# ORIENTABLE HAMILTON CYCLE EMBEDDINGS OF COMPLETE TRIPARTITE GRAPHS I: LATIN SQUARE CONSTRUCTIONS

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ABSTRACT. In an earlier paper the authors constructed a hamilton cycle embedding of $K_{n,n,n}$ in a nonorientable surface for all $n \geq 1$ and then used these embeddings to determine the genus of some large families of graphs. In this two-part series, we extend those results to orientable surfaces for all $n \neq 2$. In part I, we explore a connection between orthogonal latin squares and embeddings. A product construction is presented for building pairs of orthogonal latin squares such that one member of the pair has a certain hamiltonian property. These special squares are then used to construct embeddings of the complete tripartite graph $K_{n,n,n}$ on an orientable surface such that the boundary of every face is a hamilton cycle. This construction works for all $n \geq 1$ such that $n \neq 2$ and $n \neq 2p$ for every prime $p$. Moreover, it is shown that the latin square construction utilized to get hamilton cycle embeddings of $K_{n,n,n}$ can also be used to obtain triangulations of $K_{n,n,n}$. Part II of this series covers the case $n = 2p$ for every prime $p$ and applies these embeddings to obtain some genus results.

## 1. Introduction

In [5], the present authors constructed nonorientable hamilton cycle embeddings of $K_{n,n,n}$ for all $n \geq 2$. In this paper we extend those results to the orientable case for all $n \geq 3$ such that $n \neq 2p$ for every prime $p$. To construct these embeddings, we first establish a connection between hamilton cycle embeddings of $K_{n,n,n}$ and orthogonal latin squares. Namely, it is shown that we can construct an orientable hamilton cycle embedding of $K_{n,n,n}$ such that the vertices in one of the partite sets appear in the same order around each face from a pair of orthogonal latin squares, one of which satisfies an additional hamiltonian property. Then we build the desired latin squares by using a new method we call the "step product" construction, which is a generalization of other known product constructions.

The second part of this series [6] completes the proof that an orientable hamilton cycle embedding of $K_{n,n,n}$ exists for all $n \geq 1$, $n \neq 2$ by presenting a voltage graph construction of such embeddings for $n = 2p$ for every prime $p$. This complete result is then used to determine the genus of several families of graphs, including $K_{t,n,n,n}$ for $t \geq 2n$ and, in certain cases, $\overline{K_m} + K_n$ for $m \geq n - 1$.

A basic understanding of topological graph theory is assumed. In particular, a *surface* is a compact 2-manifold without boundary. The orientable surface $S_h$ is obtained by adding $h$ handles to a sphere, and the *genus* of a graph $G$, denoted $g(G)$, is the minimum value of $h$ for which $G$ can be embedded on $S_h$. It is well known that a cellular embedding can be characterized, up to homeomorphism, by providing a set of facial walks that double cover the edges and yield a proper rotation at each vertex. To define a proper rotation, we must introduce the *rotation graph* at a vertex $v$, denoted $R_v$. This graph has as its vertex set the neighbors of $v$, and two vertices $u_1$ and $u_2$ are joined by one edge for each occurrence of the subsequence $(\cdots u_1 v u_2 \cdots)$, or its reverse, in one of the facial walks. $R_v$ is 2-regular; we say it is *proper* if $R_v$ consists of a single cycle. This ensures that the neighborhood around each vertex is homeomorphic to a disk. The embedding is orientable if and only if the faces can be oriented so that each edge appears once in each direction. For additional details and terminology, see [14]. For further background information on hamilton cycle embeddings, see [5].

## 2. Preliminaries

Our terminology agrees with that set forth by Wanless in [17, 18]. A *latin square* of order $n$ is an $n \times n$ matrix on some $n$-set $E$ such that every row and every column contain exactly one copy of each element of

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$E$. Assume the rows and columns of $L$ are labeled using the $n$-sets $R$ and $C$, respectively; if the entry in row $r \in R$ and column $c \in C$ contains the entry $e \in E$, we say that $L$ contains the ordered triple $(r, c, e)$, or $L_{rc} = e$. A latin square is thus equivalent to a set of ordered triples. Unless otherwise noted, we will assume $R = C = E = \mathbb{Z}_n$. Two latin squares $L_1$ and $L_2$ on the sets $E_1$ and $E_2$, respectively, are called *orthogonal*, denoted $L_1 \perp L_2$, if the ordered pairs obtained by overlapping the two squares cover every element of $E_1 \times E_2$ exactly once. If $L_1 \perp L_2$ for some $L_2$, we say $L_1$ has an *orthogonal mate*.

Given an $n \times n$ latin square $L$, a *transversal* is a set of $n$ ordered triples $\{(r_i, c_i, e_i) \in L \mid i \in \mathbb{Z}_n\}$ such that $\{r_0, ..., r_{n-1}\} = \{c_0, ..., c_{n-1}\} = \{e_0, ..., e_{n-1}\} = \mathbb{Z}_n$. In other words, a transversal is a collection of cells from $L$ such that every row, column, and entry is covered exactly once. It is well known that a latin square has an orthogonal mate if and only if it can be decomposed into disjoint transversals.

For this paper, we will utilize a generalization of a transversal known as a $k$-plex. If $L$ is a latin square of order $n$, a *$k$-plex* is a set of $kn$ ordered triples $\{(r_i, c_i, e_i) \in L \mid i \in \mathbb{Z}_{kn}\}$ such that the collections $\{r_0, ..., r_{kn-1}\}$, $\{c_0, ..., c_{kn-1}\}$ and $\{e_0, ..., e_{kn-1}\}$ each cover $\mathbb{Z}_n$ $k$ times. In other words, a $k$-plex is a collection of cells from $L$ such that every row, column, and entry is covered exactly $k$ times. Thus, a transversal is a 1-plex, and an example of a 2-plex is given in Example 2.1.

**Example 2.1.** The starred entries in $L$ form a 2-plex.

$$L = \begin{pmatrix} 0^* & 1^* & 2 & 3 & 4 & 5 \\ 1 & 2^* & 3^* & 4 & 5 & 0 \\ 2 & 3 & 4^* & 5^* & 0 & 1 \\ 3 & 4 & 5 & 0^* & 1^* & 2 \\ 4 & 5 & 0 & 1 & 2^* & 3^* \\ 5^* & 0 & 1 & 2 & 3 & 4^* \end{pmatrix}$$

If $L$ can be decomposed into disjoint parts $K_1, K_2, ..., K_d$, where each $K_i$ is an $k_i$-plex, then we call this a *$(k_1, k_2, ..., k_d)$-partition* of $L$. If all the parts have the same size $k$, then we simply call this a $k$-partition. A decomposition into transversals is a 1-partition, so a latin square has an orthogonal mate if and only if it has a 1-partition.

We use $\mathbb{Z}_n$ to denote the addition table for the cyclic group of order $n$; this latin square will be a key ingredient in all of the constructions presented in this paper. In particular, the following property will be useful.

**Lemma 2.2.** If $n$ is odd, then $\mathbb{Z}_n$ admits a 1-partition. If $n$ is even, then $\mathbb{Z}_n$ admits a 2-partition.

*Proof.* For all $j \in \mathbb{Z}_n$, let $T_j = \{(i, i + j, 2i + j) \mid i \in \mathbb{Z}_n\}$. When $n$ is odd, $\{T_0, T_1, ..., T_{n-1}\}$ is a 1-partition of $\mathbb{Z}_n$; when $n$ is even, $\{T_0 \cup T_1, T_2 \cup T_3, ..., T_{n-2} \cup T_{n-1}\}$ is a 2-partition of $\mathbb{Z}_n$. □

In [5] we used the idea of an induced pair graph to aid in our slope sequence construction. We will use induced pair graphs again in this paper, albeit in a different context. Let $P = \{(s_0, t_0), (s_1, t_1), ..., (s_{n-1}, t_{n-1})\}$ be a collection of pairs such that $s_0, s_1, ..., s_{n-1}, t_0, t_1, ..., t_{n-1}$ double cover $\mathbb{Z}_n$. Form the graph $G_P$ with vertices $\{v_0, v_1, ..., v_{n-1}\}$ and $m$ edges joining distinct vertices $v_{j_1}$ and $v_{j_2}$, where $m = |\{s_{j_1}, t_{j_1}\} \cap \{s_{j_2}, t_{j_2}\}|$. We call $G_P$ the *induced pair graph* for collection of pairs $P$. This graph is 2-regular, so $G_P$ decomposes into a union of cycles.

We will denote by $\gcd(m, n)$ the greatest common divisor of integers $m$ and $n$; as usual, we say $m$ and $n$ are *relatively prime* if $\gcd(m, n) = 1$. We let $A = \{a_0, ..., a_{n-1}\}$, $B = \{b_0, ..., b_{n-1}\}$ and $C = \{c_0, ..., c_{n-1}\}$ be the vertices of $K_{n,n,n}$ so that $A, B$ and $C$ are the maximal independent sets. A hamilton cycle face of the form $(a_{j_0} b_{k_0} c_{\ell_0} a_{j_1} b_{k_1} c_{\ell_1} \cdots a_{j_{n-1}} b_{k_{n-1}} c_{\ell_{n-1}})$ is called an *ABC cycle*; when this cycle is the boundary of a face we will refer to it as an *ABC face*. We will refer to an orientable face 2-colorable hamilton cycle embedding as an *O2HC-embedding*.

## 3. O2HC-embeddings from latin squares

**Lemma 3.1.** Let $\mathcal{Z}$ be the collection of facial walks obtained from a hamilton cycle embedding of $K_{n,n,n}$. Suppose that $\mathcal{Z}$ consists of all $ABC$ faces. The following conditions are equivalent:

(i) There exist collections $\mathcal{X}, \mathcal{Y} \subset \mathcal{Z}$ such that $\mathcal{X} \cup \mathcal{Y} = \mathcal{Z}$, $\mathcal{X} \cap \mathcal{Y} = \emptyset$, and every edge of $G$ appears in a face from both $\mathcal{X}$ and $\mathcal{Y}$.

(ii) The embedding is orientable.
(iii) The embedding is face 2-colorable.

*Proof.* (i)⇒(ii) Since every edge appears once in a $\mathcal{X}$ face and once in a $\mathcal{Y}$ face, the faces admit a proper orientation (e.g., orient the $\mathcal{X}$ faces forwards as written in $ABC$ order, and orient the $\mathcal{Y}$ faces backwards as written in $ABC$ order).

(ii)⇒(i) Let $\mathcal{X}$ be the faces oriented forwards as written in $ABC$ order and $\mathcal{Y}$ be the faces oriented backwards as written in $ABC$ order. If any distinct faces $X_1, X_2 \in \mathcal{X}$ share an edge, then they cannot both be oriented forwards. Thus, no two $\mathcal{X}$ faces share an edge, so each edge is appears in at most one face from $\mathcal{X}$. An analogous argument shows that each edge appears in at most one face from $\mathcal{Y}$, and the result follows.

The equivalence (i)⇔(iii) is straightforward, so the proof is complete. □

Assume we have a collection $\mathcal{C} = \{C_0, C_1, ..., C_{n-1}\}$ of $ABC$ cycles that cover every edge of $K_{n,n,n}$ exactly once. Moreover, assume that the $A$ vertices appear in the same fixed order in each $C_i$. We can form a latin square $L_\mathcal{C}$ of order $n$ by taking each subsequence $(\cdots a_j b_k c_\ell \cdots) \in C_i$ and letting $\ell$ be the entry in row $j$ of column $k$. The fact that the $A$ vertices appear in the same fixed order in each $C_i$ ensures that each entry $\ell$ appears exactly once in each row $j$. Following this process, it is readily seen that the entries arising from $C_i$ form a transversal for all $0 \leq i \leq n-1$. Thus, $L_\mathcal{C}$ admits a 1-partition, which is equivalent to $L_\mathcal{C}$ having an orthogonal mate.

**Example 3.2.** Consider the following cycle cover of $K_{5,5,5}$:

$$\begin{array}{rl}
C_0: & (a_0b_0c_0a_1b_1c_2a_2b_2c_4a_3b_3c_1a_4b_4c_3) \\
C_1: & (a_0b_1c_1a_1b_2c_3a_2b_3c_0a_3b_4c_2a_4b_0c_4) \\
C_2: & (a_0b_2c_2a_1b_3c_4a_2b_4c_1a_3b_0c_3a_4b_1c_0) \\
C_3: & (a_0b_3c_3a_1b_4c_0a_2b_0c_2a_3b_1c_4a_4b_2c_1) \\
C_4: & (a_0b_4c_4a_1b_0c_1a_2b_1c_3a_3b_2c_0a_4b_3c_2)
\end{array}$$

From this we obtain the following latin square $L_\mathcal{C}$, where $L'_\mathcal{C}$ provides the transversals.

$$L_\mathcal{C} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 \\ 2 & 3 & 4 & 0 & 1 \\ 3 & 4 & 0 & 1 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{pmatrix}, \; L'_\mathcal{C} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 4 & 0 & 1 & 2 & 3 \\ 3 & 4 & 0 & 1 & 2 \\ 2 & 3 & 4 & 0 & 1 \\ 1 & 2 & 3 & 4 & 0 \end{pmatrix}$$

Since $\mathcal{X}$ and $\mathcal{Y}$ from Lemma 3.1 both satisfy the conditions of the preceeding paragraph, we can form two pairs of orthogonal latin squares of order $n$ from an O2HC-embedding of $K_{n,n,n}$. It is desirable to determine the conditions under which we can reverse this process and form an O2HC-embedding of $K_{n,n,n}$ from order $n$ orthogonal latin squares. Theorem 3.5 presents one such set of conditions that enables us to obtain the desired embeddings from a single pair of orthogonal latin squares; a more general construction appears in [16]. The following notation is used, where $T$ is a transversal of a latin square $L$.

$$\begin{array}{rcl}
E(L,r,c) & = & \text{entry in } L \text{ that appears in row } r \text{ of column } c; \\
C(L,r,e) & = & \text{column in } L \text{ that contains entry } e \text{ in row } r; \\
E(T,r) & = & \text{entry in } T \text{ that appears in row } r; \\
C(T,r) & = & \text{column in } T \text{ that contains entry in row } r.
\end{array}$$

In other words, $(r, c, E(L,r,c)), (r, C(L,r,e), e) \in L$ and $(r, C(T,r), E(T,r)) \in T$.

**Definition 3.3.** Let $L$ be a latin square of order $n$ and define the collection of pairs

$$P_\ell = \{(C(L,j,\ell), C(L,j,\ell-1)) \mid j \in \mathbb{Z}_n\} \text{ for all } \ell \in \mathbb{Z}_n.$$

If the induced pair graph $G_{P_\ell}$ is a hamilton cycle for all $\ell \in \mathbb{Z}_n$, then we say that $L$ is *consecutively entry hamiltonian*, or *ce-hamiltonian* for short.

For each $\ell \in \mathbb{Z}_n$, we can view $G_{P_\ell}$ as a graph with vertices corresponding to the columns of $L$ and an edge for each row between the columns in that row that contain $\ell$ and $\ell - 1$.

**Example 3.4.** The square $L$ below is ce-hamiltonian; we show how to obtain $G_{P_2}$, the induced pair graph for the entries 2 and 1. Denoting column $k$ by $c_k$, we see that row 0 yields the edge $c_2c_1$, row 1 yields $c_3c_0$, row 2 yields $c_0c_4$, row 3 yields $c_1c_3$, and row 4 yields $c_4c_2$; thus, $G_{P_2}$ is given by the cycle $(c_2c_1c_3c_0c_4)$.

$$L = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 4 & 0 & 2 & 3 \\ 2 & 0 & 3 & 4 & 1 \\ 3 & 2 & 4 & 1 & 0 \\ 4 & 3 & 1 & 0 & 2 \end{pmatrix}$$

We also note here that the square $\mathbb{Z}_n$ is ce-hamiltonian for all $n \geq 2$.

**Theorem 3.5.** If there exists a ce-hamiltonian latin square of order $n$ that admits a 1-partition, then there exists an O2HC-embedding of $K_{n,n,n}$.

*Proof.* Let $L$ be a ce-hamiltonian latin square of order $n$ that admits a 1-partition; label the transversals $T_0, T_1, ..., T_{n-1}$ in order as they appear in row 0 of $L$. In other words, $T_k$ is the transversal in $L$ that contains the entry in row 0 of column $k$.

Form the following cycles:

$$\begin{array}{rlllll} X_i: & (a_0 b_{C(T_i,0)} c_{E(T_i,0)} & \cdots & a_j b_{C(T_i,j)} c_{E(T_i,j)} & \cdots & a_{n-1} b_{C(T_i,n-1)} c_{E(T_i,n-1)}); \\ Y_i: & (a_0 b_{C(T_i,0)} c_{E(T_i,0)+1} & \cdots & a_j b_{C(T_i,j)} c_{E(T_i,j)+1} & \cdots & a_{n-1} b_{C(T_i,n-1)} c_{E(T_i,n-1)+1}). \end{array}$$

Note that each $X_i$ and $Y_i$ corresponds to the transversal $T_i$. If the entry $(j, k, \ell)$ appears in $T_i$, then the cycle $X_i$ contains the sequence $a_j b_k c_\ell$ and the cycle $Y_i$ contains the sequence $a_j b_k c_{\ell+1}$. Moreover, these sequences of length 3 are assembled row by row so that the $A$ vertices appear in increasing order. We will prove that the collections $\mathcal{X} = \{X_0, X_1, ..., X_{n-1}\}$ and $\mathcal{Y} = \{Y_0, Y_1, ..., Y_{n-1}\}$ together form an O2HC-embedding of $K_{n,n,n}$. It is not hard to show from the properties of latin squares that every $AB$ edge and every $BC$ edge is covered once by a cycle from $\mathcal{X}$ and once by a cycle from $\mathcal{Y}$. The fact that the $A$ vertices appear in the same fixed order in each cycle implies that every $CA$ edge is covered once by a cycle from $\mathcal{X}$ and once by a cycle from $\mathcal{Y}$ as well. To prove that this cycle double cover is in fact an O2HC-embedding, it remains to show that the rotation around each vertex is a single cycle of length $2n$.

Consider first the vertex $a_j$. For every $k \in \mathbb{Z}_n$, there exist $i$ such that $k = C(T_i, j)$. The cycle $X_i$ contains the sequence $c_\ell a_j b_k$, where $\ell = E(T_i, j-1)$. By construction, $Y_i$ contains the sequence $c_{\ell+1} a_j b_k$. Thus, the rotation around $a_j$ contains sequences of the form $c_\ell b_k c_{\ell+1}$. The endpoints of these sequences clearly match up to form a single cycle; thus, the rotation around $a_j$ is a single cycle of length $2n$ for every $j$.

Next, consider the vertex $b_k$. For every $j \in \mathbb{Z}_n$, we know the sequence $a_j b_k c_\ell$ appears in some cycle of $\mathcal{X}$, where $\ell = E(L, j, k)$. By construction, the sequence $a_j b_k c_{\ell+1}$ appears in some cycle of $\mathcal{Y}$. Thus, the rotation around $b_k$ contains sequences of the form $c_\ell a_j c_{\ell+1}$. The endpoints of these sequences clearly match up to form a single cycle; thus, the rotation around $b_k$ is a single cycle of length $2n$ for every $k$.

Finally, consider the vertex $c_\ell$. For every $j \in \mathbb{Z}_n$, we know the sequence $a_j b_{C(L,j,\ell)} c_\ell a_{j+1}$ appears in some cycle of $\mathcal{X}$. Similarly, the sequence $a_j b_{C(L,j,\ell-1)} c_\ell a_{j+1}$ appears in some cycle of $\mathcal{Y}$. Thus, for each $j$, the rotation around $c_\ell$ contains the sequence $b_{C(L,j,\ell)} a_{j+1} b_{C(L,j,\ell-1)}$. Because $L$ is ce-hamiltonian, the endpoints of this sequence match up to form a single cycle; thus, the rotation around $c_\ell$ is a single cycle of length $2n$ for every $\ell$.

We have shown that the rotation around every vertex is indeed a single cycle of length $2n$. Combining this with Lemma 3.1 proves that $\mathcal{X} \cup \mathcal{Y}$ forms an O2HC-embedding of $K_{n,n,n}$. □

The following construction for odd $n$ is straightforward and illustrates the usefulness of Theorem 3.5. This case can also be handled using the slope sequence construction from [5].

**Theorem 3.6.** If $n$ is odd, then there exists an O2HC-embedding of $K_{n,n,n}$ obtained from a latin square.

*Proof.* Consider the square given by $\mathbb{Z}_n$; we know $\mathbb{Z}_n$ has a 1-partition by Lemma 2.2. It remains to show that the induced pair graph $G_{P_\ell}$ is a hamilton cycle for all $\ell \in \mathbb{Z}_n$. We have that

$$P_\ell = \{(C(L, j, \ell), C(L, j, \ell-1)) \mid j \in \mathbb{Z}_n\} = \{(\ell - j, \ell - j - 1) \mid j \in \mathbb{Z}_n\} = \{(r, r-1) \mid r \in \mathbb{Z}_n\}.$$

The induced pair graph $G_{P_\ell}$ is clearly a hamilton cycle; thus, $\mathbb{Z}_n$ is ce-hamiltonian, and by Theorem 3.5 there exists the desired embedding of $K_{n,n,n}$. □

## 4. Step product construction

The construction of the required latin squares when $n$ is even is considerably more complicated. In fact, Euler famously conjectured that for $n \equiv 2 \pmod 4$, no latin square of order $n$ had a 1-partition [10]. Although Euler was wrong, it took nearly two centuries to construct a counterexample of order $4k+2$ for all $k \geq 2$ [1, 2]. We now seek to impose further structure on these squares. To accomplish this goal, we introduce a new construction called a *step product construction* that is a modification and generalization of the turn-square construction utilized by Brown and Parker [3, 4] in their hunt for large families of mutually orthogonal latin squares.

Let $L$ be a latin square of order $n$ with entries from $\mathbb{Z}_n$. For an integer $x \in \mathbb{Z}_n$, denote by $x \circ L$ the Latin square obtained by cyclically shifting the rows of $L$ down $x$ rows. Moreover, for an integer $a$ and an integer $b \in \{0, 1, ..., a-1\}$, let $aL + b$ be the latin square obtained by multiplying every entry in $L$ by $a$ and then adding $b$ to every resulting product, where the arithmetic is done in $\mathbb{Z}_{nm}$, not $\mathbb{Z}_n$.

**Example 4.1.** Let

$$L = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}.$$

Then

$$1 \circ L = \begin{pmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \end{pmatrix} \text{ and } 5L + 2 = \begin{pmatrix} 2 & 7 & 12 & 17 \\ 7 & 2 & 17 & 12 \\ 12 & 17 & 2 & 7 \\ 17 & 12 & 7 & 2 \end{pmatrix}.$$

Let $L$ and $M$ be latin squares of order $n$ and $m$, respectively, with rows, columns, and entries indexed by $\mathbb{Z}_n$ and $\mathbb{Z}_m$, respectively. Let $X = (x_{i,j})$ be an $n \times n$ matrix with entries from $\mathbb{Z}_m$. Define the *step product* $L \boxdot_X M$ to be the latin square of order $nm$ given by

$$\left(\begin{array}{c|c|c|c} x_{0,0} \circ (nM + L_{0,0}) & x_{0,1} \circ (nM + L_{0,1}) & \cdots & x_{0,n-1} \circ (nM + L_{0,n-1}) \\ \hline x_{1,0} \circ (nM + L_{1,0}) & x_{1,1} \circ (nM + L_{1,1}) & \cdots & x_{1,n-1} \circ (nM + L_{1,n-1}) \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline x_{n-1,0} \circ (nM + L_{n-1,0}) & x_{n-1,1} \circ (nM + L_{n-1,1}) & \cdots & x_{n-1,n-1} \circ (nM + L_{n-1,n-1}) \end{array}\right).$$

To help clarify this construction, we present an example.

**Example 4.2.** Let $L = \mathbb{Z}_3$, $M = \mathbb{Z}_5$, and

$$X = \begin{pmatrix} 3 & 1 & 4 \\ 0 & 0 & 2 \\ 4 & 1 & 4 \end{pmatrix}.$$

Then

$$L \boxdot_X M = \left(\begin{array}{ccccc|ccccc|ccccc} 6 & 9 & 12 & 0 & 3 & 13 & 1 & 4 & 7 & 10 & 5 & 8 & 11 & 14 & 2 \\ 9 & 12 & 0 & 3 & 6 & 1 & 4 & 7 & 10 & 13 & 8 & 11 & 14 & 2 & 5 \\ 12 & 0 & 3 & 6 & 9 & 4 & 7 & 10 & 13 & 1 & 11 & 14 & 2 & 5 & 8 \\ 0 & 3 & 6 & 9 & 12 & 7 & 10 & 13 & 1 & 4 & 14 & 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 & 0 & 10 & 13 & 1 & 4 & 7 & 2 & 5 & 8 & 11 & 14 \\ \hline 1 & 4 & 7 & 10 & 13 & 2 & 5 & 8 & 11 & 14 & 9 & 12 & 0 & 3 & 6 \\ 4 & 7 & 10 & 13 & 1 & 5 & 8 & 11 & 14 & 2 & 12 & 0 & 3 & 6 & 9 \\ 7 & 10 & 13 & 1 & 4 & 8 & 11 & 14 & 2 & 5 & 0 & 3 & 6 & 9 & 12 \\ 10 & 13 & 1 & 4 & 7 & 11 & 14 & 2 & 5 & 8 & 3 & 6 & 9 & 12 & 0 \\ 13 & 1 & 4 & 7^{*} & 10 & 14 & 2 & 5 & 8 & 11 & 6 & 9 & 12 & 0 & 3 \\ \hline 5 & 8 & 11 & 14 & 2 & 12 & 0 & 3 & 6 & 9 & 4 & 7 & 10 & 13 & 1 \\ 8 & 11 & 14 & 2 & 5 & 0 & 3 & 6 & 9 & 12 & 7 & 10 & 13 & 1 & 4 \\ 11 & 14 & 2 & 5 & 8 & 3 & 6 & 9 & 12 & 0 & 10 & 13 & 1 & 4 & 7 \\ 14 & 2 & 5 & 8 & 11 & 6 & 9 & 12 & 0 & 3 & 13 & 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 & 14 & 9 & 12 & 0 & 3 & 6 & 1 & 4 & 7 & 10 & 13 \end{array}\right).$$

**Remark 4.3.** We use the set $\mathbb{Z}_n \times \mathbb{Z}_m$ to label the rows and columns of a step product square, while the entries are from $\mathbb{Z}_{nm}$. For example, the starred entry in Example 4.2 corresponds to the ordered triple $((1,4),(0,3),7)$.

Now that we have our construction, we need to use it to create O2HC-embeddings of complete tripartite graphs. By Theorem 3.5, we simply need to find ce-hamiltonian latin squares that have a 1-partition. In Section 5 we develop conditions for the step product of two latin squares to be ce-hamiltonian. In Section 6 we determine when these squares have a 1-partition. Finally, the step product construction is used to construct ce-hamiltonian latin squares with a 1-partition in Sections 7 and 8.

## 5. Ce-hamiltonicity of step product squares

Requiring that $L$ and $M$ are ce-hamiltonian is not sufficient to ensure that the step product $L \boxdot_X M$ is ce-hamiltonian. The permutations of rows given by the entries in $X$ affect the requisite induced pair graphs. Thus, to obtain a step product square that is ce-hamiltonian, we need to have some conditions on $X$. Let $X(k)$ be the set of cells that contain the entry $k$ in $L$. In other words, $X(k) = \{(i,j) \mid L_{ij} = k\}$. Moreover, set $\sigma_k = \sum_{(i,j)\in X(k)} x_{ij}$ for all $k \in \mathbb{Z}_n$, where the addition is done in $\mathbb{Z}_m$. We call the vector $\sigma(X,L) = (\sigma_0, \sigma_1, ..., \sigma_{n-1})$ the *representative vector for $X$ over $L$*.

**Remark 5.1.** Let $L$ be a latin square of order $n$ and let $X$ be an $n \times n$ matrix with entries from $\mathbb{Z}_m$. We can permute the rows and columns of $L$ without affecting $\sigma(X,L)$ if we apply the same permutation of the rows and columns to $X$. Namely, if $\lambda_r$ and $\lambda_c$ are permutations of $\mathbb{Z}_n$ applied to the rows and columns, respectively, of $L$ and $X$, then

$$\lambda_r\lambda_c(L) \boxdot_{\lambda_r\lambda_c(X)} \mathbb{Z}_m = (\lambda_r, 1_m)(\lambda_c, 1_m)(L \boxdot_X \mathbb{Z}_m)$$

where $1_m$ is the identity permutation on $\mathbb{Z}_m$, and therefore

$$\sigma(\lambda_r\lambda_c(X), \lambda_r\lambda_c(L)) = \sigma(X,L).$$

Here and later in this paper we will need the following observation about the square $x \circ \mathbb{Z}_n$.

**Observation 5.2.** If $M = \mathbb{Z}_m = \{(r,c,r+c) \mid r,c \in \mathbb{Z}_m\}$, then $x \circ M = \{(r,c,r+c-x) \mid r,c \in \mathbb{Z}_m\}$.

We now prove sufficient conditions on $X$ for the product $L \boxdot_X Z_m$ to be ce-hamiltonian.

**Theorem 5.3.** Let $L$ be a ce-hamiltonian latin square with rows, columns and entries from $\mathbb{Z}_n$, let $X$ be an $n \times n$ matrix with entries from $\mathbb{Z}_m$, and let $\sigma(X,L) = (\sigma_0, ..., \sigma_{n-1})$ be the representative vector for $X$ over $L$. Define $\delta_k = \sigma_k - \sigma_{k+1}$ for all $0 \le k \le n-2$ and $\delta_{n-1} = \sigma_{n-1} - \sigma_0 - n$. If $(\delta_k, m) = 1$ for all $k \in \mathbb{Z}_n$, then the latin square $L \boxdot_X \mathbb{Z}_m$ is ce-hamiltonian.

*Proof.* Let $K = L \boxdot_X \mathbb{Z}_m$. Define

$$P_{\ell+1} = \{(C(K,(i,j),\ell+1), C(K,(i,j),\ell)) \mid (i,j) \in \mathbb{Z}_n \times \mathbb{Z}_m\} \text{ for all } \ell \in \mathbb{Z}_{nm}$$

as in Theorem 3.5. We need to show that the induced pair graph $G_{P_{\ell+1}}$ is a hamilton cycle for all $\ell \in \mathbb{Z}_{nm}$.

Write $\ell = an + b$, where $0 \le a \le m-1$ and $0 \le b \le n-1$; we know every occurrence of $\ell \in K$ corresponds to the entries $a \in \mathbb{Z}_m$ and $b \in L$. By Remark 5.1, permuting the rows and columns of $L$ and $X$ simultaneously does not affect $\sigma(X,L)$. It also does not affect ce-hamiltonicity of the resulting step product square $K$. Thus, since $L$ is ce-hamiltonian, we can permute the rows and columns of $L$ and $X$ simultaneously to obtain the following partial representation of $L$:

$$L = \begin{pmatrix} b & b+1 & & \\ & b & b+1 & \\ & & \ddots & \ddots \\ b+1 & & & b \end{pmatrix}.$$

For each $(i,j) \in X(b)$, write $\alpha_i = x_{ij}$. Similarly for each $(i,j) \in X(b+1)$, write $\beta_i = x_{ij}$. We have the following partial representation for $K$:

$$\left(\begin{array}{c|c|c|c}
\alpha_0 \circ (n\mathbb{Z}_m + b) & \beta_0 \circ (n\mathbb{Z}_m + b + 1) & & \\
\hline
 & \alpha_1 \circ (n\mathbb{Z}_m + b) & \beta_1 \circ (n\mathbb{Z}_m + b + 1) & \\
\hline
 & & \ddots & \ddots \\
\hline
\beta_{n-1} \circ (n\mathbb{Z}_m + b + 1) & & & \alpha_{n-1} \circ (n\mathbb{Z}_m + b)
\end{array}\right).$$

We note here that $x \circ (aM + b) = a(x \circ M) + b$ for any latin square $M$ and any integers $a$ and $b$, and we will use the latter representation in the following arguments.

Assume first that $\ell \not\equiv -1 \pmod{n}$; we know that every occurrence of $\ell + 1$ in $L$ corresponds to the entries $a \in \mathbb{Z}_m$ and $b + 1 \in L$. Assume that for some $(i,j)$ we have $C(K,(i,j),\ell+1) = (y,z)$. Then the entry $a$ appears in row $j$ and column $z$ of the square $\beta_i \circ \mathbb{Z}_m$; from Observation 5.2 we know $a = j + z - \beta_i$. We want to determine a column $\gamma$ such that $a$ appears in row $j$ and column $\gamma$ of square $\alpha_i \circ \mathbb{Z}_m$. We know that $a = j + \gamma - \alpha_i = j + z - \beta_i$, which implies that $\gamma = z + \alpha_i - \beta_i$. From this we learn that $C(K,(i,j),\ell) = (y-1, z + \alpha_i - \beta_i)$. Thus, all of the pairs in $P_{\ell+1}$ have the form $((y,z),(y-1,z+\alpha_i-\beta_i))$. From this it is clear that we can partition $P_{\ell+1}$ into subsets of the form

$$\begin{aligned} P(y) = \quad & \{((y,z),(y-1,z+\alpha_{i_0}-\beta_{i_0})), \\ & ((y-1,z+\alpha_{i_0}-\beta_{i_0}),(y-2,z+\alpha_{i_0}+\alpha_{i_1}-\beta_{i_0}-\beta_{i_1})), \ldots, \\ & ((y+1, z+\textstyle\sum_{j=0}^{n-2}\alpha_{i_j} - \sum_{j=0}^{n-2}\beta_{i_j}),(y, z+\sum_{j=0}^{n-1}\alpha_{i_j} - \sum_{j=0}^{n-1}\beta_{i_j}))\} \end{aligned}$$

for all $y \in \mathbb{Z}_n$. Using our representation of $L$ described earlier, we know $b+1$ occurs in the row above $b$ in each column of $L$, so we must have $i_{j+1} = i_j - 1$ and $\{i_0, i_1, \ldots, i_{n-1}\} = \mathbb{Z}_n$; therefore, $\sum_{j=0}^{n-1}\alpha_{i_j} = \sigma_b$ and $\sum_{j=0}^{n-1}\beta_{i_j} = \sigma_{b+1}$. Since $\sigma_b - \sigma_{b+1}$ is relatively prime to $m$, we know $z + \sigma_b - \sigma_{b+1} \neq z$, so the induced pair graph of each $P(y)$ is a path on $n$ edges. Moreover, the fact that $\sigma_b - \sigma_{b+1}$ is relatively prime to $m$ implies that the endpoints of these paths match up to form a single cycle of length $nm$.

If $\ell \equiv -1 \pmod{n}$, then $\ell = an + (n-1)$ and $\ell + 1 = (a+1)n$, so we have $b = n-1$ and $b+1 = 0$. Let $\alpha_i = x_{ij}$ for all $(i,j) \in X(n-1)$ and let $\beta_i = x_{ij}$ for all $(i,j) \in X(0)$. Assume that for some $(i,j)$ we have $C(K,(i,j),\ell+1) = (y,z)$. Then the entry $a+1$ appears in row $j$ and column $z$ of the square $\beta_i \circ \mathbb{Z}_m$; again from Observation 5.2 we know $a + 1 = j + z - \beta_i$. We want to determine a column $\gamma$ such that $a$ appears in row $j$ and column $\gamma$ of square $\alpha_i \circ \mathbb{Z}_m$. We know that $a = j + \gamma - \alpha_i = j + z - \beta_i - 1$, which implies that $\gamma = z + \alpha_i - \beta_i - 1$. From this we learn that $C(K,(i,j),\ell) = (y-1, z+\alpha_i-\beta_i-1)$. Thus, all of the pairs in $P_{\ell+1}$ have the form $((y,z),(y-1,z+\alpha_i-\beta_i-1))$. From this it is clear that we can partition $P_{\ell+1}$ into subsets of the form

$$\begin{aligned} P(y) = \quad & \{((y,z),(y-1,z+\alpha_{i_0}-\beta_{i_0}-1)), \\ & ((y-1,z+\alpha_{i_0}-\beta_{i_0}-1),(y-2,z+\alpha_{i_0}+\alpha_{i_1}-\beta_{i_0}-\beta_{i_1}-2)), \ldots, \\ & ((y+1, z+\textstyle\sum_{j=0}^{n-2}\alpha_{i_j} - \sum_{j=0}^{n-2}\beta_{i_j} - (n-1)),(y, z+\sum_{j=0}^{n-1}\alpha_{i_j} - \sum_{j=0}^{n-1}\beta_{i_j} - n))\} \end{aligned}$$

for all $y \in \mathbb{Z}_n$. From our representation of $L$ described earlier, it is again clear that $i_{j+1} = i_j - 1$, so we must have $\{i_0, i_1, \ldots, i_{n-1}\} = \mathbb{Z}_n$; therefore, $\sum_{j=0}^{n-1}\alpha_{i_j} = \sigma_{n-1}$ and $\sum_{j=0}^{n-1}\beta_{i_j} = \sigma_0$. Since $\sigma_{n-1} - \sigma_0 - n$ is relatively prime to $m$, we know $z + \sigma_{n-1} - \sigma_0 - n \neq z$, so each $P(y)$ induces a path on $n$ edges. Moreover, the fact that $\sigma_{n-1} - \sigma_0 - n$ is relatively prime to $m$ implies that the endpoints of these paths match up to form a single cycle of length $nm$.

Since $G_{P_{\ell+1}}$ is a hamilton cycle for all $\ell \in \mathbb{Z}_{nm}$, the step product square $K$ is ce-hamiltonian. □

## 6. Decomposing step product squares into transversals

We want to present conditions for lifting $k$-plexes in $L$ and $\mathbb{Z}_m$ to transversals in $L \boxdot_X \mathbb{Z}_m$. Before we proceed, a deeper exploration of the step product construction is required.

**Observation 6.1.** Let $(r_1, c_1, e_1) \in L$ and $(r_2, c_2, e_2) \in \mathbb{Z}_m$; furthermore, let $x_{r_1,c_1} \in X$ be the entry found in row $r_1$ and column $c_1$ of the matrix $X$. Utilizing Observation 5.2, the step product square $K = L \boxdot_X \mathbb{Z}_m$ has the following form, where here and elsewhere in this chapter we write $[f(x)]_m$ to denote that the expression $f(x)$ is evaluated in $\mathbb{Z}_m$ and assume all other arithmetic for $K$ is done in $\mathbb{Z}_{nm}$.

$$\begin{array}{cc} & \begin{array}{ccc} \phantom{\cdots\cdots} & (c_1,c_2) & \phantom{\cdots\cdots} \end{array} \\ (r_1,r_2) & \left[\begin{array}{c|c|c} \ddots & \vdots & \cdot^{\cdot^{\cdot}} \\ \hline \cdots & n[e_2 - x_{r_1,c_1}]_m + e_1 & \cdots \\ \hline \cdot^{\cdot^{\cdot}} & \vdots & \ddots \end{array}\right] \end{array},$$

where we note that $e_2 = r_2 + c_2$. Thus, each pair of triples $(r_1, c_1, e_1) \in L$ and $(r_2, c_2, e_2) \in M$ defines a unique triple $((r_1, r_2), (c_1, c_2), n[e_2 - x_{r_1,c_1}]_m + e_1) \in L$. We use this to define a binary operation $\boxdot_X$ on entries of $L$ and $M$ such that $(r_1, c_1, e_1) \boxdot_X (r_2, c_2, e_2) = ((r_1, r_2), (c_1, c_2), n[e_2 - x_{r_1,c_1}]_m + e_1)$. Furthermore, for any subset of ordered triples $T_1 \subseteq L$ and $T_2 \subseteq M$, we define $T_1 \boxdot_X T_2 = \{(r_1, c_1, e_1) \boxdot_X (r_2, c_2, e_2) \mid (r_i, c_i, e_i) \in T_i, i = 1, 2\}$. We extend this definition to include $(r_1, c_1, e_1) \boxdot_X T_2$ and $T_1 \boxdot_X (r_2, c_2, e_2)$ in the obvious way.

**Observation 6.2.** For every entry $((r_1, r_2), (c_1, c_2), e) \in L \boxdot_X M$, there exist unique $(r_1, c_1, e_1) \in L$ and $(r_2, c_2, e_2) \in \mathbb{Z}_m$ such that $(r_1, c_1, e_1) \boxdot_X (r_2, c_2, e_2) = ((r_1, r_2), (c_1, c_2), e)$. Thus, if $S \cap S' = \emptyset$ or $T \cap T' = \emptyset$, then $(S \boxdot_X T) \cap (S' \boxdot_X T') = \emptyset$.

The following lemma explains how transversals in $L$ and $\mathbb{Z}_m$ are lifted to transversals in $L \boxdot_X \mathbb{Z}_m$.

**Lemma 6.3.** Let $L$ be a latin square of order $n$, and let $S$ be a transversal in $L$. Furthermore, let $T$ be a transversal in $\mathbb{Z}_m$. For any $n \times n$ matrix $X$ on $\mathbb{Z}_m$, the collection $S \boxdot_X T$ forms a transversal in $K = L \boxdot_X \mathbb{Z}_m$.

*Proof.* Let $(r_1, r_2) \in \mathbb{Z}_n \times \mathbb{Z}_m$ be any row in $K$. Since $S$ is a transversal in $L$, there is a unique triple $(r_1, c_1, e_1) \in S$ covering the row $r_1$. Since $T$ is a transversal in $\mathbb{Z}_m$, there is a unique triple $(r_2, c_2, e_2) \in T$ covering the row $r_2$. The element $(r_1, c_1, e_1) \boxdot_X (r_2, c_2, e_2) \in S \boxdot_X T$ covers the row $(r_1, r_2)$. An analogous argument shows that $S \boxdot_X T$ covers every column.

Let $e \in \mathbb{Z}_{nm}$ be any entry in $K$. Define $e_1$ and $e_2$ to be the unique integers such that $e = e_2 n + e_1$, with $0 \le e_2 \le m - 1$ and $0 \le e_1 \le n - 1$. Since $S$ is a transversal in $L$, there is a unique triple $(r_1, c_1, e_1) \in S$ covering the entry $e_1$. Let $x = x_{r_1,c_1} \in X$ be the entry found in row $r_1$ and column $c_1$ of the matrix $X$; since $T$ is a transversal in $\mathbb{Z}_m$, there is a unique triple $(r_2, c_2, [e_2 + x]_m) \in T$ covering the entry $[e_2 + x]_m$. The element $(r_1, c_1, e_1) \boxdot_X (r_2, c_2, [e_2 + x]_m) \in S \boxdot_X T$ covers the entry $e$.

We have shown that every row, column, and entry in $K$ is covered *at least once* by $S \boxdot_X T$. Since $|S \boxdot_X T| = nm$, it must be true that every row, column, and entry in $K$ is covered *exactly once* by $S \boxdot_X T$; therefore, $S \boxdot_X T$ is a transversal in $K$. □

While the preceding lemma shows how to lift transversals in $L$ to transversals in $L \boxdot_X \mathbb{Z}_m$ for *any* matrix $X$, the following results prove only that *there exists* a matrix $X$ that allows $m$-plexes in $L$ to be lifted to transversals in $L \boxdot_X \mathbb{Z}_m$.

Before we state and prove this lemma, however, some additional ideas are needed. Given an $m$-plex $S$ in $L$, let $\pi$ and $\tau$ be maps from all row and column pairs in $S$ to the group $\mathbb{Z}_m$; with a slight abuse of notation, we write $\pi, \tau : S \to \mathbb{Z}_m$. For convenience we will use $\pi + g$ to denote the map given by $(r, c) \mapsto [\pi(r, c) + g]_m$; similar conventions will be used for $\tau$. If we also have an $n \times n$ matrix $X$ with entries from $\mathbb{Z}_m$, then we define

$$\begin{aligned} T(S, \pi, \tau, X) &= \{(r, c, e) \boxdot_X (\pi(r, c), \tau(r, c), [\pi(r, c) + \tau(r, c)]_m) \mid (r, c, e) \in S\} \\ &= \{((r, \pi(r, c)), (c, \tau(r, c)), n[\pi(r, c) + \tau(r, c) - x_{r,c}]_m + e \mid (r, c, e) \in S\} \\ &\subset S \boxdot_X \mathbb{Z}_m. \end{aligned}$$

**Lemma 6.4.** Let $L$ be a latin square of order $n$, and let $S$ be an $m$-plex in $L$. Suppose there exist functions $\pi, \tau : S \to \mathbb{Z}_m$ and an $n \times n$ matrix $X$ with entries from $\mathbb{Z}_m$ such that:

(i) for every fixed row $r$ of $L$, $\Psi(\pi, r) = \{\pi(r, c) \mid (r, c, e) \in S\} = \mathbb{Z}_m$;
(ii) for every fixed column $c$ of $L$, $\Psi(\tau, c) = \{\tau(r, c) \mid (r, c, e) \in S\} = \mathbb{Z}_m$;

(iii) for every fixed entry $e$ of $L$, $\Psi(\pi,\tau,e) = \{[\pi(r,c)+\tau(r,c)-x_{r,c}]_m \mid (r,c,e) \in S\} = \mathbb{Z}_m$.

Then $T = T(S,\pi,\tau,X)$ is a transversal in $K = L \boxdot_X \mathbb{Z}_m$.

*Proof.* For each row $r$ of $L$, condition (1) implies that the set $\{(r,\pi(r,c)) \mid (r,c,e) \in S\}$ covers every row of $K$ with first coordinate $r$ exactly once; since this holds for all $r$, $T$ covers every row of $K$ exactly once. An analogous argument based on condition (2) implies that $T$ covers every column of $K$ exactly once. Finally, for each entry $e$ of $L$, condition (3) implies that the set $\{n[\pi(r,c)+\tau(r,c)-x_{r,c}]_m + e \mid (r,c,e) \in S\}$ covers every entry of $K$ that is congruent to $e$ modulo $n$ exactly once; since this holds for all $e$, $T$ covers every entry of $K$ exactly once. Thus, $T$ is a set of triples that covers every row, column and entry of $K$ exactly once; we simply need to show all of these triples are actually in $K$. Note that the triples in $T$ are of the form described in Observation 6.1, so $T$ is indeed a transversal in $K$. □

**Corollary 6.5.** Let $L$ be a latin square of order $n$, and let $S$ be an $m$-plex in $L$. Suppose the conditions of Lemma 6.4 hold for some functions $\pi$ and $\tau$ and some matrix $X$. Then there exists a family of $m^2$ disjoint transversals in $K = L \boxdot_X \mathbb{Z}_m$ covering all of $S \boxdot_X \mathbb{Z}_m$.

*Proof.* Define $T_{g,h} = T(S,\pi+g,\tau+h,X)$ and set $\mathcal{T} = \{T_{g,h} \mid (g,h) \in \mathbb{Z}_m \times \mathbb{Z}_m\}$. Assume $(g,h) \neq (g',h')$; without loss of generality we can assume $g \neq g'$. Since $\pi+g$ differs from $\pi+g'$ on all of $S$, it is clear that $T_{g,h}$ and $T_{g',h'}$ are disjoint. It remains to show that $T_{g,h}$ is a transversal for any $(g,h) \in \mathbb{Z}_m \times \mathbb{Z}_m$. By Lemma 6.4, it will suffice to show that $\pi+g$ and $\tau+h$ satisfy conditions (1), (2) and (3). We obtain the sets $\Psi(\pi+g,r)$, $\Psi(\tau+h,c)$ and $\Psi(\pi+g,\tau+h,e)$ by adding the fixed amounts $g$, $h$ and $g+h$ to the sets $\Psi(\pi,r)$, $\Psi(\tau,c)$ and $\Psi(\pi,\tau,e)$, respectively, and reducing modulo $m$. It is clear that these three sets also cover the entire group $\mathbb{Z}_m$, so each $T_{g,h}$ is a transversal. Thus, the collection $\mathcal{T}$ forms a collection of $m^2$ disjoint transversals contained in $S \boxdot_X \mathbb{Z}_m$; since $\mathcal{T}$ and $S \boxdot_X \mathbb{Z}_m$ both cover $nm^3$ entries, we must have that $\mathcal{T}$ covers all of $S \boxdot_X \mathbb{Z}_m$. □

We now want to show that for any $\pi$ and $\tau$ satisfying conditions (1) and (2) of the previous lemma, we can always find a matrix $X$ such that condition (3) is satisfied as well.

**Lemma 6.6.** Let $L$ be a latin square of order $n$, and let $S$ be an $m$-plex in $L$. Suppose there exist functions $\pi,\tau : S \to \mathbb{Z}_m$ such that:

(i) for every row $r$ of $S$, $\Psi(\pi,r) = \{\pi(r,c) \mid (r,c,e) \in S\} = \mathbb{Z}_m$;

(ii) for every column $c$ of $S$, $\Psi(\tau,c) = \{\tau(r,c) \mid (r,c,e) \in S\} = \mathbb{Z}_m$.

Then we can assign values to the cells in an $n \times n$ matrix $X$ that correspond to $S$ such that for every entry $e$ of $S$, $\Psi(\pi,\tau,e) = \{[\pi(r,c)+\tau(r,c)-x_{r,c}]_m \mid (r,c,e) \in S\} = \mathbb{Z}_m$.

*Proof.* Since any distinct $(r,c,e),(r',c',e') \in S$ must satisfy $(r,c) \neq (r',c')$, we can define $x_{r,c}$ independently for each $(r,c,e) \in S$ such that $\Psi(\pi,\tau,e) = \mathbb{Z}_m$ for every entry $e$ of $S$. □

If $\pi$ is simply the projection of each pair $(r,c) \in S$ to its order by column amongst all pairs in row $r$ of $S$, then we say $\pi$ is the *canonical row projection*. Similarly, if $\tau$ is simply the projection of each pair $(r,c) \in S$ to its order by row amongst all pairs in column $c$ of $S$, then we say $\tau$ is the *canonical column projection*. If $\pi$ or $\tau$ is the projection of each pair to its reverse order by row or column, then we say $\pi$ or $\tau$ is the *reverse canonical row projection* or *reverse canonical column projection*, respectively.

**Remark 6.7.** The canonical projections and reverse canonical projections both satisfy the conditions of Lemma 6.6.

Finally, we put the previous lemmas together to get two decomposition theorems. Theorems 6.8 and 6.9 allow us to build latin squares with a 1-partition from two smaller latin squares that do not necessarily admit such a partition.

**Theorem 6.8.** Let $n$ and $m$ be integers with $m$ odd, and let $L$ be a latin square of order $n$ that admits an $(m,m,...,m,1,1,...1)$-partition. Then there exists an $n \times n$ matrix $X$ on $\mathbb{Z}_m$ such that $K = L \boxdot_X \mathbb{Z}_m$ admits a 1-partition.

*Proof.* Because $m$ is odd, we know $\mathbb{Z}_m$ has a 1-partition by Lemma 2.2. Assume $L$ can be decomposed into $p$ transversals and $q$ labeled $m$-plexes, all of which are mutually disjoint. By counting the number of entries covered by each transversal or $m$-plex, we know $p + qm = n$. By Lemma 6.3, each of the $p$ transversals in $L$

combines with each of the $m$ transversals in $\mathbb{Z}_m$ to yield a transversal in $K$ for any matrix $X$, providing $pm$ total disjoint transversals in $K$. For each $m$-plex $S$, let $\pi$ and $\tau$ be any maps that satisfy the conditions of Lemma 6.6. By that result, we can assign values to the cells in an $n \times n$ matrix $X$ that correspond to each $m$-plex $S$ such that the conditions of Corollary 6.5 are satisfied. Thus, each of the $q$ $m$-plexes in $L$ yields $m^2$ disjoint transversals in $K$, providing $qm^2$ total disjoint transversals in $K$. Because all of the underlying transversals and $m$-plexes in $L$ are mutually disjoint, we know from Observation 6.2 that the $pm + qm^2$ transversals in $K$ are all mutually disjoint. Moreover, $pm + qm^2 = (p + qm)m = nm$, so we have the desired 1-partition of $K$. □

We can relax the restriction that $m$ is odd if $L$ has a complete decomposition into $m$-plexes.

**Theorem 6.9.** Let $L$ be a latin square of order $n$ that admits an $m$-partition. Then there exists an $n \times n$ matrix $X$ on $\mathbb{Z}_m$ such that $K = L \boxdot_X \mathbb{Z}_m$ admits a 1-partition.

*Proof.* For each $m$-plex $S$, let $\pi$ and $\tau$ be any maps that satisfy the conditions of Lemma 6.6. By that result and Corollary 6.5, there exists an $n \times n$ matrix $X$ such that each of the $\frac{n}{m}$ $m$-plexes in $L$ yields $m^2$ disjoint transversals in $K$, providing $nm$ total disjoint transversals in $K$. □

## 7. Construction for $n \equiv 2 \pmod 4$

Our goal is to build ce-hamiltonian latin squares that admit a 1-partition. In Section 6, we proved the existence of a matrix $X$ so that for an appropriate choice of $L$ and $M$ the step product $L \boxdot_X M$ has a 1-partition. Moreover, we showed in Section 5 that if $X$ satisfies certain properties, this step product square will be ce-hamiltonian. In this section, we determine some squares $L$ and $M$ for which the $X$ from Theorem 6.8 can be modified to also meet the conditions of Theorem 5.3. The result will be a family of latin squares that correspond to O2HC-embeddings of complete tripartite graphs.

To accomplish this goal, we need to use the step product on two different levels. First, we are going to find a matrix $X$ so that $K_{2p} = \mathbb{Z}_p \boxdot_X \mathbb{Z}_2$ has the desired properties of $L$. In particular, we want $K_{2p}$ to be ce-hamiltonian and to have a $(q, q, 1, 1, ..., 1)$-partition for any fixed odd $q$, $3 \le q \le p$. Then, we are going to find a matrix $Y$ so that $K_{2p} \boxdot_Y \mathbb{Z}_q = (\mathbb{Z}_p \boxdot_X \mathbb{Z}_2) \boxdot_Y \mathbb{Z}_q$ is a ce-hamiltonian latin square of order $2pq$ that admits a 1-partition.

**Lemma 7.1.** Let $p$ and $q$ be odd integers with $q$ prime and $p \ge q \ge 3$. There exists a $p \times p$ matrix $X$ on $\mathbb{Z}_2$ such that $K_{2p} = \mathbb{Z}_p \boxdot_X \mathbb{Z}_2$ is ce-hamiltonian and admits a $(q, q, 1, 1, ..., 1)$-partition.

*Proof.* Let $X$ initially be the $p \times p$ matrix of all 0's; we will modify $X$ to get appropriate values. Since the union of two disjoint transversals is a 2-plex, Lemma 2.2 implies that we can find $\frac{1}{2}(p - q)$ disjoint 2-plexes in $\mathbb{Z}_p$; call them $S_0, ..., S_{\frac{1}{2}(p-q)-1}$. Using the canonical row and column projections together with Lemma 6.6 and Corollary 6.5, we can assign values to $X$ so that the collection $\mathcal{S} = \{S_0 \boxdot_X \mathbb{Z}_2, ..., S_{\frac{p-q}{2}-1} \boxdot_X \mathbb{Z}_2\}$ can be partitioned into $2(p - q)$ disjoint transversals. Note that we only assigned values to the cells in $X$ corresponding to the cells of $\mathbb{Z}_p$ covered by the $S_i$'s.

The previous paragraph required combining $p - q$ disjoint transversals in $\mathbb{Z}_p$ to get $\frac{1}{2}(p - q)$ disjoint 2-plexes. Thus, there are $q$ disjoint transversals in $\mathbb{Z}_p$ remaining; call them $T_0, ..., T_{q-1}$. We need to assign values to $X$ so the collection $\{T_0 \boxdot_X \mathbb{Z}_2, ..., T_{q-1} \boxdot_X \mathbb{Z}_2\}$ can be partitioned into 2 disjoint $q$-plexes. For all $(r, c, e) \in T_0$, we keep $x_{r,c} = 0$ and form the collection

$$\begin{aligned} W_1 &= \{(r, c, e) \boxdot_X (0, 0, 0), (r, c, e) \boxdot_X (0, 1, 1), (r, c, e) \boxdot_X (1, 0, 1) \mid (r, c, e) \in T_0\} \\ &= \{((r, 0), (c, 0), e), ((r, 0), (c, 1), p + e), ((r, 1), (c, 0), p + e) \mid (r, c, e) \in T_0\}. \end{aligned}$$

Note that $W_1$ covers every row of the form $(r, 0)$ twice and every row of the form $(r, 1)$ once, every column of the form $(c, 0)$ twice and every column of the form $(c, 1)$ once, and every entry of the form $e$ once and every entry of the form $p + e$ twice, where $r, c, e \in \mathbb{Z}_p$. For all $(r, c, e) \in T_1$ we set $x_{r,c} = 1$ and form the collection

$$\begin{aligned} W_2 &= \{(r, c, e) \boxdot_X (1, 1, 0) \mid (r, c, e) \in T_1\} \\ &= \{((r, 1), (c, 1), p + e) \mid (r, c, e) \in T_1\}. \end{aligned}$$

Note that $W_2$ covers every row of the form $(r, 1)$ once, every column of the form $(c, 1)$ once, and every entry of the form $p + e$ once, where $r, c, e \in \mathbb{Z}_p$. The collection $W_1 \cup W_2$ covers every row and column of $K_{2p}$ exactly twice, covers every entry of the form $e$ once, and covers every entry of the form $p + e$ three times, where $e \in \mathbb{Z}_p$.

Thus far, we have assigned values to all entries $x_{r,c} \in X$ except if $(r, c, e) \in T_i$ for some $i = 2, 3, ..., q-1$. Let $\sigma(X, L) = (\sigma_0, \sigma_1, ..., \sigma_{p-1})$ be the representative vector for $X$ over $L$ as defined in Section 5. Enumerate $T_2$ as $\{(r_0, c_0, 0), (r_1, c_1, 1), ..., (r_{p-1}, c_{p-1}, p-1)\}$. We will assign values to $x_{r_i,c_i}$ for all $i \in \mathbb{Z}_p$ such that $\sigma(X, L)$ satisfies the conditions of Theorem 5.3. Since every $x_{r_i,c_i}$ and $\sigma_i$ is simply 0 or 1, we assign values to $x_{r_i,c_i}$ so that $\sigma_i = 1$ if $i$ is even, and $\sigma_i = 0$ otherwise. Thus, for all $i = 0, 1, ..., p-2$ we have $\sigma_i - \sigma_{i+1} \equiv 1 \pmod 2$, and $\sigma_{p-1} - \sigma_0 - p = 1 - 1 - p \equiv 1 \pmod 2$ as well, so the conditions of Theorem 5.3 are satisfied. It remains to show that we can find the partition into 2 disjoint $q$-plexes without changing any more entries in $X$.

Let $E_0 = \{(0, 0, 0), (1, 1, 0)\} \subset \mathbb{Z}_2$ and $E_1 = \{(0, 1, 1), (1, 0, 1)\} \subset \mathbb{Z}_2$. Form the collection $W_3 = \{(r, c, e) \boxdot_X E_{x_{r,c}} \mid (r, c, e) \in T_2\}$; elements of $W_3$ can take two forms depending on the value of $x_{r,c}$. For all $(r, c, e) \in T_2$, if $x_{r,c} = 0$ then $((r, 0), (c, 0), e), ((r, 1), (c, 1), e) \in W_3$, and if $x_{r,c} = 1$, then $((r, 0), (c, 1), e)$, $((r, 1), (c, 0), e) \in W_3$. Note that in either case the rows $(r, 0)$ and $(r, 1)$ and the columns $(c, 0)$ and $(c, 1)$ are covered once, while the entry $e$ is covered twice. Thus, $W_3$ covers every row and column in $K_{2p}$ exactly once and every entry $e$ exactly twice, where $e \in \mathbb{Z}_p$. Therefore, the collection $W_1 \cup W_2 \cup W_3$ forms a 3-plex. We now have $q - 3$ remaining unused transversals, which are $T_3, ..., T_{q-1}$. Recalling that $x_{r,c} = 0$ for any $(r, c, e) \in T_i$ with $i = 3, ..., q-1$, let

$$\begin{aligned} W_{even} &= \{T_i \boxdot_X E_0 \mid i \text{ even}\} \\ &= \{((r, 0), (c, 0), e), ((r, 1), (c, 1), e) \mid (r, c, e) \in T_i, i \text{ even}\} \end{aligned}$$

and

$$\begin{aligned} W_{odd} &= \{T_i \boxdot_X E_1 \mid i \text{ odd}\} \\ &= \{((r, 0), (c, 1), p+e), ((r, 1), (c, 0), p+e) \mid (r, c, e) \in T_i, i \text{ odd}\}. \end{aligned}$$

Note that $W_{even}$ covers every row and column of $K_{2p}$ exactly $\frac{1}{2}(q-3)$ times. Additionally, $W_{even}$ covers every entry $e$ exactly $q-3$ times, where $e \in \mathbb{Z}_p$. Similarly, $W_{odd}$ covers every row and column of $K_{2p}$ exactly $\frac{1}{2}(q-3)$ times. Additionally, $W_{odd}$ covers every entry $p+e$ exactly $q-3$ times. Thus, $W_{even} \cup W_{odd}$ forms a $(q-3)$-plex, and the collection $W = W_1 \cup W_2 \cup W_3 \cup W_{even} \cup W_{odd}$ forms a $q$-plex. Let $V = K_{2p} \setminus (\mathcal{S} \cup W)$; since $\mathcal{S}$ and $W$ together cover every row, column, and entry of $K_{2p}$ exactly $2(p-q) + q = 2p - q$ times, $V$ must cover every row, column, and entry of $K_{2p}$ exactly $2p - (2p - q) = q$ times. Therefore, $V$ is also a $q$-plex, and we have our desired partition of $K_{2p}$ into $2(p-q)$ transversals and 2 $q$-plexes, all of which are mutually disjoint. As mentioned before, the matrix $X$ we constructed satisfies the conditions of Theorem 5.3, so $K_{2p}$ is also ce-hamiltonian. □

We want to use $K_{2p}$ as the first ingredient of a step product construction with $\mathbb{Z}_q$; in this step product we use a matrix $Y$ to form the product $K_{2p} \boxdot_Y \mathbb{Z}_q$. We need to show that the matrix $Y$ that we obtain from Theorem 6.8 can be modified to satisfy the conditions of Theorem 5.3 without destroying the partition into transversals. To do so, we will apply Theorem 6.8 to $K_{2p}$ and $\mathbb{Z}_q$ with one $q$-plex labeled with the canonical projections and the other $q$-plex labeled with the reverse canonical projections. This will yield a matrix $Y$ such that $K_{2p} \boxdot_Y \mathbb{Z}_q$ has a 1-partition. We will then alter $\pi$ and $\tau$ for each $q$-plex and show that making appropriate changes in $Y$ yields a matrix such that $K_{2p} \boxdot_Y \mathbb{Z}_q$ still admits a 1-partition, but is ce-hamiltonian as well.

Before we accomplish this goal, it will now be helpful to identify some properties of the $q$-plexes created in Lemma 7.1. Observe that when $(r, c, e) \in T_0$, 3 of the 4 entries of $K_{2p}$ obtained from $(r, c, e) \boxdot_X \mathbb{Z}_2$ are in $W$. Similarly when $(r, c, e) \in T_1$, 1 of the 4 entries of $K_{2p}$ obtained from $(r, c, e) \boxdot_X \mathbb{Z}_2$ is in $W$. We assume for the sake of presentation that $T_0$ and $T_1$ are found along the first two diagonals of $\mathbb{Z}_p$. We then have the following representation for $K_{2p}$, where the superscript denotes whether each entry is in the $q$-plex $W$ or $V$:

$$\left(\begin{array}{cc|cc|cc|cc}
0^W & p^W & p+1^V & 1^V & & & & \\
p^W & 0^V & 1^V & p+1^W & & & & \\
\hline
 & & 2^W & p+2^W & p+3^V & 3^V & & \\
 & & p+2^W & 2^V & 3^V & p+3^W & & \\
\hline
 & & & & & \ddots & & \ddots \\
\hline
2p-1^V & p-1^V & & & & & p-2^W & 2p-2^W \\
p-1^V & 2p-1^W & & & & & 2p-2^W & p-2^V
\end{array}\right).$$

Assume that $W$ is labeled with the canonical projections $\pi_W = \pi$ and $\tau_W = \tau$. The triple $(r, c, e) \in T_0$ gives rise to the triples $((r,0),(c,0),e), ((r,0),(c,1),p+e), ((r,1),(c,0),p+e) \in W$. Assume $\pi$ and $\tau$ take the following values:

$$\begin{array}{rclrcl}
\pi((r,0),(c,0)) & = & \rho_1, & \tau((r,0),(c,0)) & = & \gamma_1, \\
\pi((r,0),(c,1)) & = & \rho_2, & \tau((r,0),(c,1)) & = & \gamma_2, \\
\pi((r,1),(c,0)) & = & \rho_3, & \tau((r,1),(c,0)) & = & \gamma_3.
\end{array}.$$

From the way $\pi$ and $\tau$ are defined, we know $\rho_2 = \rho_1 + 1$ and $\gamma_3 = \gamma_1 + 1$. We want to switch the values of $\pi$ and $\tau$ for these triples while preserving condition (3) of Lemma 6.4. To do so, we will simultaneously need to change the matrix $Y$. First, consider the elements $a_1 = [\rho_1 + \gamma_1 - x_{(r,0),(c,0)}]_q$ and $a_2 = [\rho_2 + \gamma_2 - x_{(r,0),(c,1)}]_q$. If we switch $\rho_1$ and $\rho_2$, then we obtain the elements $a_1' = [\rho_2 + \gamma_1 - x_{(r,0),(c,0)}]_q$ and $a_2' = [\rho_1 + \gamma_2 - x_{(r,0),(c,1)}]_q$. To ensure $\{a_1', a_2'\} = \{a_1, a_2\}$, we also need to increase $x_{(r,0),(c,0)}$ by one and decrease $x_{(r,0),(c,1)}$ by one. This increases $\sigma_e$ by one while decreasing $\sigma_{p+e}$ by one. Such a switch is called a *$W$-row switch on $e$*. Now consider $a_3 = [\rho_3 + \gamma_3 - x_{(r,1),(c,0)}]_q$. If we switch $\gamma_1$ and $\gamma_3$, then we obtain the elements $a_1'' = [\rho_1 + \gamma_3 - x_{(r,0),(c,0)}]_q$ and $a_3'' = [\rho_3 + \gamma_1 - x_{(r,1),(c,0)}]_q$. To ensure $\{a_1'', a_3''\} = \{a_1, a_3\}$, we also need to increase $x_{(r,0),(c,0)}$ by one and decrease $x_{(r,1),(c,0)}$ by one. This again increases $\sigma_e$ by one while decreasing $\sigma_{p+e}$ by one. Such a switch is called a *$W$-column switch on $e$*.

Given any $e \in \mathbb{Z}_p$, we can perform just a $W$-row switch on $e$ or we can independently perform both a $W$-row switch and a $W$-column switch on $e$. Thus, we can increase $\sigma_e$ by either one or two, respectively, while decreasing $\sigma_{p+e}$ by that same amount. We want to define similar switches for the triples in $V$ that are obtained from $T_1$; however, if we want the switches to affect $\sigma_e$ and $\sigma_{p+e}$ in the same way, we need to assume that $V$ is initially labeled with the reverse canonical projections $\pi_V$ and $\tau_V$. It is not hard to show that $V$-row and $V$-column switches on $e$ also increase $\sigma_e$ by one or two while decreasing $\sigma_{p+e}$ by that same amount. Thus, we can increase $\sigma_e$ by up to four while decreasing $\sigma_{p+e}$ by that same amount without affecting the transversals that $W$ and $V$ yield through Corollary 6.5.

The following lemma supplies a special case in the proof of Theorem 7.3 below, while also providing an example of both Lemma 7.1 and switches.

**Lemma 7.2.** There exists a ce-hamiltonian latin square of order 18 that admits a 1-partition.

*Proof.* A matrix $X'$ guaranteed by Lemma 7.1 so that $K_6 = \mathbb{Z}_3 \boxdot_{X'} \mathbb{Z}_2$ is ce-hamiltonian and can be partitioned into 2 disjoint 3-plexes is shown below. Let $\pi_W$ and $\tau_W$ be the canonical projections for $W$, and let $\pi_V$ and $\tau_V$ be the reverse canonical projections for $V$; a matrix $X$ guaranteed by Theorem 6.8 so that $K_6 \boxdot_X \mathbb{Z}_3$ has a 1-partition is shown below, along with $K_6$. For each $(r, c, e) \in K_6$, the superscript denotes whether that entry is in the 3-plex $W$ or $V$ and provides the ordered pair $\pi_W(r,c), \tau_W(r,c)$ or $\pi_V(r,c), \tau_V(r,c)$, and the subscript provides $x_{r,c}$.

$$X' = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, K_6 = \left(\begin{array}{cc|cc|cc}
0_0^{W,0,0} & 3_0^{W,1,0} & 4_2^{V,2,2} & 1_0^{V,1,2} & 2_0^{W,2,0} & 5_0^{V,0,2} \\
3_2^{W,0,1} & 0_2^{V,2,2} & 1_0^{V,1,1} & 4_0^{W,1,0} & 5_1^{V,0,2} & 2_2^{W,2,0} \\ \hline
4_0^{V,2,2} & 1_0^{W,0,1} & 2_0^{W,1,0} & 5_0^{W,2,1} & 3_0^{V,1,1} & 0_0^{V,0,1} \\
1_0^{W,0,2} & 4_0^{V,2,1} & 5_0^{W,1,1} & 2_2^{V,1,1} & 0_0^{V,0,0} & 3_0^{W,2,1} \\ \hline
5_0^{V,2,1} & 2_0^{V,1,0} & 0_0^{W,0,2} & 3_0^{V,0,0} & 1_2^{W,1,1} & 4_1^{W,2,2} \\
2_0^{V,2,0} & 5_1^{W,0,2} & 3_0^{V,1,0} & 0_2^{W,1,2} & 4_2^{W,2,2} & 1_2^{V,0,0}
\end{array}\right).$$

At this point, we have $\sigma(X, K_6) = (1,1,1,2,2,2)$. This does not satisfy the conditions of Theorem 5.3, so we perform the following $W$-row switch on 1. The notation below is similar to that used above; however, we only label the entries involved in the switch.

$$K_6 : \left(\begin{array}{cc|cc|cc}
0 & 3 & 4 & 1 & 2 & 5 \\
3 & 0 & 1 & 4 & 5 & 2 \\ \hline
4 & 1 & 2 & 5 & 3 & 0 \\
1 & 4 & 5 & 2 & 0 & 3 \\ \hline
5 & 2 & 0 & 3 & 1_2^{W,1,1} & 4_1^{W,2,2} \\
2 & 5 & 3 & 0 & 4 & 1
\end{array}\right) \Longrightarrow \left(\begin{array}{cc|cc|cc}
0 & 3 & 4 & 1 & 2 & 5 \\
3 & 0 & 1 & 4 & 5 & 2 \\ \hline
4 & 1 & 2 & 5 & 3 & 0 \\
1 & 4 & 5 & 2 & 0 & 3 \\ \hline
5 & 2 & 0 & 3 & 1_0^{W,2,1} & 4_0^{W,1,2} \\
2 & 5 & 3 & 0 & 4 & 1
\end{array}\right).$$

This switch increases $\sigma_1$ by one while decreasing $\sigma_4$ by one; we now have $\sigma(X, K_6) = (1, 2, 1, 2, 1, 2)$. This satisfies the conditions of Theorem 5.3, so the square $K_6 \boxdot_X \mathbb{Z}_3$ is ce-hamiltonian and has a 1-partition. $\square$

We are now ready to provide the main construction for most values of $n$ such that $n \equiv 2 \pmod 4$.

**Theorem 7.3.** Let $p$ and $q$ be odd integers with $q$ prime and $p \geq q \geq 3$. There exists a $2p \times 2p$ matrix $X$ such that $K_{2p} \boxdot_X \mathbb{Z}_q$ is a ce-hamiltonian latin square of order $2pq$ that admits a 1-partition.

*Proof.* Lemma 7.1 gives us a partition of $K_{2p}$ into $2(p-q)$ transversals and 2 $q$-plexes $W$ and $V$; we can assume $W$ is labeled with the canonical projections and $V$ is labeled with the inverse of the canonical projections. By Theorem 6.8, there exists a $2p \times 2p$ matrix $X$ such that $K_{2p} \boxdot_X \mathbb{Z}_q$ has a 1-partition. We need to show that this $X$ can be altered to meet the conditions of Theorem 5.3.

If $p > q$, then our partition of $K_{2p}$ contains at least one transversal, call it $T$. Since $q$ is prime, $\gcd(y, q) = 1$ for all $0 \neq y \in \mathbb{Z}_q$. Moreover, since $q \geq 3$, there exists some $\gamma \in \mathbb{Z}_q$ such that $1 - \gamma$ and $\gamma - 1 - 2p$ are both nonzero. Since Lemma 6.3 works for any $X$, and Theorem 6.8 allows arbitrary values in entries of $X$ corresponding to transversals in $K_{2p}$, we can assign arbitrary values to $x_{r,c}$ when $(r, c, e) \in T$. Enumerate $T$ as $\{(r_0, c_0, 0), ..., (r_{2p-1}, c_{2p-1}, 2p-1)\}$; just as in the proof of Lemma 7.1, both $x_{r_i,c_i}$ and $\sigma_i$ are integers modulo $q$, so we simply assign values to $x_{r_i,c_i}$ so that $\sigma_i = 1$ if $i$ is even, $\sigma_i = 0$ if $i = 1, 3, ..., 2p-3$, and $\sigma_{2p-1} = \gamma$. Thus, for $i = 0, 2, ..., 2p-4$ we have $\sigma_i - \sigma_{i+1} \equiv 1 \pmod q$, for $i = 1, 3, ..., 2p-3$ we have $\sigma_i - \sigma_{i+1} \equiv -1 \pmod q$, $\sigma_{2p-2} - \sigma_{2p-1} = 1 - \gamma$, and $\sigma_{2p-1} - \sigma_0 - 2p = \gamma - 1 - 2p$. Since $-1, 1, 1-\gamma$ and $\gamma - 1 - 2p$ are all relatively prime to $q$, the conditions of Theorem 5.3 are satisfied.

Now assume $p = q \geq 5$, and define $\delta_k = \sigma_k - \sigma_{k+1}$ for $0 \leq k \leq 2p-2$ and $\delta_{2p-1} = \sigma_{2p-1} - \sigma_0 - 2p$ just as in Theorem 5.3. Set $D = \{\delta_0, \delta_1, ..., \delta_{2p-1}\}$; our goal is to ensure $(d, q) = 1$ for all $d \in D$. Since $q$ is assumed to be prime, we simply need to ensure that every element in $D$ is nonzero.

Choose any $0 \leq e \leq p-1$ such that at least one member of $D(e) = \{\delta_{e-1}, \delta_e, \delta_{e+p-1}, \delta_{e+p}\}$ is zero. Since $q \geq 5$, there exists some $\lambda \in \{1, 2, 3, 4\} \subset \mathbb{Z}_q$ such that $\delta_{e-1} - \lambda$, $\delta_e + \lambda$, $\delta_{e+p-1} + \lambda$, and $\delta_{e+p} - \lambda$ are all nonzero. Now, we simply perform enough $W$- and $V$-row and column switches on $e$ so that $\sigma_e$ is increased by $\lambda$ and $\sigma_{p+e}$ is decreased by $\lambda$. Since switching on $e$ in this way will never affect any $\delta_k \notin D(e)$, we can repeat this process until every element in $D$ is nonzero and $\sigma(X, K_{2p})$ satisfies the conditions of Theorem 5.3.

The final case $p = q = 3$ is covered by Lemma 7.2. $\square$

## 8. Construction for $n \equiv 0 \pmod 4$

In the case when $n \equiv 0 \pmod 4$, we will use $L = \mathbb{Z}_{\frac{n}{2}}$ and $M = \mathbb{Z}_2$. Before we provide the general constructions, we modify the switching procedure described for $K_{2p}$ to work on the square $\mathbb{Z}_{\frac{n}{2}}$. In the 2-partition guaranteed by Lemma 2.2, every row $r$ of each 2-plex $S$ contains two entries $(r, c, e)$ and $(r, c+1, e+1)$, where $e = r + c$ is even. Let $\pi_S = \pi$ be defined so that $\pi(r, c) = 0$ and $\pi(r, c+1) = 1$ for each row $r$; this is equivalent to the canonical row projection except when $c = \frac{n}{2} - 1$. We again want to switch the values of $\pi$ for these triples without changing the entries they yield in $S \boxdot_X \mathbb{Z}_2$. As before, this switch results in increasing $x_{r,c}$ by one and decreasing $x_{r,c+1}$ by one; in $\mathbb{Z}_2$ this is equivalent to flipping the values $\sigma_e$ and $\sigma_{e+1}$ from 0 to 1 or from 1 to 0. This will similarly be called a *$S$-row switch on $e$*. In an analogous fashion, we note that every column of $S$ contains two entries $(r, c, e-1)$ and $(r+1, c, e)$, where again $e = r + c + 1$ is even. Let $\tau_S = \tau$ be defined so that $\tau(r, c) = 0$ and $\tau(r+1, c) = 1$ for each column $c$; this is equivalent to the canonical column projection except when $r = \frac{n}{2} - 1$. By switching the $\tau$ values of these triples, we flip the values $\sigma_{e-1}$ and $\sigma_e$ in $\mathbb{Z}_2$; this is a *$S$-column switch on $e$*. By performing both a $S$-row switch on $e$ and a $S$-column switch on $e$, we leave the value of $\sigma_e$ unchanged while flipping both $\sigma_{e-1}$ and $\sigma_{e+1}$ in $\mathbb{Z}_2$. The following lemma will be needed.

**Lemma 8.1.** Let $p$ be even, and let $S$ be a 2-plex in $\mathbb{Z}_p$. If $\pi_S = \pi, \tau_S = \tau$ and $X$ satisfy the conditions of Lemma 6.4, then the set $\{x_{r,c} \mid (r, c, e) \in S, x_{r,c} = 1\}$ contains an even number of elements.

*Proof.* For each entry $e \in Z_p$, let $(r_1, c_1, e), (r_2, c_2, e) \in S$ be the two triples of $S$ containing $e$; condition (3) of Lemma 6.4 implies that we must have

$$\{\pi(r_1, c_1) + \tau(r_1, c_1) - x_{r_1,c_1}, \pi(r_2, c_2) + \tau(r_2, c_2) - x_{r_2,c_2}\} = \{0, 1\}.$$

Thus, since $p$ is even,

$$\sum_{(r,c,e)\in S} (\pi(r,c) + \tau(r,c) - x_{r,c}) = 0,$$

where the addition is taken in $\mathbb{Z}_2$. Again since $p$ is even,

$$\sum_{(r,c,e)\in S} \pi(r,c) = \sum_{(r,c,e)\in S} \tau(r,c) = 0$$

in $\mathbb{Z}_2$, so we must have

$$\sum_{(r,c,e)\in S} x_{r,c} = 0$$

as well. It follows that the set $\{x_{r,c} \mid (r,c,e) \in S, x_{r,c} = 1\}$ has even order. $\square$

We are now ready to prove the main construction for $n \equiv 0 \pmod 8$.

**Theorem 8.2.** Let $n \equiv 0 \pmod 8$. There exists an $\frac{n}{2} \times \frac{n}{2}$ matrix $X$ such that $\mathbb{Z}_{\frac{n}{2}} \boxdot_X \mathbb{Z}_2$ is a ce-hamiltonian latin square that admits a 1-partition.

*Proof.* Set $b = \frac{n}{4}$; note that $b$ is even. By Lemma 2.2 there exists a 2-partition $\mathcal{S}$ of $\mathbb{Z}_{2b}$. For each 2-plex $S$ we use $\pi_S$ and $\tau_S$ as defined at the beginning of this section; by Theorem 6.9 there exists a $2b \times 2b$ matrix $X$ such that $\mathbb{Z}_{2b} \boxdot_X \mathbb{Z}_2$ has a 1-partition. We need to show that this $X$ can be modified to meet the conditions of Theorem 5.3. According to Lemma 8.1, each 2-plex $S \in \mathcal{S}$ has an even number of corresponding 1's in $X$. Thus, if $\sigma(X, \mathbb{Z}_{2b}) = (\sigma_0, \sigma_1, ..., \sigma_{2b-1})$, then

$$\sum_{i=0}^{2b-1} \sigma_i = \sum_{x_{r,c}\in X} x_{r,c} = \sum_{S\in\mathcal{S}} \left( \sum_{(r,c,e)\in S} x_{r,c} \right) = 0,$$

and the representative vector $\sigma(X, \mathbb{Z}_{2b})$ has an even number of 1's. Let $v(i)$ be the vector of length $2b$ with 1's in position $i$ and $i+1$ and 0's everywhere else; the set $\{v(i) \mid 0 \le i \le 2b-2\}$ forms a basis over $\mathbb{Z}_2$ for all length $2b$ vectors with an even number of 1's. Furthermore, any $S$-row or $S$-column switch on $e$ will simply add $v(e)$ or $v(e-1)$, respectively, to $\sigma(X, \mathbb{Z}_{2b})$. Since $b$ is even, the vector $(0,1,0,1,...,0,1)$ contains an even number of 1's; we now perform the required $S$-row and $S$-column switches to ensure that $\sigma(X, \mathbb{Z}_{2b}) = (0,1,0,1,...,0,1)$. It is now clear that $\sigma(X, \mathbb{Z}_{2b})$ meets the conditions of Theorem 5.3. $\square$

When $n \equiv 4 \pmod 8$, we require a special construction. Again set $b = \frac{n}{4}$, where $b$ is now odd, and let $X$ be the $2b \times 2b$ matrix with $x_{2b-1,c} = 1$ for all even $c$, and $x_{r,c} = 0$ otherwise. We form the square $J_n = \mathbb{Z}_{2b} \boxdot_X \mathbb{Z}_2$ and show that it has the required properties.

**Theorem 8.3.** Let $n \equiv 4 \pmod 8$ with $n \ge 12$. The latin square $J_n$ is ce-hamiltonian and admits a 1-partition.

*Proof.* It is readily seen that $\sigma(X, \mathbb{Z}_{2b}) = (0,1,0,1,...,0,1)$; thus, $J_n$ is ce-hamiltonian by Theorem 5.3. It remains to show that $J_n$ admits a 1-partition.

For a 2-plex $S$ in $\mathbb{Z}_{2b}$ and any maps $\pi_S = \pi$ and $\tau_S = \tau$, we say $(r,c,e) \in S$ has a *uniform label* if $\pi(r,c) = \tau(r,c)$; otherwise we say $(r,c,e) \in S$ has a *mixed label.* Let $S$ be the following 2-plex in $\mathbb{Z}_{2b}$, where the superscript on each entry $(r,c,e)$ defines $\pi_S(r,c), \tau_S(r,c)$.

$$S = \left( \begin{array}{cc|ccc|ccc}
0^{0,0} & 1^{1,1} & & & & & & \\
\hline
 & & 3^{0,0} & & & (b+2)^{1,0} & & \\
 & & & \ddots & & & \ddots & \\
 & & & & (2b-1)^{0,0} & & & (b-2)^{1,0} \\
\hline
b^{0,1} & (b+1)^{1,0} & & & & & & \\
\hline
 & & (b+3)^{0,1} & & & 2^{1,1} & & \\
 & & & \ddots & & & \ddots & \\
 & & & & (b-1)^{0,1} & & & (2b-2)^{1,1}
\end{array} \right)$$

Set $T_{g,h} = T(S, \pi_S + g, \tau_S + h, X)$; we will show that the conditions of Lemma 6.4 are met. It is readily seen from their definition that $\pi_S$ takes both values 0 and 1 in every row of $S$ and $\tau_S$ takes both values 0 and 1 in every column of $S$, so conditions (1) and (2) are met for any $g, h \in \mathbb{Z}_2$. Finally, $x_{r,c} = 0$ for all $(r, c, e) \in S$, and each entry in $S$ has one uniform label and one mixed label, so $\Psi(\pi_S + g, \tau_S + h, e) = \{0, 1\}$ for each entry $e \in S$, and condition (3) is met for any $g, h \in \mathbb{Z}_2$ as well. By Lemma 6.4, $T_{0,0}$ is a transversal in $J_n$. As in the proof of Corollary 6.5, $T_{0,1}$ is also a transversal in $J_n$. Additionally, let $S + y$ be the 2-plex with the same labels as $S$ that is obtained by shifting the cells of $S$ $y$ columns to the right, and let $T_{g,h} + y = T(S + y, \pi_S + g, \tau_S + h, X)$. If $y$ is even, then the entries in row $2b-1$ of $S+y$ are all even, so the corresponding entries in $X$ are 0 and it is readily seen that $T_{0,0} + y$ and $T_{0,1} + y$ are also transversals. We claim the collection $\mathcal{S} = \{T_{0,0}, T_{0,0} + 2, .., T_{0,0} + (2b-2), T_{0,1}, T_{0,1} + 2, ..., T_{0,1} + (2b-2)\}$ forms $2b$ mutually disjoint transversals in $J_n$. Indeed, the 2-plexes $S + y$ and $S + z$ are disjoint unless $z = y + b - 1$ (or $y = z + b - 1$). In that case, the 2-plexes $S + y$ and $S + (y + b - 1)$ overlap in every row except row 0 and $b$. However, for every overlapped entry, the transversals $T_{0,0} + y$ and $T_{0,1} + y$ use the entries in row 0 of the corresponding subsquare $\mathbb{Z}_2$, while the transversals $T_{0,0} + (y + b - 1)$ and $T_{0,1} + (y + b - 1)$ use the entries in row 1 of the corresponding subsquare $\mathbb{Z}_2$. Thus, the members of $\mathcal{S}$ are mutually disjoint, and $|\mathcal{S}| = 2b$.

We now present a similar 2-plex that we can use to cover the remaining entries in $\mathbb{Z}_{2b}$. Let $S'$ be the following 2-plex, with superscripts again defining $\pi_{S'}, \tau_{S'}$.

$$S' = \left(\begin{array}{c|cc|cccc|ccc}
 & 1^{0,0} & 2^{1,1} & & & & & & & \\ \hline
 & & & 4^{0,0} & & & & (b+3)^{1,0} & & \\
 & & & & \ddots & & & & \ddots & \\
 & & & & & (2b-2)^{0,0} & & & & (b-3)^{1,0} \\
(b-1)^{1,0} & & & & & & 0^{0,0} & & & \\ \hline
 & (b+1)^{0,1} & (b+2)^{1,0} & & & & & & & \\ \hline
 & & & (b+4)^{0,1} & & & & 3^{1,1} & & \\
 & & & & \ddots & & & & \ddots & \\
 & & & & & (b-2)^{0,1} & & & & (2b-3)^{1,1} \\ \hline
(2b-1)^{0,1} & & & & & & b^{1,1} & & &
\end{array}\right)$$

Note that $b$ has two uniform labels and $2b-1$ has two mixed labels; these correspond to nonzero entries in the last row of $X$. Every other entry has both a uniform and a mixed label, and the corresponding entries in $X$ are 0. Lemma 6.4 again implies that $T' = T(S', \pi_{S'}, \tau_{S'}, X)$ is a transversal. By defining $S' + y$ and $T'_{g,h} + y$ in a manner similar to $S + y$ and $T_{g,h} + y$ in the previous paragraph and using an analogous argument, we learn that $\mathcal{S} = \{T'_{0,0}, T'_{0,0} + 2, .., T'_{0,0} + (2b-2), T'_{0,1}, T'_{0,1} + 2, ..., T'_{0,1} + (2b-2)\}$ forms $2b$ mutually disjoint transversals in $J_n$.

It remains to show that the transversals in $\mathcal{S}$ are disjoint from the transversals in $\mathcal{S}'$. For even $y$ and $z$, it is clear that each 2-plex $S+y$ is disjoint from each 2-plex $S'+z$ in every row except row 0 or $b$, because $S+y$ and $S' + z$ use columns of different parity in all rows except 0 and $b$. But for any even overlapped entry in row 0, $T_{0,0} + y$ and $T_{0,1} + y$ use the entries found in row 0 of the corresponding subsquare $\mathbb{Z}_2$, while $T'_{0,0} + z$ and $T'_{0,1} + z$ use the entries found in row 1 of the corresponding subsquare $\mathbb{Z}_2$. In a similar fashion for any odd overlapped entry in row 0, $T_{0,0} + y$ and $T_{0,1} + y$ use the entries found in row 1 of the corresponding subsquare $\mathbb{Z}_2$, while $T'_{0,0} + z$ and $T'_{0,1} + z$ use the entries found in row 0 of the corresponding subsquare $\mathbb{Z}_2$. It is similarly shown that $T_{0,0} + y$, $T_{0,1} + y$, $T'_{0,0} + z$ and $T'_{0,1} + z$ are disjoint in row $b$. The collection $\mathcal{S} \cup \mathcal{S}'$ contains $4b = n$ mutually disjoint transversals and is the desired 1-partition of $J_n$. □

## 9. Summary of results

We finish this paper by summarizing our results; the following theorem unifies the latin square constructions.

**Theorem 9.1.** If $n \geq 1$ such that $n \neq 2$ and $n \neq 2p$ for every prime $p$, then there exists a ce-hamiltonian latin square of order $n$ that admits a 1-partition.

*Proof.* If $n$ is odd, then the required square is given by Theorem 3.6. If $n \equiv 2 \pmod 4$ such that $n \geq 18$ and $n \neq 2p$ for every prime $p$, then we can write $n = pq$ where $p$ and $q$ are odd, $q$ is prime, and $p \geq q \geq 3$; the required square is given by Theorem 7.3. If $n \equiv 0 \pmod 4$ and $n \geq 8$, then either $n \equiv 0 \pmod 8$ or $n \equiv 4 \pmod 8$, so the required square is given by Theorem 8.2 or 8.3, respectively. □

From this we obtain the desired embeddings of $K_{n,n,n}$; the following corollary will be used in [6].

**Corollary 9.2.** If $n \geq 1$ such that $n \neq 2$ and $n \neq 2p$ for every prime $p$, then there exists an orientable face 2-colorable hamilton cycle embedding of $K_{n,n,n}$ in which every face is an $ABC$ face.

*Proof.* The desired embedding is derived from Theorem 9.1 by way of Theorem 3.5. □

## 10. Extension to triangulations

We have shown a correspondence between orientable hamilton cycle embeddings of $K_{n,n,n}$ and orthogonal latin squares; there is a well-known correspondence between orientable triangulations of $K_{n,n,n}$ and latin squares. An orientable triangulation of $K_{n,n,n}$ – equivalently an orientable hamilton cycle embedding of $K_{n,n}$ – yields a pair of latin squares, and this process can be reversed if the pair of latin squares satisfies certain conditions; this is known as a *biembedding of latin squares*. In fact, Grannell, Griggs and Knor proved that biembeddings of latin squares comprise all orientable triangulations of $K_{n,n,n}$ [12, Proposition 1].

The following definition is a reformulation of Definition 2.1 in [11].

**Definition 10.1.** Let $L$ be a latin square of order $n$ and define the collection of pairs

$$P_j = \{(E(L,j,\ell), E(L,j+1,\ell)) \mid \ell \in \mathbb{Z}_n\} \text{ for all } j \in \mathbb{Z}_n.$$

If the induced pair graph $G_{P_j}$ is a hamilton cycle for all $j \in \mathbb{Z}_n$, then we say that $L$ is *consecutively row hamiltonian*, or *cr-hamiltonian* for short.

A latin square $L$ is *conjugate* to $L'$ if $L'$ can be obtained from $L$ by permuting the roles of the rows, columns and entries. It is not hard to show that a ce-hamiltonian latin square is conjugate to a cr-hamiltonian latin square. Therefore, the squares constructed in this paper – and the many additional ce-hamiltonian squares that can be obtained via the step product construction – can be used in conjunction with the following theorem to obtain orientable triangulations of $K_{n,n,n}$.

**Theorem 10.2** (Grannell and Griggs, Lemma 2.1 [11])**.** If $L$ is cr-hamiltonian, then there exists a biembedding of $L$ with a copy of itself.

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