1) - V(D_1)$, $\partial N = \partial N'$, and $N$ is a tight ladder net with respect to $\partial N[u, v]$.

## 5 Paths and 2-walks

We will now find a 1-way infinite Tutte path $P$ and an SDR of the nontrivial $P$-bridges in every 3-connected 2-indivisible infinite planar graph. Using the SDR of the $P$-bridges, we then detour into each bridge to build our 1-way infinite 2-walk.

Let $G$ be a 2-indivisible plane graph with a net $N = (C_1, C_2, C_3, \ldots)$. Then a $uv$-path $P$ in $G$ is a forward $uv$-path if whenever vertices $u, x, y, v$ occur on $P$ in this order, there is no $i \in \{1, 2, 3, \ldots\}$ such that $x \in V(C_{i+2}) - V(C_{i+1})$ and $y \in V(C_i)$. A forward path may move “backward” a little, but only to a limited extent. We will find forward Tutte paths in finite portions of nets, and then use the fact that these paths are forward to show (using a variation of König’s Lemma) that they converge to a 1-way infinite Tutte path.

We first focus on radial nets. The following theorem is similar to [6, (3.4)], but we assume that the graph is $(3, C_1)$-connected, and we find an SDR of the $P$-bridges.

**Theorem 5.1.** Let $G$ be a 2-indivisible plane graph with a tight radial net $N = (C_1, C_2, C_3, \ldots)$ such that $G$ is $(3, C_1)$-connected, and let $u \in V(C_1)$. Then for every $k \in \{1, 2, 3, \ldots\}$, there exist a vertex $v \in V(C_k)$, a forward $C_1$-Tutte $uv$-path $P$ in $I(C_k)$, and an SDR $S$ of the $P$-bridges.

**Proof.** The proof is by induction on $k$. For $k = 1$, the one-vertex path $v = u$ suffices, with $S = \{u\}$.

So we can assume that $k > 1$ and that the statement holds for positive integers less than $k$. Let $H$ be the block of $I(C_k) - V(C_1)$ containing $C_k$, with the embedding inherited from $G$. Since $N$ is tight and $G$ is nicely embedded, $C_2$ bounds a face of $H$ containing $C_1$, and every $(H \cup C_1)$-bridge has at most one vertex of attachment, called a tip, in $C_2$. 

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Let $t_1, t_2, \ldots, t_n \in V(C_2)$ be all tips of $(H \cup C_1)$-bridges of $I(C_k)$, listed in clockwise cyclic order on $C_2$. Since $G$ is $(3, C_1)$-connected, $n \geq 3$. For $i = 1, 2, \ldots, n$, let $L_i$ be the union of all $(H \cup C_1)$-bridges that attach at $t_i$. The cycle $C_1$ has a collection of pairwise edge-disjoint segments $\{P(t_i)\}^n_{i=1}$ in clockwise cyclic order such that, for every $i = 1, 2, \ldots, n$, $P(t_i)$ contains $V(L_i) \cap V(C_1)$ and the ends of $P(t_i)$ are in $V(L_i)$. Since $n \geq 3$ the collection $\{P(t_i)\}^n_{i=1}$ is well-defined. For each $i = 1, 2, \ldots, n$, let $p_i$ and $q_i$ be the endpoints of $P(t_i)$ so that $p_1, q_1, p_2, q_2, \ldots, p_n, q_n$ occur in $C_1$ in clockwise order. Let $u_1$ be the clockwise neighbor of $u$ on $C_1$. We may assume that $uu_1 \in C_1[q_n, q_1]$; hence $u \neq q_1$.

Now $(C_2, C_3, C_4, \ldots)$ is a tight radial net in $H$. Since $G$ is $(3, C_1)$-connected, $H$ is $(3, C_2)$-connected by Lemma 2.1(g). Therefore, by induction, there is a vertex $v \in V(C_k)$, a forward $C_2$-Tutte $t_1 v$-path $P'$ in $H$, and an SDR $S'$ of the $P'$-bridges. Since $P'$ is $C_2$-Tutte and $G$ is 2-connected, every $P'$-bridge $D$ containing an edge of $C_2$ has exactly two attachments, which are on $C_2$; hence $V(D) \cap V(C'_j) = \emptyset$ for $j \geq 3$.

We divide the $(H \cup C_1)$-bridges into three types: (i) those with no tip, (ii) those with a tip on $P'$, and (iii) those with a tip not on $P'$. We will form collections of these bridges, along with portions of $C_1$ that they span, to form subgraphs where we can apply our standard pieces. If $t_i \in V(P')$, the subgraph $K_{t_i}$ collects together (is the union of) $L_i, P(t_i)$, and every type (i) bridge with all attachments in $P(t_i)$. If $t_i \in V(H) - V(P')$, then any bridge $J$ with tip $t_i$ is of type (iii). The $P'$-bridge $D$ in $H$ containing $t_i$ has exactly two attachments, $c_D$ and $d_D$, which are in $C_2$. One of $c_D$ or $d_D$ is the representative of $D$ in $S'$. Suppose that $t_i, t_{i+1}, \ldots, t_m$ are the tips in $V(D) - \{c_D, d_D\}$. The subgraph $K_D$ collects together $D, L_i, L_{i+1}, \ldots, L_m, C[p_i, q_m]$, and every type (i) bridge with all attachments in $C_1[p_i, q_m]$.

For each $K_{t_i}$ with $i \neq 1$, $K_{t_i}$ is $(3, P(t_i) \cup \{t_i\})$-connected by Lemma 2.1(g), so by Lemma 2.3(b) we may apply SDR SP2 to obtain a $p_i q_i$-path $P_i$ in $K_{t_i}$ avoiding $t_i$ such that $P_i \cup \{t_i\}$ is a $P(t_i)$-Tutte subgraph and an SDR $S_{t_i}$ of the $(P_i \cup \{t_i\})$-bridges with $q_i, t_i \notin S_{t_i}$ (this is valid even if $p_i = q_i$).

For each $K_{t_i}$, label $c_D$ and $d_D$ so that $c_D \in S', d_D \notin S'$. By Lemma 2.1(g), $K_D$ is $(3, C_1[p_i, q_m] \cup \{c_D, d_D\})$-connected. So SDR SP3 gives a $p_i q_m$-path $P_D$ in $K_D$ avoiding $c_D$ and $d_D$ such that $P_D \cup \{c_D, d_D\}$ is a $C_1[p_i, q_m]$-Tutte subgraph, and an SDR $S_D$ of the $(P_D \cup \{c_D, d_D\})$-bridges with $q_m, d_D \notin S_D$. Then replacing $D$ and its representative $c_D$ by the $P_D$-bridges in $K_D$ and $S_D$ does not use a representative more than once.

For every segment $C_1[q_i, p_{i+1}], i \neq n$ and $q_i \neq p_{i+1}$, not already included in some $K_D$, we also collect $C_1[q_i, p_{i+1}]$ and every type (i) bridge with all attachments in $C_1[q_i, p_{i+1}]$ to form a subgraph $K_i$. By Lemmas 2.1(g) and 2.3(a) we may apply SDR SP1 to $K_i$ to obtain a $C_1[q_i, p_{i+1}]$-Tutte $q_i p_{i+1}$-path $P_i$ in $K_i$ and an SDR $S_i$ of the $P_i$-bridges with $p_{i+1} \in S_i$.

Let $P''$ be the union of $P_i$ for every $K_i$ considered above, $P_D$ for every $K_D$ considered above, and $P_i$ for every $K_i$ considered above. Let $S''$ be the union of every $S_{t_i}$, every $S_D$, and every $S_i$. Let $S_0'$ be the set of vertices $c_D$ for every $K_D$ considered above. Then $P''$ is a $q_n q_m$-path, and by the Jigsaw Principle $P' \cup P''$ is a $C_1[q_1, q_n]$-Tutte subgraph and $(S' - S_0') \cup S''$ is an SDR of the $P' \cup P''$-bridges with $q_n \notin (S' - S_0') \cup S''$.

Finally, we use the bridges with tip $t_1$ to find vertex-disjoint $q_n u$- and $t_1 q_1$-paths, and a corresponding SDR. Let $F$ be the union of $C_1[q_n, q_1]$ and every $(H \cup C_1)$-bridge all of whose attachments on $C_1$ belong to $C_1[q_n, q_1]$. Then $F$ contains $K_{t_1}$, and possibly some additional types (i) bridges; $F$ also contains $u$. By Lemma 2.1(g), $F$ is $(3, C_1[q_1, q_n] \cup \{t_1\})$-connected. So, by
Lemma 2.3(b), $F - t_1$ is a plane chain of circuit blocks $(b_0 = q_n, B_1, b_1, B_2, \ldots, b_{m-1}, B_m, b_m = q_1)$ with $m \geq 1$. Let $Y$ be the $q_nq_1$-path $X_{F - t_1}[q_n, q_1]$.

Set $z, 1 \leq z \leq m$, so that $u \in V(B_z) - \{b_z\}$ (recall that $u \neq q_1$). Let $w$ be the first vertex in $Y[b_{z-1}, q_1]$, other than $b_{z-1}$, that is a neighbor of $t_1$ in $F$. Since $B_m - b_{m-1}$ has such a vertex, $w$ is defined. Let $B_r$ be the block such that $w \in V(B_r) - \{b_{r-1}\}$. Then $z \leq r \leq m$.

Let $H_1 = B_1 \cup B_2 \cup \cdots \cup B_{z-1}$, $H_2 = B_z \cup B_{z+1} \cup \cdots \cup B_r$, and $H_3 = B_{r+1} \cup B_{r+2} \cup \cdots \cup B_m$. Apply SDR SP1 to $H_1$ to find an $X_{H_1}$-Tutte $q_n, b_{z-1}$-path $Q_1$ in $H_1$ and an SDR $T_1$ of the $Q_1$-bridges with $b_{z-1} \notin T_1$. Similarly, apply SDR SP1 to $H_3$ to find an $X_{H_3}$-Tutte $b_{z-1}q_1$-path $Q_3$ in $H_3$ and an SDR $T_3$ of the $Q_3$-bridges with $q_1 \notin T_3$.

Let $H'_2 = H_2 \cup b_{z-1}w$, and $Z'_2 = X_{H'_2} = b_{z-1}w \cup Y[w, b_r] \cup C_1[b_{z-1}, b_r]$. By choice of $r$, $Z'_2$ is a cycle. Now $A_G(H_2) \subseteq V(Z'_2)$, so by Lemma 2.1(d), (e), and (g), $H_2$ and hence $H'_2$ are $(3, V(Z'_2))$-connected. Thus, $(H'_2, Z'_2)$ is a circuit graph. Apply Corollary 3.3 to $H'_2$ to find a $Z'_2$-Tutte $ub_r$-path $Q_2$ through $b_{z-1}w$ and an SDR $T_2$ of the $Q'_2$-bridges with $b_r \notin T_2$. Let $Q_2 = Q'_2 - b_{z-1}w$ and $Z_2 = Z'_2 - b_{z-1}w$, then $Q_2$ is $Z_2$-Tutte, consisting of vertex-disjoint $b_{z-1}w$- and $wv$-paths, and an SDR of the $Q_2$-bridges in $H_2$.

If any $Q_2$-bridge $B$ in $H_2$ has three attachments and contains any edges of $X_{H_2}$, then these edges are in $Y[b_{z-1}, w]$. By choice of $w$, no internal vertex of $B$ is a neighbor of $t_1$ in $F$. Thus $B$ is a $Q_2$-bridge in $F$, still with only three attachments. Applying this and Remark 3.4, $Q = Q_1 \cup Q_2 \cup t_1w \cup Q_3 = C_1[q_n, q_1]$-Tutte in $F$, consisting of vertex-disjoint $q_nu$- and $t_1q_1$-paths, with an SDR $T = T_1 \cup T_2 \cup T_3$ of the $Q$-bridges such that $t_1, q_1 \notin T$.

Let $P = P' \cup P'' \cup Q$ and $S = (S' - S'_0) \cup S'' \cup T$. Then $P$ is a $C_1$-Tutte $uv$-path in $I(C_k)$, and $S$ is an SDR of the $P$-bridges. We must show that $P$ is a forward $uv$-path. Suppose that $u, x, y, v$ are vertices in that order along $P$, with $x \notin V(C_i+2)$ for some $i \in \{1, 2, \ldots, k-2\}$. Now $H \cap (P - V(P'))$ is contained in the union of $P'$-bridges of $H$ that contain an edge of $C_2$, and these bridges are vertex-disjoint from $C_i+2$, so $x \notin V(P')$. But then $y \in V(P')$, so $y \notin V(C_i)$ because $P'$ is forward and $V(P') \cap V(C_i) = \emptyset$. Thus $P$ is a forward $uv$-path. \qed

Now let $G$ be a 2-indivisible plane graph with a ladder net $N = (C_1, C_2, C_3, \ldots)$. Suppose that $V(C_1) \neq V(D_1)$, so that we may choose a subpath $C_0 = D_0$ of $C_1 - V(D_1)$. For each $i \geq 0$, suppose the path $D_i$ has ends $x_i$ and $y_i$, where $\ldots, x_2, x_1, x_0, y_0, y_1, y_2, \ldots$ occur along $\partial N$ in this order. By construction, all of these vertices are distinct, except we might have $x_0 = y_0$. Set $I(C_0) = C_0 = D_0$. See Figure 4. Let $r$ and $s$ be integers such that $0 \leq r \leq s$. The $(r, s)$-truncation of $G$ relative to $N$ is the graph $G_{r,s}$ obtained from $I(C_s)$ by deleting the vertices of $I(C_r) - V(D_r)$.

The following lemma is similar to Claim 2 in the proof of [23, (3.6)], but we assume that $G$ is $(3, \partial N)$-connected, and find an SDR of the $P$-bridges.

**Theorem 5.2.** Let $G$ be a 2-indivisible plane graph with a ladder net $N = (C_1, C_2, C_3, \ldots)$ such that $G$ is $(3, \partial N)$-connected. Let $\ldots, x_2, x_1, x_0, y_0, y_1, y_2, \ldots$ be as above, and assume that $N$ is tight with respect to $D_0 = \partial N[x_0, y_0]$. Suppose that $0 \leq r \leq s$. Then there exist an $X_{G_{r,s}}[x_r, y_s]$-Tutte path $P$ in $G_{r,s}$ from $x_r$ to $y_s$ if $s - r$ is even (to $x_s$ if $s - r$ is odd) such that $\{x_r, x_{r+1}, \ldots, x_s, y_r, y_{r+1}, \ldots, y_s\} \subseteq V(P)$ and $P$ is a forward path in $G$, and an SDR $S$ of the $P$-bridges in $G_{r,s}$ such that $y_r \notin S$. Symmetrically, there is also such a path from $y_r$ to $x_s$ ($y_s$) and a corresponding SDR of its bridges.

**Proof.** The proof is by induction on $s - r$. If $s = r$, then $P = D_s$ and $S = \emptyset$ suffice.
So assume that $r < s$. Let $H = G_{r,s}$ and $H' = G_{r+1,s}$. By induction, there is an $X_{H'}[x_r, y_s]$-Tutte path $P'$ in $H'$ from $y_{r+1}$ to $y_s$ if $s - r$ is even ($x_s$ if $s - r$ is odd) such that $\{x_{r+1}, x_{r+2}, \ldots, x_s, y_{r+1}, y_{r+2}, \ldots, y_s\} \subseteq V(P')$ and $P'$ is a forward path in $G$ from $y_{r+1}$ to $y_s$ (if $x_s$), and an SDR $S'$ of the $P'$-bridges such that $x_{r+1} \notin S'$. By Lemma 2.1(g), since $G$ is $(3, \partial N)$-connected, $H$ is $(3, X_H)$-connected. Since $X_H$ is a cycle, $(H, X_H)$ is a circuit graph.

Since $N$ is tight, every $(H' \cup D_r)$-bridge in $G_{r,s}$ has at most one attachment in $D_{r+1}$. Let $t_1, t_2, \ldots, t_n \in V(D_{r+1})$ be all of the tips of $(H' \cup D_r)$-bridges in $G_{r,s}$, listed in order in $X_H[x_{r+1}, y_r]$. Then $n \geq 2$, $t_1 = x_{r+1}$, and $t_n = y_{r+1}$. So we may proceed similarly to the proof of Theorem 5.1. Define subpaths $P(t_i)$ of $D_r$ with ends $p_i$ and $q_i$, so that $p_1 = x_r, q_1, p_2, q_2, \ldots, p_n, q_n = y_r$ occur in this order along $D_r$. Collect together subgraphs $K_{t_1}$, $K_D$, and $K_i$, and use the SDR standard pieces to to find a $q_1p_n$-path $P''$ such that $P' \cup P''$ is a $D_r[q_1, p_n]$-Tutte subgraph and an SDR $S''$ of the $(P' \cup P'')$-bridges such that $p_n \notin S''$ (for each piece, we ensure that the vertex in $D_r$ closest to $q_n$ is not in its SDR).

We must still deal with $K_{t_1}$ and $K_{t_n}$.

Consider $K_{t_1}$. Let $v$ be the clockwise neighbor of $t_1 = x_{r+1}$ in $X_H$. By Lemmas 2.1(g) and 2.3(b), $K_{t_1} - t_1$ is a plane chain of circuit blocks $A' = (a_0 = v, A_1, a_1, A_2, \ldots, a_{m-1}, A_m, a_m = q_1)$ with $m \geq 0$ along $X_H[v, q_1]$. Let $A = (a_0, A_{a+1}, a_{a+1}, \ldots, a_m)$ be the subchain of $A'$ along $X_H[x_r, q_1]$ (with $a_0$ as in $A'$). By SDR SP1 there is an $X_{A'}$-Tutte path $x_r q_1$ through $a_0$ in $A$ and an SDR $S_1'$ of the $P_1$-bridges with $q_1 \notin S_1'$. If $\alpha > 0$ there is a nontrivial $(P_1 \cup \{t_1\})$-bridge containing $t_1v$ with attachments $t_1$ and $a_\alpha$, so take $t_1 = x_{r+1}$ as its representative and set $S_1 = S'_1 \cup \{x_{r+1}\}$; otherwise set $S_1 = S'_1$. By Remark 3.4, $P_1 \cup \{t_1\}$ is an $X_H[x_{r+1}, q_1]$-Tutte subgraph of $K_{t_1}$; $S_1$ is an SDR of the $(P_1 \cup \{t_1\})$-bridges with $q_1 \notin S_1$.

Finally, consider $K_{t_n}$. Let $w$ be the counterclockwise neighbor of $t_n = y_{r+1}$ in $X_H$. By Lemmas 2.1(g) and 2.3(b), $K_{t_n} - t_n$ is a plane chain of circuit blocks $(b_0 = p_n, B_1, b_1, B_2, \ldots, b_{z-1}, B_z, b_z = w)$ with $z \geq 0$. If $z = 0$, then $p_n = y_r = w$; let $P_n = y_r t_n$ and $S_n = \emptyset$. If $z > 0$ let $B_\beta$ be any block with $y_r \in V(B_\beta)$. For each $j$, apply Theorem 3.2 to find an $X_{B_j}$-Tutte path $b_{j-1}b_j$-path $R_j$ in $B_j$ through $y_r$ if $j = \beta$, and an SDR $U_j$ of the $R_j$-bridges such that $b_j \notin U_j$ if $j < \beta$, $y_r \notin U_j$ if $j = \beta$, and $b_{j-1} \notin U_j$ if $j > \beta$. Set $P_n = \left( \bigcup_{j=1}^{z} R_j \right) \cup wt_n$ and $S_n = \bigcup_{j=1}^{z} U_j$. In either case, $P_n$ is an $X_H[p_n, y_{r+1}]$-Tutte path $p_n y_{r+1}$-path in $K_{t_n}$ (using Remark 3.4 if $z > 0$) and $S_n$ is an SDR of the $P_n$-bridges with $y_r, y_{r+1} \notin S_n$ (but $y_{r+1}$ may

Figure 4: Ladder net notation — the second graph is $G_{r,s}$.
already be in \( S' \).

Let \( P = P' \cup P'' \cup P_1 \cup P_n \), and \( S = S' \cup S'' \cup S_1 \cup S_n \). Then \( P \) is an \( X_H[ x_s, y_s ] \)-Tutte path in \( H = G_{r,s} \) from \( x_r \) to \( y_s ( x_s ) \) such that \( \{ x_r, x_{r+1}, \ldots, x_s, y_r, y_{r+1}, \ldots, y_s \} \subseteq V(P) \), and \( S \) is an SDR of the \( P \)-bridges such that \( y_s \notin S \). By an argument similar to the one in Theorem 5.1, \( P \) is a forward path from \( x_r \) to \( y_s ( x_s ) \).

We will build our 1-way infinite Tutte path from a sequence of finite forward Tutte paths. The following lemma will help to compare bridges of different subgraphs in different graphs.

**Lemma 5.3.** Let \( G \) be a 2-indivisible plane graph with a net \( N = ( C_1, C_2, C_3, \ldots ) \). Let \( i \) be a positive integer and \( w \in V(D_{i+2}) \). For \( k = 1, 2 \) suppose that \( I(C_{i+1}) \subseteq G_k \subseteq G \), and that \( R_k \subseteq G_k \) is a forward path in \( G \) from \( u \) through \( w \). Suppose that \( R_1[u, w] = R_2[u, w] \) and \( J \subseteq I(C_i) \). Then \( J \) is an \( R_1 \)-bridge in \( G_1 \) if and only if \( J \) is an \( R_2 \)-bridge in \( G_2 \), and \( J \) has the same attachments in both cases.

**Proof.** For \( J \) to be a bridge, it must be connected and have at least two vertices. Given this, \( J \) is an \( R_k \)-bridge in \( G_k \) with attachment set \( S \) if and only if (i) \( E(J) \subseteq E(G_k) - E(R_k) \), (ii) \( A_{G_k}(J) \subseteq V(R_k) \), (iii) \( J - V(R_k) \) is connected or empty, and (iv) \( V(J) \cap V(R_k) = S \). Since \( R_k \) is forward, after \( w \) it contains no vertex of \( I(C_i) \) and hence no vertex or edge of \( J \), so we may replace \( R_k \) in (i)–(iv) with \( R_k[u, w] \). Since every edge of \( G \) incident with a vertex of \( J \) belongs to \( I(C_{i+1}) \) and \( I(C_{i+1}) \subseteq G_k \subseteq G \), we may replace \( G_k \) in (i)–(iv) with \( I(C_{i+1}) \). But then the conditions are identical for \( k = 1 \) or 2.

We are now ready to find a 1-way infinite Tutte path and an SDR of its bridges in every 3-connected 2-indivisible infinite plane graph. We actually prove another similar result first, for graphs that have a net but satisfy a somewhat weaker connectivity condition. The proof of this theorem is a variation of König’s Lemma and is similar to \([6, (3.7) \text{ and } (3.8)]\), \([23, (3.5)]\), and \([24, \text{Lemma 5.2}]\).

**Theorem 5.4.** Let \( G \) be a 2-indivisible infinite plane graph with a net \( N = ( C_1, C_2, C_3, \ldots ) \). If \( N \) is a radial net, let \( \Delta = C_1 \) and \( u = v \in V(C_1) \). If \( N \) is a ladder net, let \( \Delta = \partial N \) and \( u, v \in \partial N \). Assume that \( G \) is \((3, \Delta)\)-connected. Then there exist a 1-way infinite \( \Delta \)-Tutte path \( P \) in \( G \) from \( u \) through \( v \) and an SDR \( S \) of the \( P \)-bridges.

**Proof.** In either case, \( G \) is locally finite since the neighbors of any vertex belong to a finite graph \( I(C_n) \) for large enough \( n \). Also, \( G \) is 2-connected by Lemma 2.1(h), taking \( H \) to be \( C_1 \) when \( N \) is a radial net, and the 2-connected subgraph \( \bigcup_{i=1}^{\infty} C_i \) when \( N \) is a ladder net. Therefore, by Lemma 4.7 or 4.8, we may assume that \( N \) is a tight net, and that if \( N \) is a ladder net then \( u, v \in V(C_1) - V(D_1) \) and \( N \) is a tight ladder net with respect to \( D_0 = \partial N[u, v] \).

Throughout this paper, a system of distinct representatives has been represented by a set of vertices. For our infinite limiting argument in this proof, we must also keep track of the specific assignment of vertices as representatives of different bridges. If \( P \) is a subgraph of \( G \), \( \mathcal{B} \) is the set of nontrivial \( P \)-bridges, and \( \sigma : \mathcal{B} \to V(G) \) is an injection such that, for every \( B \in \mathcal{B} \), \( \sigma(B) \in V(B) \cap V(P) \), then we say that \( \sigma \) is an assignment function for \( \mathcal{B} \).
The range of $\sigma$ in $V(G)$ is an SDR $S$ of the $P$-bridges in $B$, and the existence of an SDR $S$ guarantees the existence of $\sigma$. If we restrict the codomain of $\sigma$ to $S$, then $\sigma$ is a bijection.

If $N$ is a radial net, then by Theorem 5.1 for $n = 1, 2, 3, \ldots$ we can find a forward $C_1$-Tutte path $Q_n$ in $I(C_n)$ from $u$ to some vertex in $D_n$ (in this case, $D_n = C_n$) and an SDR of the $Q_n$-bridges in $I(C_n)$. If $N$ is a ladder net, then by Theorem 5.2 for $n = 1, 2, 3, \ldots$ we can find a forward $(\partial N \cap C_n)$-Tutte path $Q_n$ in $G_{n,v} = I(C_n)$ from $u$ through $v$ to some vertex in $D_n$ and an SDR of the $Q_n$-bridges in $I(C_n)$. In either case, each $Q_n$ is a forward $(\Delta \cap I(C_n))$-Tutte path in $I(C_n)$ from $u$ through $v$. Let $T_n$ be the collection of nontrivial $Q_n$-bridges in $I(C_n)$, and let $\alpha_n : T_n \rightarrow V(G)$ be an assignment function for $T_n$.

We now construct an infinite sequence of paths and assignment functions in $G$, which will converge to the path $P$ and to an assignment function giving an SDR of the $P$-bridges, respectively. For subgraphs $H, K$ of a graph $F$, let $\text{NB}(H, F; K)$ denote the set of nontrivial $H$-bridges in $F$ that are subgraphs of $K$.

Let $A_0 = \{1, 2, 3, \ldots \}$. Suppose $i \geq 1$ and we have an infinite set $A_{i-1}$ of positive integers. Let $P_i$ be a path in $I(C_{i+2})$ from $u$ to a vertex of $D_{i+2}$ such that, for an infinite set $A' \subseteq A_{i-1}$ of values of $n$, $Q_n$ has $P_i$ as an initial segment. Such a $P_i$ exists because $A_{i-1}$ is infinite, $I(C_{i+2})$ is finite and each $Q_n, n \in A_{i-1}$, uses a vertex of $D_{i+2}$. (There may be more than one such path, but we fix just one as $P_i$.) Suppose $n \in A'_i$; necessarily $n \geq i + 2$. Since $Q_n$ is a forward path from $u$ through $v$, $P_i$ is also a forward path from $u$ through $v$. By Lemma 5.3, if we define $B_i = \text{NB}(P_i, I(C_{i+2}); I(C_i))$, then also $B_i = \text{NB}(Q_n, I(C_n); I(C_i)) \subseteq T_n$. Let $\sigma_i : B_i \rightarrow V(G)$ be an assignment function for $B_i$ such that, for an infinite set $A_i \subseteq A'_i$ of values of $n$, $\alpha_n|_{B_i} = \sigma_i$. Such a $\sigma_i$ exists because $A'_i$ is infinite and $B_i$ is finite.

Constructing $A_i$ from $A_{i-1}$ in this way gives an infinite sequence of sets $A_0 = \{1, 2, 3, \ldots \} \supseteq A_1 \supseteq A_2 \ldots$. Suppose $i < j$ and choose $n \in A_j \subseteq A_i$. Since $P_i$ is an initial segment of $Q_n$ from $u$ to $D_{i+2}$ in $I(C_{i+2})$, and $P_j$ is an initial segment of $Q_n$ from $u$ to $D_{j+2}$ in $I(C_{j+2})$, we have $P_i \subseteq P_j$. Hence $P = \bigcup_{i=1}^{n} P_i$ is a 1-way infinite path; since each $P_i$ is a forward path from $u$ through $v$, $P$ is also a forward path from $u$ through $v$. Also, $B_i = \text{NB}(Q_n, I(C_n); I(C_i)) \subseteq \text{NB}(Q_n, I(C_n); I(C_j)) = B_j$. Hence $B_1 \subseteq B_2 \subseteq B_3 \subseteq \ldots \subseteq B = \bigcup_{i=1}^{\infty} B_i$. Furthermore, since $\sigma_i = \alpha_n|_{B_i}$ and $\sigma_j = \alpha_n|_{B_j}$, $\sigma_j$ is an extension of $\sigma_i$. Hence we can define a function $\sigma : B \rightarrow V(G)$ where $\sigma|_{B_i} = \sigma_i$ for all $i$. Since each $\sigma_i$ is injective, $\sigma$ is injective, and its range $S$ is an SDR of $B$.

Now we show that each $P$-bridge $J$ in $G$ is finite. Suppose not. Then $J - V(P)$ is infinite, and by planarity it intersects infinitely many $D_i$. In particular, for some $i \geq 4$, $J - V(P)$ contains a path $M$ from $D_{i-3}$ to $D_i$; by terminating $M$ at its first vertex in $D_i$, we may assume that $M \subseteq I(C_i)$. Choose $n \in A_i$. Then $M \subseteq I(C_i) = I(C_{i+2}) - V(P) = I(C_i) - V(Q_n) \subseteq I(C_n) - V(Q_n)$, so $M$ is contained in some $Q_n$-bridge $J'$ in $I(C_n)$. Since $Q_n$ is a Tutte path in $I(C_n)$, $J'$ has at most three attachments on $Q_n$. But since $M \subseteq J'$, $J'$ must have at least one attachment on $Q_n$ in each of $D_{i-3}, D_{i-2}, D_{i-1}$, and $D_i$, a contradiction.

By Lemma 5.3, for each $i \geq 1$ we have $B_i = \text{NB}(P_i, I(C_{i+2}); I(C_i)) = \text{NB}(P, G; I(C_i))$. Therefore, if $J \in B$ then $J \in B_i$ for some $i$ and hence $J$ is a $P$-bridge in $G$. Conversely, if $J$ is a nontrivial $P$-bridge in $G$ then, because $J$ is finite, $J \subseteq I(C_i)$ for some $i$, and hence $J \in B_i \subseteq B$. So $B$ is precisely the set of nontrivial $P$-bridges in $G$.

For each $P$-bridge $J$ in $G$, with $J \subseteq I(C_i)$ and $n \in A_i$, Lemma 5.3 gives that the number of attachments of $J$ on $P$ in $G$ is the same as the number of attachments of $J$ on $Q_n$ in
shows that $P$ edges of $G$.

We conclude that $P$ is a $\Delta$-Tutte path in $G$, and $S$ is an SDR of the $P$-bridges. □

**Theorem 5.5.** Let $G$ be a 3-connected 2-indivisible infinite plane graph, and $F$ the set of vertices of infinite degree in $G$. If $F = \emptyset$, then $G$ has a net $N$; let $u = v \in V(G)$ be arbitrary if $N$ is a radial net, and let $u, v \in V(\partial N)$ if $N$ is a ladder net. If $F = \{f_1\}$, let $u = v = f_1$. If $F = \{f_1, f_2\}$, let $u = f_1$ and $v = f_2$. Then there exist a 1-way infinite Tutte path $P$ in $G$ from $u$ through $v$ and an SDR $S$ of the $P$-bridges.

**Proof.** If $F = \emptyset$ then we can use Lemma 4.4 to find a net in $G$, Lemma 4.7 to adjust a radial net so that $u = v \in V(C_1)$, Lemma 2.1(f) to show that $G$ is $(3, C_1)$- or $(3, \partial N)$-connected, and Theorem 5.4 to find $P$ and $S$.

So suppose that $|F| \geq 1$. Assume first that $G - F$ has no infinite block. By Lemma 4.6, $|F| = 2$ and $G$ has a spanning 1-way infinite plane chain of finite circuit blocks $H = (B_1, b_1, B_2, b_2, \ldots )$ such that $F \subseteq V(X_{B_1}) - \{b_1\}$. If $i > 1$ then by Theorem 3.2 we find an $X_{B_i}$-Tutte $b_{i-1}b_i$-path $P_i$ in $B_i$ and an SDR $S_i$ of the $P_i$-bridges in $B_i$ such that $b_{i-1} \notin S_i$.

In $B_1$, by Theorem 3.2, we find an $X_{B_1}$-Tutte $ub_1$-path $P_1$ in $B_1$ through $v$ and an SDR $S_1$ of the $P_1$-bridges in $B_1$ such that $u \notin S_1$. Let $P = \bigcup_{i=1}^{\infty} P_i$ and $S = \bigcup_{i=1}^{\infty} S_i$, then $P$ is an $X_H$-Tutte path from $u$ through $v$ in $H$ with an SDR $S$ of the $P$-bridges in $H$. Now assume that $G - F$ has an infinite block. By Lemma 4.5, $G - F$ has a spanning subgraph $H$ with a ladder net $N$ so that $H$ is $(3, \partial N)$-connected. Apply Theorem 5.4 to $H$ to obtain a $\partial N$-Tutte path $P$ from $u$ through $v$ in $H$ with an SDR $S$ of the $P$-bridges in $H$. In both cases, all edges of $G$ not in $H$ are incident with at least one vertex of $F$, and applying Remark 3.4 shows that $P$ is a Tutte path in $G$ and $S$ is still an SDR of the $P$-bridges in $G$. □

We now find 1-way infinite 2-walks, using a 1-way infinite Tutte path $P$ provided by Theorem 5.4 or 5.5 as the skeleton of each 2-walk. To detour into each nontrivial $P$-bridge to visit the remaining vertices, we use some results of Gao, Richter and Yu from the proof of [8, Theorem 6]. Lemma 5.6 is the more technical result from which their Theorem 6 follows, modified to include the case of a trivial block. Gao, Richter and Yu also examine the structure of a bridge $L$ of a Tutte subgraph in a circuit graph. Parts (a) and (b) of Lemma 5.6 correspond to when $L$ has three or two vertices of attachment, respectively; part (b) is just a special case of our Lemma 2.3(b).

For a plane graph $G$, an *internal 3-cut* is a 3-cut $A$ of $G$ such that $G - A$ has a component vertex-disjoint from $X_G$. Let $N_G(v)$ denote the set of vertices adjacent to $v$ in $G$.

**Lemma 5.6 ([8, proof of Theorem 6]).** Let $(G, X_G)$ be a circuit block, and let $x, y \in V(X_G)$ with $x \neq y$. Then $G$ contains a closed 2-walk visiting $x$ and $y$ exactly once, such that every vertex visited twice is either in an internal 3-cut $A$ of $G$, or in a 2-cut $A$ of $G$ with $A \subseteq V(X_G[x, y])$ or $A \subseteq V(X_G[y, x])$.

**Lemma 5.7 ([8, proof of Theorem 6]).** Let $L$ be a plane graph.

(a) Suppose $L$ is $(3, \{a,b,c\})$-connected, where $a, b, c$ are distinct vertices each appearing once on $X_L$, in that clockwise order. Then $L - \{b,c\}$ is a plane chain of circuit blocks $K = (a = b_0, B_1, b_1, \ldots, b_{k-1}, B_k, b_k = d)$ where $N_L(b) \subseteq V(X_K[a, d]) \cup \{c\}$ and $N_L(c) \subseteq V(X_K[d, a]) \cup \{b\}$.
(b) Suppose \( L \) is \((3, X_L[a, b])\)-connected, where \( a \neq b \). Then \( L - b \) is a plane chain of circuit blocks \( K = (a = b_0, B_1, b_1, \ldots, b_{k-1}, B_k, b_k = d) \) where \( d \) is the neighbor of \( b \) in \( X_L[a, b] \), and \( N_L(b) \subseteq V(X_K[d, a]) \).

**Theorem 5.8.** Let \( G \) be a 2-indivisible infinite plane graph with a net \( N = (C_1, C_2, C_3, \ldots) \). If \( N \) is a radial net, let \( \Delta = C_1 \). If \( N \) is a ladder net, let \( \Delta = \partial N \). Assume that \( u \in V(\Delta) \) and that \( G \) is \((3, \Delta)\)-connected. Then \( G \) contains a 1-way infinite 2-walk beginning at \( u \) such that every vertex used more than once belongs to a 2- or 3-cut of \( G \).

**Proof.** By Theorem 5.4, \( G \) contains a 1-way infinite \( \Delta \)-Tutte path \( P \) beginning at \( u \) and an SDR \( S \) of the \( P \)-bridges. Our walk \( W \) will traverse \( P \), beginning at \( u \), until we reach a vertex \( a \in S \) that is a representative of a nontrivial \( P \)-bridge \( L \).

If \( L \) has three attachments \( a, b, c \), then \( L \) cannot contain an edge of \( \Delta \), so \( L \) is \((3, \{a, b, c\})\)-connected by Lemma 2.1(g). Then \( L - \{b, c\} \) is a plane chain of circuit blocks \( K = (a = b_0, B_1, b_1, \ldots, b_{k-1}, B_k, b_k = d) \) as in Lemma 5.7(a), and we can apply Lemma 5.6 to each block \( B_i \) to get a 2-walk using \( b_{i-1} \) and \( b_i \) only once. Combining these 2-walks yields a 2-walk \( W_a \) in \( K \). Each vertex used twice by \( W_a \) is (i) some \( b_i \), \( i \geq 1 \), which is in a 3-cut \( \{b_i, b, c\} \) of \( G \), (ii) in an internal 3-cut of \( K \), which is also a 3-cut of \( G \), or (iii) in a 2-cut \( A \) of \( K \) contained in either \( X_K[a, d] \) or \( X_K[d, a] \), so that one of \( A \cup \{b\} \) or \( A \cup \{c\} \) is a 3-cut of \( G \).

If \( L \) has two attachments \( a, b \) then necessarily \( a, b \in V(\Delta) \), and one of \( X_L[a, b] \) or \( X_L[b, a] \), call it \( R \), is a subpath of \( \Delta \). By Lemma 2.1(g), \( L \) is \((3, R)\)-connected. Then \( L - b \) is a plane chain of circuit blocks \( K \) as in Lemma 5.7(b) (or its mirror image), and we can apply Lemma 5.6 to each block, combining the resulting 2-walks to obtain a 2-walk \( W_a \). Each vertex used twice by \( W_a \) is (i) some \( b_i \), \( i \geq 1 \), which is in a 2-cut \( \{b_i, b\} \) of \( G \), (ii) in an internal 3-cut of \( K \), which is also a 3-cut of \( G \), or (iii) in a 2-cut \( A \) of \( K \), which is either a 2-cut of \( G \) or such that \( A \cup \{b\} \) is a 3-cut of \( G \).

Splicing \( W_a \) into \( P \) for every representative \( a \in S \) gives the required 2-walk \( W \). The vertices used twice by \( W \) are the vertices used twice by each \( W_a \) and the vertices \( a \in S \) themselves, each of which lies in a 2- or 3-cut of \( G \). □

The following result can be proved in the same way as Theorem 5.8, using Theorem 5.5 instead of Theorem 5.4. In fact, the proof is simpler, because now \( P \) has no bridges with two attachments. Our main result, Theorem 1.3, follows immediately from this.

**Theorem 5.9.** Let \( G \) be a 3-connected 2-indivisible infinite plane graph, and let \( F \) be the set of vertices of infinite degree in \( G \). If \( F = \emptyset \), then \( G \) has a net \( N \). Let \( u \in V(G) \) be arbitrary if \( N \) is a radial net, and let \( u \in V(\partial N) \) if \( N \) is a ladder net. Otherwise, let \( u \in F \). Then \( G \) contains a 1-way infinite 2-walk beginning at \( u \) such that every vertex used more than once belongs to a 3-cut of \( G \).

### 6 Infinite Prisms

In this final section, we discuss spanning paths in prisms over infinite planar graphs. The **prism** over a graph \( G \) is the Cartesian product \( G \square K_2 \) of \( G \) with the complete graph \( K_2 \). In [3],
we showed that prisms over bipartite circuit graphs and near-triangulations are hamiltonian. A near-triangulation is a finite plane graph where every face is a triangle, except for possibly the outer face, which is bounded by a cycle. Here we extend these results to infinite graphs by showing that if \( G \) is a 2-indivisible infinite analog of a bipartite circuit graph or near-triangulation then \( G \square K_2 \) has a 1-way infinite spanning path. Tracing this path in the original graph \( G \) gives a 1-way infinite 2-walk. So for these classes of graphs we can strengthen the existence of a 1-way infinite 2-walk in a somewhat different way from what we did in Theorems 5.8 and 5.9, where we controlled the location of the vertices used twice.

In the prism \( G \square K_2 \), we may identify \( G \) with one of its two copies in the prism. Let \( v \) be a vertex in \( G \). In the prism, we let \( v \) denote the copy of the vertex in the graph that is identified with \( G \), and we let \( v^* \) denote the other copy. We use the same notation for edges. An edge of the form \( vv^* \) is called a vertical edge. Below we often take \( u \in V(G) \) and find a path in \( G \square K_2 \) beginning at \( u \); then there is also a symmetric path beginning at \( u^* \).

Our results for infinite bipartite graphs depend on the following result for finite graphs.

**Lemma 6.1** ([3, Theorem 2.4]). Let \( (G, X_G) \) be a bipartite circuit graph and let \( u, v \) be two distinct vertices in \( X_G \). Then there is a hamilton cycle in \( G \square K_2 \) that uses the vertical edges at \( u \) and \( v \).

**Theorem 6.2.** Let \( G \) be a 2-indivisible infinite bipartite plane graph with a net \( N = (C_1, C_2, C_3, \ldots) \). If \( N \) is a radial net, let \( \Delta = C_1 \). If \( N \) is a ladder net, let \( \Delta = \partial N \). Assume that \( u \in V(\Delta) \) and that \( G \) is \( (3, \Delta) \)-connected. Then \( G \square K_2 \) contains a 1-way infinite spanning path beginning at \( u \).

**Proof.** By Theorem 5.4, \( G \) contains a 1-way infinite Tutte path \( P = v_1v_2v_3v_4\ldots \) with \( v_1 = u \). Let \( P' = v_1v_1^*v_2^*v_2v_3v_3^*v_4^*v_4\ldots \) be the 1-way infinite spanning path of \( P \square K_2 \) starting at \( u \) and using every vertical edge.

We modify \( P' \) to detour into \( L \square K_2 \) for each nontrivial \( P \)-bridge \( L \), as follows. As in the proof of Theorem 5.8, \( L \) is decomposed using Lemma 5.7. Instead of using Lemma 5.6 to obtain 2-walks in circuit blocks that we splice together, we use Lemma 6.1 (extended to allow blocks that are edges) to find hamilton cycles in the prisms over circuit blocks, which we splice together at vertical edges (deleting those vertical edges) and then splice into \( P' \).

The following result can be proved similarly, using Theorem 5.5 instead of Theorem 5.4.

**Theorem 6.3.** Let \( G \) be a 3-connected 2-indivisible infinite bipartite plane graph, and let \( F \) be the set of vertices of infinite degree in \( G \). If \( F = \emptyset \), then \( G \) has a net \( N \). Let \( u \in V(G) \) be arbitrary if \( N \) is a radial net, and let \( u \in V(\partial N) \) if \( N \) is a ladder net. Otherwise, let \( u \in F \). Then \( G \square K_2 \) contains a 1-way infinite spanning path beginning at \( u \).

We also want to give results for infinite versions of triangulations or near-triangulations. These are based on the following result for finite graphs.

**Lemma 6.4** ([3, Theorem 2.7]). Let \( G \) be a finite near-triangulation and let \( u, v \) be two distinct vertices in \( X_G \). Then there is a hamilton cycle in \( G \square K_2 \) that uses the vertical edges at \( u \) and \( v \).
Theorem 6.5. Let $G$ be a connected 2-indivisible infinite nicely embedded plane graph with a net $N = (C_1, C_2, C_3, \ldots)$. If $N$ is a radial net, let $\Delta = C_1$ and suppose that every face is bounded by a triangle except perhaps one face bounded by $C_1$. If $N$ is a ladder net, let $\Delta = \partial N$ and suppose that every finite walk bounding a face is a triangle. Then for every $u \in V(\Delta)$, $G \Box K_2$ contains a 1-way infinite spanning path beginning at $u$.

Proof. We show that $G$ is $(3, \Delta)$-connected. First notice that if $N$ is a radial net and $v$ is any vertex, or if $N$ is a ladder net and $v \notin \partial N$, then $v \in I(C_i) - V(C_i)$ for some $i$. This follows from the definition of a net. Consequently, every edge incident with $v$ is incident with two faces bounded by finite walks, since they are also faces of the finite graph $I(C_i)$.

Now suppose there is $T \subseteq V(G)$ with $|T| \leq 2$ so that $G - T$ has a component $K$ with no vertex of $\Delta$. Since $G$ is connected, $|T| \geq 1$.

Suppose first that $|T| = 1$, with $T = \{t_1\}$. Then $t_1$ has neighbors $v_0, v_1$ consecutive in its rotation with $v_0 \notin V(K)$ and $v_1 \in V(K)$. Let $f_0$ be the face containing $v_0t_1v_1$. If $N$ is a radial net, then $f_0$ is not bounded by $C_1$, since $v_1 \notin V(C_1)$. If $N$ is a ladder net, then $f_0$ is a face bounded by a finite walk since $v_1 \notin \partial N$. In either case, $f_0$ is a triangle. Since $t_1$ is the only place where we can change between vertices in $K$ and not in $K$, $f_0 = t_1 \ldots v_0t_1v_1 \ldots t_1$. Hence, $f_0$ has length at least 4, and has a repeated vertex, either of which is a contradiction.

Now suppose that $|T| = 2$, with $T = \{t_1, t_2\}$. Then there is a sequence $v_0, v_1, \ldots, v_k, v_{k+1}$ of neighbors of $t_1$ in clockwise order with $k \geq 1$, $v_1, v_2, \ldots, v_k \in K$, and $v_0, v_{k+1} \notin K$; these vertices are distinct except possibly $v_0 = v_{k+1}$. Let $f_0$ be the face containing $v_0t_1v_1$ and $f_k$ the face containing $v_kt_1v_{k+1}$. By the same reasoning as in the case $|T| = 1$, $f_0$ is a triangle. Now since $t_1$ and $t_2$ are the only places we can change between vertices in $K$ and not in $K$, $f_0 = t_1 \ldots v_0t_1v_1 \ldots t_1$ where possibly $v_0 = t_i$, but $v_1 \neq t_i$. Since $f_0$ is a triangle we must have $v_0 = t_2$ and $f_0 = t_2t_1v_1t_2$. By similar reasoning, $v_{k+1} = t_2$ and $f_k = t_2t_1v_kt_2$. Since we cannot have two edges $t_1t_2$, the neighbors of $t_1$ must be exactly $t_2, v_1, v_2, \ldots, v_k$. Since, from above, $\{t_2\}$ cannot isolate a component with no vertex of $\Delta$, $t_1 \in V(\Delta)$. But $\Delta$ is 2-regular, which means one of $v_1, v_2, \ldots, v_k$ is in $\Delta$, a contradiction.

Hence no such $T$ exists, and $G$ is $(3, \Delta)$-connected.

We now proceed as in the proof of Theorem 6.2, using Theorem 5.4, with Lemma 6.4 instead of Lemma 6.1. This works because our conditions on faces guarantee that every circuit graph that we consider as a subgraph of a bridge $L$ of the 1-way infinite walk $P$ is actually a near-triangulation. \qed

We can also prove the following, using Theorem 5.5 and Lemma 6.4. We say a face has bounded extent if it is bounded in the metric space sense.

Theorem 6.6. Let $G$ be a 3-connected 2-indivisible infinite plane graph with a nice embedding in which every finite walk bounding a face is a triangle. Let $F$ be the set of vertices of infinite degree in $G$. If $F = \emptyset$, then $G$ has a net $N$. Let $u \in V(G)$ be arbitrary if $N$ is a radial net, and let $u \in V(\partial N)$ if $N$ is a ladder net. Otherwise, let $u \in F$. Then $G \Box K_2$ contains a 1-way infinite spanning path beginning at $u$.

As a special case, suppose $G$ is a locally finite 2-indivisible infinite plane graph in which every edge is incident with two faces of bounded extent, each bounded by a triangle. Then for every $u \in V(G)$, $G \Box K_2$ contains a 1-way infinite spanning path beginning at $u$. 

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Proof. In the special case we can prove that the graph is 3-connected by the same reasoning we used in the previous theorem. Therefore there is a net, which must be a radial net, because for a ladder net $N$ the edges of $\partial N$ fail the condition. The embedding must be nice because the side of any cycle with finitely many vertices partitions into a finite number of triangles, each of which is bounded in the metric space sense.

References


[22] Xingxing Yu, Infinite paths in planar graphs, I, graphs with radial nets, *J. Graph Theory* 47 No. 2 (2004), 147–162.

[23] Xingxing Yu, Infinite paths in planar graphs, II, structures and ladder nets, *J. Graph Theory* 48 No. 4 (2005), 247–266.

