Nonorientable hamilton cycle embeddings of complete tripartite graphs

M. N. Ellingham[a,1,*], Justin Z. Schroeder[a,1]

[a]*Department of Mathematics, 1326 Stevenson Center, Vanderbilt University, Nashville, TN 37240, USA*

Abstract

A cyclic construction is presented for building embeddings of the complete tripartite graph $K_{n,n,n}$ on a nonorientable surface such that the boundary of every face is a hamilton cycle. This construction works for several families of values of n, and we extend the result to all n with some methods of Bouchet and others. The nonorientable genus of $K_{t,n,n,n}$, for $t \geq 2n$, is then determined using these embeddings and a surgical method called the 'diamond sum' technique.

Keywords: complete tripartite graph, graph embedding, nonorientable genus, hamilton cycle
2000 MSC: 05C10

1. Introduction

An important topic in topological graph theory is embeddings of graphs on surfaces of minimum and maximum genera. Embeddings of minimum genus generally have faces as small as possible, while embeddings of maximum genus have faces as large as possible. Embeddings where the boundary of every face is a hamilton cycle serve both ends. A hamilton cycle embedding of a graph G, if it exists, is necessarily an embedding of G on a surface of maximum genus over all closed 2-cell embeddings of G. Additionally, a hamilton cycle embedding of G with m faces corresponds to a triangular embedding of $\overline{K_m} + G$, the join of the edgeless graph $\overline{K_m}$ with G. This triangulation is necessarily a minimum genus embedding of $\overline{K_m} + G$.

Some minimum genus results can be interpreted as hamilton cycle embeddings of familiar graphs. In 1970 Ringel and Youngs [15] determined the orientable genus of the complete tripartite graph $K_{n,n,n}$ for all n. The triangulations that achieve this genus correspond to orientable hamilton cycle embeddings of the complete bipartite graph $K_{n,n}$. More recently the first author, together with Stephens and Zha [7], determined the nonorientable genus of complete tripartite graphs $K_{\ell,m,n}$, where $\ell \geq m \geq n$. For $n \geq 4$, the embeddings constructed for the case $\ell = m = n$ correspond to nonorientable hamilton cycle embeddings of $K_{n,n}$.

Going in the other direction, the first author and Stephens [5, 6] constructed hamilton cycle embeddings of K_n and used them to obtain minimum genus embeddings of $\overline{K_m} + K_n$ for $m \geq n-1$. Hamilton cycle embeddings of $K_{n,n}$ also played a role in [6].

[*]Corresponding author
Email addresses: mark.ellingham@vanderbilt.edu (M. N. Ellingham), justin.z.schroeder@vanderbilt.edu (Justin Z. Schroeder)

[1]Supported by National Security Agency grant H98230-09-1-0065. The United States Government is authorized to reproduce and distribute reprints notwithstanding any copyright notation herein.

Hamilton cycle embeddings have also been related to minimum genus embeddings in a different way. Grannell, Griggs and Širáň [8] derived hamilton cycle embeddings of K_n from triangulations (hence minimum genus embeddings) of K_n.

In this paper we extend hamilton cycle embedding results to the complete tripartite graph $K_{n,n,n}$. Then, in Section 5 we show that hamilton cycle embeddings of $K_{n,n,n}$ can be used to obtain minimum genus embeddings of $K_{t,n,n,n}$ for $t \geq 2n$. Constructing the hamilton cycle embeddings requires several steps. First, Theorem 2.1 provides a general cyclic construction using "slope sequences" with certain properties. Next, slope sequences exhibiting these properties are given for several families of values. Finally, a connection to triangulations of quadripartite graphs and some covering triangulation results due to Bouchet and others [1, 2, 4] are used to obtain the general result. All the embeddings we construct are nonorientable, although our techniques (slope sequences, in particular) can also be used to obtain orientable embeddings.

A basic understanding of topological graph theory is assumed. In particular, a *surface* is a compact 2-manifold without boundary. The nonorientable surface N_k is obtained by adding k crosscaps to a sphere, and the *nonorientable genus* of a nonplanar graph G, denoted $\tilde{g}(G)$, is the minimum value of k for which G can be embedded on N_k. For a planar graph G, we use the convention that $\tilde{g}(G) = 0$. It is well known that a cellular embedding can be characterized, up to homeomorphism, by providing a set of facial walks that double cover the edges and yield a proper rotation at each vertex. To define a proper rotation, we must introduce the *rotation graph* at a vertex v, denoted R_v. This graph has as its vertex set the neighbors of v, and two vertices u_1 and u_2 are joined by one edge for each occurrence of the subsequence $(\cdots u_1 v u_2 \cdots)$, or its reverse, in one of the facial walks. R_v is 2-regular; we say it is *proper* if R_v consists of a single cycle. This ensures that the neighborhood around each vertex is homeomorphic to a disk. The embedding is orientable if and only if the faces can be oriented so that each edge appears once in each direction. For additional details and terminology, see [9].

We let $A = \{a_0, ..., a_{n-1}\}$, $B = \{b_0, ..., b_{n-1}\}$ and $C = \{c_0, ..., c_{n-1}\}$ be the vertices of $K_{n,n,n}$ so that A, B and C are the maximal independent sets. A hamilton cycle face of the form $(a_{j_0} b_{k_0} c_{\ell_0} a_{j_1} b_{k_1} c_{\ell_1} \cdots a_{j_{n-1}} b_{k_{n-1}} c_{\ell_{n-1}})$ is called an *ABC cycle*.

2. Slope sequence construction

In this section we describe the general construction on which the proofs in Section 3 are based. Some preliminary definitions are required. Let $S = ((s_0, t_0), (s_1, t_1), ..., (s_{n-1}, t_{n-1}))$. If $s_j \neq t_j$ for all $j \in \mathbb{Z}_n$ and the collection $\{s_0, ..., s_{n-1}, t_0, ..., t_{n-1}\}$ covers every element of $\mathbb{Z}_n$ twice, we say S is a *slope sequence*. Form the graph G_S with vertices $\{v_0, v_1, ..., v_{n-1}\}$ and m edges joining distinct vertices v_{j_1} and v_{j_2}, where $m = |\{s_{j_1}, t_{j_1}\} \cap \{s_{j_2}, t_{j_2}\}|$. We call G_S the *induced pair graph* for the slope sequence S. This graph is 2-regular, so G_S decomposes into a union of cycles. As Theorem 2.1 shows, it will be desirable to have induced pair graphs that consist of a single cycle.

Theorem 2.1. *Suppose $S = ((s_0, t_0), (s_1, t_1), ..., (s_{n-1}, t_{n-1}))$ is a slope sequence such that the following hold:*

(i) $\{j + s_j \mid j \in \mathbb{Z}_n\} = \{j + t_j \mid j \in \mathbb{Z}_n\} = \mathbb{Z}_n$*;*
(ii) $t_j - s_j$ *is relatively prime to n for all $j \in \mathbb{Z}_n$;*
(iii) *the induced pair graph G_S consists of a single cycle of length n.*

Then the collection of cycles $\mathcal{X} = \{X_i \mid i \in \mathbb{Z}_n\}$ and $\mathcal{Y} = \{Y_i \mid i \in \mathbb{Z}_n\}$, given by

$$\begin{aligned} X_i: &\quad (a_0 b_i c_{i+s_0} a_1 b_{i+1} c_{i+1+s_1} \cdots a_j b_{i+j} c_{i+j+s_j} \cdots a_{n-1} b_{i+n-1} c_{i+n-1+s_{n-1}}), \\ Y_i: &\quad (a_0 b_i c_{i+t_0} a_1 b_{i+1} c_{i+1+t_1} \cdots a_j b_{i+j} c_{i+j+t_j} \cdots a_{n-1} b_{i+n-1} c_{i+n-1+t_{n-1}}), \end{aligned}$$

form a hamilton cycle embedding of $K_{n,n,n}$ with all ABC cycle faces.

Proof. First, we must show that X_i and Y_i are indeed hamilton cycles. It is clear that every A and B vertex appears in every X_i and Y_i. Since $j + s_j$ covers $\mathbb{Z}_n$, it follows that $i + j + s_j$ also covers $\mathbb{Z}_n$, so every C vertex appears in X_i. The same argument with $j + t_j$ shows that every C vertex also appears in Y_i. By construction, these cycles are all ABC cycles.

Next, we show that these hamilton cycles form a double cover of $K_{n,n,n}$. The cycles X_{k-j} and Y_{k-j} both cover the edge $a_j b_k$ for all $j, k \in \mathbb{Z}_n$. Similarly the cycles $X_{\ell-(j-1)-s_{j-1}}$ and $Y_{\ell-(j-1)-t_{j-1}}$ both cover the edge $c_\ell a_j$ for all $j, \ell \in \mathbb{Z}_n$. Finally, consider an edge $b_k c_\ell$. We know from S being a slope sequence that there exist j' and j'' such that one of the following holds: (1) $s_{j'} = t_{j''} = \ell - k$, (2) $s_{j'} = s_{j''} = \ell - k$, or (3) $t_{j'} = t_{j''} = \ell - k$. These cases correspond to the following: (1) the cycles $X_{k-j'}$ and $Y_{k-j''}$ both cover the edge $b_k c_\ell$, (2) the cycles $X_{k-j'}$ and $X_{k-j''}$ both cover the edge $b_k c_\ell$, or (3) the cycles $Y_{k-j'}$ and $Y_{k-j''}$ both cover the edge $b_k c_\ell$. This holds for all $k, \ell \in \mathbb{Z}_n$; therefore, $\mathcal{X} \cup \mathcal{Y}$ forms a double cover of $K_{n,n,n}$.

To show that these hamilton cycles can be sewn together along common edges to yield an embedding of $K_{n,n,n}$, it remains to prove that the rotation graph around each vertex is a single cycle of length $2n$. Since this collection consists of all ABC faces, we know that the rotation graph around a vertex $a_j \in A$ will be bipartite with alternating B and C vertices. If all of the C vertices appear in the same component of R_{a_j}, then all of the B vertices must be in the same component as well. Thus, it will suffice to prove that the C vertices are contained in the same cycle in the rotation graph around every A vertex. Similarly, it will suffice to prove that the A vertices are contained in the same cycle in the rotation graph around every B and C vertex.

Consider the vertex a_j. We know the cycle $X_{\ell-(j-1)-s_{j-1}}$ contains the sequence

$$(\cdots c_\ell a_j b_{\ell+1-s_{j-1}} \cdots)$$

and the cycle $Y_{\ell-(j-1)-s_{j-1}}$ contains the sequence

$$(\cdots c_{\ell-s_{j-1}+t_{j-1}} a_j b_{i+j} \cdots).$$

Thus the vertex $c_{\ell-s_{j-1}+t_{j-1}}$ follows the vertex c_ℓ in the rotation graph around a_j. Continuing this argument, we find the C vertices form the cyclic sequence

$$(c_k c_{k+(t_{j-1}-s_{j-1})} c_{k+2(t_{j-1}-s_{j-1})} \cdots c_{k+(n-1)(t_{j-1}-s_{j-1})})$$

in the rotation graph around a_j. Since $t_{j-1} - s_{j-1}$ is relatively prime to n, this includes every C vertex.

Consider the vertex b_k. We know the cycle X_{k-j} contains the sequence

$$(\cdots a_j b_k c_{k+s_j} \cdots).$$

Since S double covers $\mathbb{Z}_n$, there exists j' such that either (1) $s_{j'} = s_j$ or (2) $t_{j'} = s_j$. In either case we know the vertex v_j arising from the pair (s_j, t_j) is adjacent in the slope graph G_S to the vertex $v_{j'}$ arising from the pair $(s_{j'}, t_{j'})$. Since G_S is a single cycle of length n, we write

$$G_S = (v_j v_{\delta(j)} v_{\delta^2(j)} \cdots v_{\delta^{n-1}(j)}),$$

where $\delta(j) = j'$. In case (1), the cycle $X_{k-j'}$ contains the sequence

$$(\cdots a_{j'} b_k c_{k+s_{j'}} \cdots).$$

Likewise in case (2), the cycle $Y_{k-j'}$ contains the sequence

$$(\cdots a_{j'} b_k c_{k+t_{j'}} \cdots).$$

Since either (1) $k + s_{j'} = k + s_j$ or (2) $k + t_{j'} = k + s_j$, we have that $a_{j'} = a_{\delta(j)}$ follows a_j in the rotation graph around b_k. Repeating this argument, we see that the A vertices form the cyclic sequence

$$(a_j a_{\delta(j)} a_{\delta^2(j)} \cdots a_{\delta^{n-1}(j)})$$

in the rotation graph around b_k, which includes every A vertex. An analogous argument shows that the A vertices form the cyclic sequence

$$(a_{j+1} a_{\delta(j)+1} a_{\delta^2(j)+1} \cdots a_{\delta^{n-1}(j)+1})$$

lying in a single component in the rotation graph around c_ℓ. □

3. Applications of slope sequence construction

j	s_j	t_j	$t_j - s_j$	j	s_j	t_j	$t_j - s_j$
0	1	2	1	$2r+2$	$-2r+2$	$-2r$	-2
1	-1	-2	-1	$2r+3$	$-2r$	$2r-5$	-6
2	1	-2	-3	$2r+4$	$2r-5$	$2r-7$	-2
3	-1	$2r$	$2r+1$	$2r+5$	$2r-7$	$2r-9$	-2
4	$2r$	$2r-2$	-2	⋮	⋮	⋮	⋮
5	$2r-2$	$2r-4$	-2	$3r$	5	3	-2
⋮	⋮	⋮	⋮	$3r+1$	3	$-2r+3$	$2r+1$
$r+1$	6	4	-2	$3r+2$	$-2r+3$	$-2r+1$	-2
$r+2$	4	0	-4	$3r+3$	$-2r+1$	-3	$2r-4$
$r+3$	0	$2r-1$	$2r-1$	$3r+4$	-3	-5	-2
$r+4$	$2r-1$	$2r-3$	-2	$3r+5$	-5	-7	-2
$r+5$	$2r-3$	-4	$2r$	⋮	⋮	⋮	⋮
$r+6$	-4	-6	-2	$4r-2$	$-2r+9$	$-2r+7$	-2
$r+7$	-6	-8	-2	$4r-1$	$-2r+7$	$-2r+5$	-2
⋮	⋮	⋮	⋮	$4r$	$-2r+5$	2	$2r-3$

Table 1: Slope sequences for nonorientable hamilton cycle embeddings of $K_{n,n,n}$ where $n = 4r + 1$, $r \geq 4$.

Lemma 3.1. *There exists a nonorientable hamilton cycle embedding of $K_{n,n,n}$ for all $n \equiv 1$ (mod 4) such that $n \geq 5$ and $3, 7 \nmid n$.*

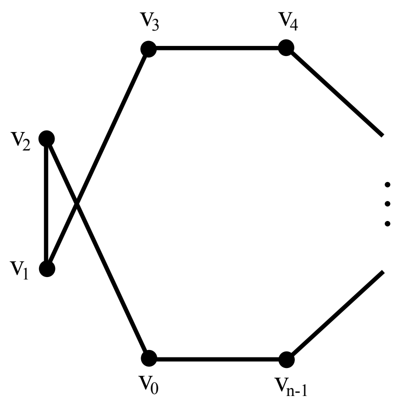

Figure 1: The slope graph G_S for the slope sequences given in Table 1 and Table 2.

Proof. Table 4 in Appendix A gives the necessary slope sequences for $n = 5$ and 13. It is a straightforward exercise to show that these sequences meet all the required conditions of Theorem 2.1, and that the resulting embeddings are nonorientable.

Table 1 gives the necessary slope sequences for $n = 4r + 1$, $r \geq 4$. It is easy to see that the collection $\{s_0, ..., s_{n-1}, t_0, ..., t_{n-1}\}$ double covers $\mathbb{Z}_n$. The slope graph G_S consists of edges $v_j v_{j+1}$ for all $3 \leq j \leq n-1$, along with the edges $v_0 v_2$, $v_2 v_1$, and $v_1 v_3$. This is a cycle of length n, as seen in Figure 1. Let $D = \{t_j - s_j \mid j \in \mathbb{Z}_n\}$. From the table we see that

$$\begin{aligned} D &= \{-6, -4, -3, -2, -1, 1, 2r-4, 2r-3, 2r-1, 2r, 2r+1\} \\ &= \left\{-6, -4, -3, -2, -1, 1, \tfrac{n-9}{2}, \tfrac{n-7}{2}, \tfrac{n-3}{2}, \tfrac{n-1}{2}, \tfrac{n+1}{2}\right\}. \end{aligned}$$

Since $2, 3, 7 \nmid n$, we know n is relatively prime to every element of D. The last condition we must prove is that $\{j + s_j \mid j \in \mathbb{Z}_n\} = \{j + t_j \mid j \in \mathbb{Z}_n\} = \mathbb{Z}_n$. Note that for every j we have $s_j = k \Leftrightarrow s_{j+k} = -k$ and $t_j = k \Leftrightarrow t_{j+k} = -k$. Let $i \in \mathbb{Z}_n$, and set $k = s_i$ and $j = i + k$. It follows that $j + s_j = i + k + s_{i+k} = i + k - k = i$. Since i was arbitrary, we know $\{j + s_j \mid j \in \mathbb{Z}_n\} = \mathbb{Z}_n$. The same argument shows that $\{j + t_j \mid j \in \mathbb{Z}_n\} = \mathbb{Z}_n$. Applying Theorem 2.1 yields a hamilton cycle embedding of $K_{n,n,n}$. To determine the orientability of this embedding, consider the following three cycles:

$$\begin{aligned} X_1 &: \quad (a_0 b_1 c_2 a_1 b_2 c_1 a_2 b_3 c_4 \cdots), \\ Y_0 &: \quad (a_0 b_0 c_2 a_1 b_1 c_{n-1} a_2 b_2 c_0 \cdots), \\ Y_1 &: \quad (a_0 b_1 c_3 a_1 b_2 c_0 a_2 b_3 c_1 \cdots). \end{aligned}$$

Assume this embedding admits an orientation, with X_1 oriented forwards. Note that Y_0 and X_1 share the edge $c_2 a_1$ and Y_1 and X_1 share the edge $a_0 b_1$, so both Y_0 and Y_1 must be oriented backwards. However, Y_0 and Y_1 share the edge $b_2 c_0$, so they must have different orientations. This is a contradiction, so this embedding is nonorientable. □

Lemma 3.2. *There exists a nonorientable hamilton cycle embedding of $K_{n,n,n}$ for all $n \equiv 3$ (mod 4) such that $3, 7 \nmid n$.*

Proof. Table 4 in Appendix A gives the necessary slope sequence for $n = 11$. It is a straightforward exercise to show that this sequence meets all the required conditions of Theorem 2.1, and that the resulting embedding is nonorientable.

j	s_j	t_j	$t_j - s_j$	j	s_j	t_j	$t_j - s_j$
0	1	2	1	$2r+4$	$-2r+1$	$-2r-1$	-2
1	-1	-2	-1	$2r+5$	$-2r-1$	$2r-4$	-6
2	1	-2	-3	$2r+6$	$2r-4$	$2r-6$	-2
3	-1	$2r+1$	$2r+2$	$2r+7$	$2r-6$	$2r-8$	-2
4	$2r+1$	$2r-1$	-2	$\vdots$	$\vdots$	$\vdots$	$\vdots$
5	$2r-1$	$2r-3$	-2	$3r+1$	6	4	-2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$3r+2$	4	$-2r+2$	$2r+1$
$r+1$	7	5	-2	$3r+3$	$-2r+2$	$-2r$	-2
$r+2$	5	3	-2	$3r+4$	$-2r$	0	$2r$
$r+3$	3	$2r$	$2r-3$	$3r+5$	0	-4	-4
$r+4$	$2r$	$2r-2$	-2	$3r+6$	-4	-6	-2
$r+5$	$2r-2$	-3	$2r+2$	$3r+7$	-6	-8	-2
$r+6$	-3	-5	-2	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$r+7$	-5	-7	-2	$n-2$	$-2r+6$	$-2r+4$	-2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$n-1$	$-2r+4$	2	$2r-2$

Table 2: Slope sequences for nonorientable hamilton cycle embeddings of $K_{n,n,n}$ where $n = 4r + 3$, $r \geq 3$.

Table 2 gives the necessary slope sequences for $n = 4r + 3$, $r \geq 3$. It is again easy to see that the collection $\{s_0, ..., s_{n-1}, t_0, ..., t_{n-1}\}$ double covers $\mathbb{Z}_n$. The slope graph G_S (Figure 1) is identical to the slope graph constructed for the slope sequence in Table 1. Let D again be the set of differences; from the table we see that

$$\begin{aligned} D &= \{-6, -4, -3, -2, -1, 1, 2r-3, 2r-2, 2r, 2r+1, 2r+2\} \\ &= \left\{-6, -4, -3, -2, -1, 1, \tfrac{n-9}{2}, \tfrac{n-7}{2}, \tfrac{n-3}{2}, \tfrac{n-1}{2}, \tfrac{n+1}{2}\right\}. \end{aligned}$$

This is the same D as in the proof of Lemma 3.1, so again we know n is relatively prime to every element of D. We also have $s_j = k \Leftrightarrow s_{j+k} = -k$ and $t_j = k \Leftrightarrow t_{j+k} = -k$ as in the proof of Lemma 3.1, which implies that $\{j + s_j \mid j \in \mathbb{Z}_n\} = \{j + t_j \mid j \in \mathbb{Z}_n\} = \mathbb{Z}_n$. Applying Theorem 2.1 yields a hamilton cycle embedding of $K_{n,n,n}$. Because s_0, s_1, s_2, t_0, t_1, and t_2 are the same in Tables 1 and 2, analyzing X_1, Y_0 and Y_1 in the same way as in the proof of Lemma 3.1 shows that this embedding is nonorientable. □

j	0	1	2	3	4	$\cdots$	$n-3$	$n-2$	$n-1$
s_j	1	1	3	3	5	$\cdots$	$n-3$	$n-1$	$n-1$
t_j	0	2	2	4	4	$\cdots$	$n-2$	$n-2$	0
$t_j - s_j$	-1	1	-1	1	-1	$\cdots$	1	-1	1

Table 3: Slope sequences for a nonorientable embedding of $K_{n,n,n}$ where $n \equiv 2 \pmod 4$.

Lemma 3.3. *There exists a nonorientable hamilton cycle embedding of $K_{n,n,n}$ for all $n \equiv 2$ (mod 4).*

Proof. Table 3 gives the necessary slope sequences for $n \equiv 2 \pmod 4$. Since $t_j - s_j = (-1)^{j+1}$, we know $t_j - s_j$ is relatively prime to n for all $j \in \mathbb{Z}_n$. Since G_S consists of the edges $v_j v_{j+1}$ for all $j \in \mathbb{Z}_n$, it is clearly a single cycle of length n. Finally, note that $j + s_j = 2j + 1$ if j is even and $j + s_j = 2j$ if j is odd. Since $n \equiv 2 \pmod 4$, this implies $\{j + s_j \mid j \in \mathbb{Z}_n, j \text{ even}\}$ covers all the odd values of $\mathbb{Z}_n$ and $\{j + s_j \mid j \in \mathbb{Z}_n, j \text{ odd}\}$ covers all the even values of $\mathbb{Z}_n$. Thus, $\{j + s_j \mid j \in \mathbb{Z}_n\} = \mathbb{Z}_n$. Using the fact that $j + t_j = 2j$ if j is even and $j + t_j = 2j + 1$ if j is odd, we derive that $\{j + t_j \mid j \in \mathbb{Z}_n\} = \mathbb{Z}_n$ as well. Applying Theorem 2.1 provides a hamilton cycle embedding of $K_{n,n,n}$. To determine the orientability of this embedding, consider the following three cycles:

$$\begin{aligned} X_0 &: \quad (a_0 b_0 c_1 a_1 b_1 c_2 a_2 b_2 c_5 \cdots), \\ Y_0 &: \quad (a_0 b_0 c_0 a_1 b_1 c_3 a_2 b_2 c_4 \cdots), \\ Y_1 &: \quad (a_0 b_1 c_1 a_1 b_2 c_4 a_2 b_3 c_5 \cdots). \end{aligned}$$

Assume this embedding admits an orientation, with X_0 oriented forwards. Note that Y_0 and X_0 share the edge $a_0 b_0$ and Y_1 and X_0 share the edge $c_1 a_1$, so both Y_0 and Y_1 must be oriented backwards. However, Y_0 and Y_1 share the edge $b_2 c_4$, so they must have different orientations. This is a contradiction, so this embedding is nonorientable. □

4. Covering triangulations

In a series of papers [1, 2, 4] in the 1970's and 1980's, Bouchet – together with Bénard and Fouquet – developed several methods for lifting triangulations of a graph G to triangulations of $G[\overline{K_m}]$, the lexicographic product of G with the empty graph $\overline{K_m}$. These methods, which Bouchet calls *covering triangulations*, are especially useful when G is the complete multipartite graph $K_{n_1,\ldots,n_q}$, because the lexicographic product $G[\overline{K_m}]$ is the complete multipartite graph $K_{mn_1,\ldots,mn_q}$. Thus, these lifts yield a product construction for triangulations of complete multipartite graphs. The following results are special cases of constructions presented in Bouchet's papers and will be needed in Section 5.

Corollary 4.1. *If there exists a nonorientable triangulation of $K_{2n,n,n,n}$ with n even, then there exists a nonorientable triangulation of $K_{2mn,mn,mn,mn}$ for every integer $m \geq 1$.*

Proof. Every vertex in $K_{2n,n,n,n}$ has even degree, thus it is an eulerian graph. The result follows from Theorem 4 in [2]. □

Corollary 4.2. *If there exists a nonorientable triangulation of $K_{2n,n,n,n}$, then there exists a nonorientable triangulation of $K_{2mn,mn,mn,mn}$ for every integer $m \geq 1$ such that $2, 3, 5 \nmid m$.*

Proof. This follows from the second corollary on page 324 of [4]. □

Corollary 4.3. *If there exists a nonorientable triangulation of $K_{2n,n,n,n}$, then for every integer $p \geq 0$ there exists a nonorientable triangulation of $K_{2mn,mn,mn,mn}$, where $m = 3^p$.*

Proof. Providing each independent set with a different color, it is easy to see that $K_{2n,n,n,n}$ is 4-colorable. The result follows from Corollary 4.3 (and Lemma 3.1) in [1]. □

5. Genus of some complete quadripartite graphs

Here we develop the connection between hamilton cycle embeddings of $K_{n,n,n}$ and triangulations of $K_{2n,n,n,n}$ and utilize the covering triangulations from Section 4. The following result is the nonorientable counterpart to Lemma 4.1 in [6].

Lemma 5.1. *The following are equivalent.*

(1) *There exists a nonorientable hamilton cycle embedding of G with p faces.*
(2) *There exists a nonorientable triangulation of $\overline{K_p} + G$.*

Moreover, if either (1) or (2) holds, then G is p-regular.

Lemma 5.1 leads to the following theorem; recall that we use the convention that the nonorientable genus of a planar graph is zero.

Theorem 5.2. *For all $n \geq 1$, $\tilde{g}(K_{2n,n,n,n}) = (n-1)(3n-2)$.*

Proof. $K_{2,1,1,1}$ is planar, so we will assume $n \geq 2$. We know from [14] that $g(K_{2n,3n}) = (3n-2)(n-1)$. Since $K_{2n,3n} \subset K_{2n,n,n,n}$, we have $\tilde{g}(K_{2n,n,n,n}) \geq (n-1)(3n-2)$. From Euler's formula, an embedding that achieves this genus must be a triangulation, so it will suffice to find a nonorientable triangulation of $K_{2n,n,n,n}$.

If n is odd, write $n = 3^p 7^q m$, where $3, 7 \nmid m$. If $m \neq 1$, then Lemmas 3.1 and 3.2 imply the existence of a nonorientable hamilton cycle embedding of $K_{m,m,m}$. Lemma 5.1 yields a triangulation of $K_{2m,m,m,m}$. Applying Corollary 4.3 provides a triangulation of $K_{2(3^p m),3^p m,3^p m,3^p m}$, and applying Corollary 4.2 gives us the desired triangulation of $K_{2n,n,n,n}$. If $m = 1$, then we use a nonorientable hamilton cycle embedding of either $K_{3,3,3}$ or $K_{7,7,7}$ from Appendix A as our starting point before applying Lemma 5.1 and the results of Section 4.

If n is even, write $n = 2^p 2m$, where m is odd. By Lemma 3.3 there exists a nonorientable hamilton cycle embedding of $K_{2m,2m,2m}$. Lemma 5.1 yields a triangulation of $K_{4m,2m,2m,2m}$, and applying Corollary 4.1 gives us the desired triangulation of $K_{2n,n,n,n}$. This completes the proof. $\square$

The construction of the necessary triangulations for $n \geq 2$ in the proof of Theorem 5.2 leads directly to the following result.

Corollary 5.3. *There exists a nonorientable hamilton cycle embedding of $K_{n,n,n}$ for all $n \geq 2$.*

Unfortunately, the hamilton cycle faces in the embeddings of $K_{n,n,n}$ obtained from Bouchet's covering triangulations of $K_{2n,n,n,n}$ are not, in general, *ABC* cycles.

The following extension of Theorem 5.2 is obtained using the 'diamond sum' technique. This surgical technique was introduced in dual form by Bouchet [3], reinterpreted by Magajna, Mohar and Pisanski [11], developed further by Mohar, Parsons, and Pisanski [12], and generalized by Kawarabayashi, Stephens and Zha [10]. In particular, the diamond sum construction allows us to combine minimum genus embeddings of $K_{t_1,n,n,n}$ and $K_{t_2,3n}$ to get a minimum genus embedding of $K_{t_1+t_2-2,n,n,n}$. This is achieved by removing a disk containing a vertex of degree $3n$ and all of its incident edges from each embedding and identifying the boundaries of the resulting holes in a suitable fashion. For similar applications of the diamond sum, see [5, 6, 7], and for more information on this technique, see [13, pages 117–118].

Corollary 5.4. *For all $n \geq 1$ and all $t \geq 2n$, $\tilde{g}(K_{t,n,n,n}) = \left\lceil \frac{(t-2)(3n-2)}{2} \right\rceil = \tilde{g}(K_{t,3n})$.*

Proof. We know that $K_{t,3n} \subseteq K_{t,n,n,n}$, and from [14] we know $\tilde{g}(K_{t,3n}) = \left\lceil \frac{(t-2)(3n-2)}{2} \right\rceil$, so $\tilde{g}(K_{t,n,n,n}) \geq \left\lceil \frac{(t-2)(3n-2)}{2} \right\rceil$. We now apply the diamond sum construction to minimum genus nonorientable embeddings of $K_{2n,n,n,n}$ and $K_{t-2n+2,3n}$. By Theorem 5.2 we know $\tilde{g}(K_{2n,n,n,n}) = (n-1)(3n-2)$, and again by [14] we know $\tilde{g}(K_{t-2n+2,3n}) = \left\lceil \frac{(t-2n)(3n-2)}{2} \right\rceil$. Via the diamond sum construction, we learn that $\tilde{g}(K_{t,n,n,n}) \leq (n-1)(3n-2) + \left\lceil \frac{(t-2n)(3n-2)}{2} \right\rceil = \left\lceil \frac{(t-2)(3n-2)}{2} \right\rceil$, and the result follows. □

Remark 5.5. Corollary 5.4 implies that for any graph G satisfying $\overline{K_{3n}} \subseteq G \subseteq K_{n,n,n}$ and for all $t \geq 2n$, the nonorientable genus of $\overline{K_t} + G$ is the same as the nonorientable genus of $K_{t,3n}$. In other words, $\tilde{g}(\overline{K_t} + G) = \left\lceil \frac{(t-2)(3n-2)}{2} \right\rceil$. Moreover, in the special case $t = 2n$, we also get $\tilde{g}(G + H) = (n-1)(3n-2)$ for graphs G and H satisfying $\overline{K_{3n}} \subseteq G \subseteq K_{2n,n}$ and $\overline{K_{2n}} \subseteq H \subseteq K_{n,n}$.

Appendix A. Special case constructions

This appendix presents the required nonorientable hamilton cycle embeddings of $K_{n,n,n}$ when $n \in \{3, 5, 7, 11, 13\}$.

By checking all possible cases, we know there does not exist a slope sequence construction for a nonorientable embedding of $K_{3,3,3}$. The desired embedding is given by the following facial boundaries:

$$\begin{array}{ll} (a_0b_0c_0a_1b_1c_1a_2b_2c_2), & (a_0b_0c_1a_1b_1c_2a_2b_2c_0), \\ (a_0b_1c_1a_1b_2c_2a_2b_0c_0), & (a_0b_2c_0a_2b_1c_2a_1b_0c_1), \\ (a_0b_2c_1a_2b_1c_0a_1b_0c_2), & (a_0b_1c_0a_2b_0c_2a_1b_2c_1). \end{array}$$

For $n \in \{5, 7, 11, 13\}$, Table 4 provides a slope sequence that yields a nonorientable hamilton cycle embedding of $K_{n,n,n}$. To show that these embeddings are indeed nonorientable, in the same way as in the proof of Lemma 3.1, consider the following sequences of faces and edges, where $F\ e\ F'$ implies F and F' share the edge e:

$$\begin{array}{llllllll} n = 5: & X_2 & a_0b_2 & Y_2 & b_3c_1 & Y_1 & c_4a_4 & X_2; \\ n = 7: & X_0 & b_0c_1 & X_4 & a_0b_4 & Y_4 & c_3a_5 & X_0; \\ n = 11: & X_0 & a_0b_0 & Y_0 & c_8a_6 & X_9 & b_0c_1 & X_0; \\ n = 13: & X_0 & a_0b_0 & Y_0 & c_8a_5 & X_{11} & b_0c_1 & X_0. \end{array}$$

References

[1] D. Bénard, A. Bouchet and J.L. Fouquet, (m)-covering of a triangulation, Discrete Math. 62 (1986), 261-270.

[2] A. Bouchet, Triangular imbeddings into surfaces of a join of equicardinal sets following an eulerian graph, Theory and Applications of Graphs (Proc. Inter. Conf., West. Mich. Univ., Kalamazoo, Mich., 1976), 86-115, Lecture Notes in Math. No. 642, Springer, Berlin, 1978.

[3] A. Bouchet, Orientable and nonorientable genus of the complete bipartite graph, J. Combin. Theory Ser. B 24 (1978), 24-33.

[4] A. Bouchet, Constructing a covering triangulation by means of a nowhere-zero dual flow, J. Combin. Theory Ser. B 32 (1982), 316-325.

[5] M.N. Ellingham and D. Christopher Stephens, The nonorientable genus of joins of complete graphs with large edgeless graphs, J. Combin. Theory Ser. B 97 (2007), 827-845.

[6] M.N. Ellingham and D. Christopher Stephens, The orientable genus of some joins of complete graphs with large edgeless graphs, Discrete Math. 309 (2009), 1190-1198.

[7] M.N. Ellingham, D. Christopher Stephens and Xiaoya Zha, The nonorientable genus of complete tripartite graphs, J. Combin. Theory Ser. B 96 (2006), 529-559.

n	j	0	1	2	3	4	5	6	7	8	9	10	11	12
5	s_j	1	4	1	4	0								
	t_j	2	3	3	0	2								
	$t_j - s_j$	1	4	2	1	2								
7	s_j	1	6	4	1	6	0	3						
	t_j	2	4	5	0	2	3	5						
	$t_j - s_j$	1	5	1	6	3	3	2						
11	s_j	1	10	1	10	4	5	3	0	7	8	6		
	t_j	2	9	9	4	5	3	0	7	8	6	2		
	$t_j - s_j$	1	10	8	5	1	9	8	7	1	9	7		
13	s_j	1	12	1	12	6	4	0	5	3	9	7	10	8
	t_j	2	11	11	6	4	0	5	3	9	7	10	8	2
	$t_j - s_j$	1	12	10	7	11	9	5	11	6	11	3	11	7

Table 4: Slope sequences for $n \in \{5, 7, 11, 13\}$.

[8] M.J. Grannell, T.S. Griggs and J. Širáň, Hamiltonian embeddings from triangulations, Bull. of the London Math. Soc. 39 (2007), 447452.

[9] J.L. Gross and T.W. Tucker, Topological Graph Theory, Wiley Interscience, New York, 1987.

[10] K. Kawarabayashi, D. Christopher Stephens and Xiaoya Zha, Orientable and nonorientable genera for some complete tripartite graphs, SIAM J. Discrete Math. 18 (2005), 479-487.

[11] Z. Magajna, B. Mohar and T. Pisanski, Minimal ordered triangulations of surfaces, J. Graph Theory 10 (1986), 451-460.

[12] B. Mohar, T.D. Parsons and T. Pisanski, The genus of nearly complete bipartite graphs, Ars Combin. 20B (1985), 173-183.

[13] B. Mohar and C. Thomassen, Graphs on Surfaces, Johns Hopkins University Press, Baltimore, 2001.

[14] G. Ringel, Der vollständige paare Graph auf nichtorientierbaren Flächen, J. Reine Angew. Math. 220 (1965), 88-93.

[15] G. Ringel and J.W.T. Youngs, Das Geschlecht des symmetrischen vollständigen dreifärbbaren Graphen, Comment. Math. Helv. 45 (1970), 152-158.