LINKAGE FOR THE DIAMOND AND THE PATH WITH FOUR VERTICES

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Abstract. Given graphs $G$ and $H$, we say $G$ is $H$-linked if for every injective mapping $\ell: V(H) \to V(G)$ we can find a subgraph $H'$ of $G$ that is a subdivision of $H$, with $\ell(v)$ being the vertex of $H'$ corresponding to each vertex $v$ of $H$. In this paper we prove two results on $H$-linkage for 4-vertex graphs $H$. Goddard showed that 4-connected planar triangulations are 4-ordered, or in other words $C_4$-linked. We strengthen this by showing that 4-connected planar triangulations are $(K_4 - e)$-linked. X. Yu characterized certain graphs related to $P_4$-linkage. We use his characterization to show that every 7-connected graph is $P_4$-linked, and to construct 6-connected graphs that are not $P_4$-linked.

1. Introduction

A graph is $k$-linked if for any $k$ pairs of vertices $\{u_i, v_i\}$, $1 \leq i \leq k$, there is a $k$-linkage, namely $k$ internally disjoint paths $\Pi_1, \Pi_2, \ldots, \Pi_k$ such that $\Pi_i$ joins $u_i$ and $v_i$. Graph linkage is a very important tool in studying graph minors.

If $G$ and $H$ are graphs, then an $H$-subdivision in $G$ is a subgraph $H'$ of $G$ isomorphic to a subdivision of $H$. There is an associated map $\ell: V(H) \to V(G)$, where $\ell(v)$ (called a branch vertex) is the vertex of $H'$ corresponding to each vertex $v$ of $H$. We say $H'$ is consistent with $\ell$. We say $G$ is $H$-linked if for every injection $\ell: V(H) \to V(G)$ there is a consistent $H$-subdivision.

Properties related to $H$-subdivisions were first studied by Jung [6] in the 1970s. This idea was recently re-introduced by Kostochka and Yu [9], and independently by Ferrara, Gould, Tansey, and Whalen [2]. Special cases of $H$-linkage include being $k$-linked ($kK_2$-linked), $k$-connected ($K_{1,k}$-linked, or $(K_2 \cup (k - 1)K_1)$-linked), and $k$-ordered ($C_k$-linked). Sufficient degree conditions for a graph to be $H$-linked were extensively studied in [2, 5, 8, 9, 10, 11]. In [13], implications among linkage properties in graphs were studied.

The study of $f(k)$, the minimum $t$ such that $t$-connected graphs are $k$-linked, has a long history. After a series of papers by Jung [6], Larman and Mani [12], Mader [14], and Robertson and Seymour [16], the first linear upper bound for $f$, namely, $f(k) \leq 22k$ was proved by

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Bollobás and Thomason [1]. This was improved by Kawarabayashi, Kostochka and Yu [7] to \( f(k) \leq 12k \), and Thomas and Wollan [19] showed that \( f(k) \leq 10k \).

Jung [6] proved that \( f(2) = 6 \) and showed that every 4-connected non-planar graph is 2-linked. Later, Seymour [17] and Thomassen [18] independently characterized all non-2-linked graphs. Thomas and Wollan [20] showed that \( f(3) \leq 10 \), but this bound is not known to be best possible.

In this paper we examine linkage for two small graphs. Let \( K_4 - e \) be the graph obtained from \( K_4 \) by removing one edge, which can also be described as \( K_{1,1,2} \), and is sometimes referred to as the diamond. It is clear that a \( K_4 \)-linked graph is \( (K_4 - e) \)-linked, a \( (K_4 - e) \)-linked graph is \( C_4 \)-linked, a \( C_4 \)-linked graph is \( P_4 \)-linked, and a \( P_4 \)-linked graph is 2-linked. There are examples showing that none of these implications can be reversed. We will investigate \( (K_4 - e) \)-linkage for planar graphs, and \( P_4 \)-linkage in general.

Planarity can provide barriers to linkage properties. For example, a 2-connected planar graph with a face of degree 4 or more is not 2-linked (and hence not \( H \)-linked for \( H = P_4, C_4, K_4 - e \) or \( K_4 \)): there is no 2-linkage for vertices \( u_1, u_2, v_1, v_2 \) in order around the face. So, what positive results can be given for \( H \)-linkage in planar graphs? One (difficult) approach is to characterize structures that prevent \( H \)-linkage; X. Yu [21] did this for \( K_4 \)-linkage in 4-connected planar graphs. Another approach is to restrict ourselves to graphs without obvious barriers to \( H \)-linkage: in particular, to avoid faces of degree 4 or more we may consider triangulations. Goddard [4] showed that 4-connected planar triangulations are \( C_4 \)-linked. A linkage property somewhat different from the ones we examine in this paper was investigated by Mori [15], who showed that 4-connected planar triangulations are \((3,3)\)-linked: for all disjoint 3-subsets \( S_1 \) and \( S_2 \) of vertices, there are vertex-disjoint connected subgraphs \( H_1 \) and \( H_2 \) with \( S_1 \subseteq V(H_1) \) and \( S_2 \subseteq V(H_2) \).

Here we strengthen Goddard’s result as follows.

**Theorem 1.1.** Any 4-connected planar triangulation is \((K_4 - e)\)-linked.

The proof of Theorem 1.1 occupies most of Section 2. We cannot replace ‘4-connected’ here by ‘3-connected,’ ‘\((K_4 - e)\)-linked’ by ‘\( K_4 \)-linked,’ or ‘planar triangulation’ by ‘planar graph’ (even if we increase the connectivity to 5): details are given at the end of Section 2. However, if we just increase the connectivity, then it is known that every 5-connected planar triangulation is \( K_4 \)-linked. This follows from X. Yu’s results; see [21, Cor. 4.3].

Motivated by trying to extend the results of [21] from 4-connected planar graphs to more general settings, X. Yu characterized a family of graphs called obstructions [22, 23, 24]. Let \( G \) be a graph, \( \{a,b,c\} \subseteq V(G) \), and \( \{a',b',c'\} \subseteq V(G) \), such that \( \{a,b,c\} \neq \{a',b',c'\} \). \((G, \{a,c\}, \{a',c'\}, (b,b')) \) is an obstruction if for every set of three vertex disjoint paths from \( \{a,b,c\} \) to \( \{a',b',c'\} \) in \( G \), one path is from \( b \) to \( b' \).

The problem of characterizing obstructions was posed by Robertson and Seymour (see [22, p. 90]). In [24], Yu stated a characterization of obstructions, and investigated their connectivity.
Theorem 1.2 (X. Yu [24]). Let \( (G, \{a, c\}, \{a', c'\}, (b, b')) \) be an obstruction. If \( \{a, c\} \neq \{a', c'\} \) and \( b \neq b' \), then \( G \) is at most 7-connected.

In the same paper, Yu constructed a class of obstructions which he claimed were 7-connected. Upon studying these graphs, however, we found that each is only 6-connected. In fact, we will show that there are essentially no 7-connected obstructions.

Theorem 1.3. Let \( (G, \{a, c\}, \{a', c'\}, (b, b')) \) be an obstruction. If \( \{a, c\} \neq \{a', c'\} \) and \( b \neq b' \), then \( G \) is at most 6-connected.

(Any graph \( G \) with \( a, b, c, a', b', c' \) chosen so that \( \{a, c\} = \{a', c'\} \) or \( b = b' \) is an obstruction, so there is no upper bound on the connectivity of these trivial types of obstructions.)

Seymour (see [24, p. 245]) has pointed out a connection between obstructions and the existence of \( P_4 \)-subdivisions in a graph, which we will state in Section 3. Using this and Theorem 1.3, it follows that every 7-connected graph is \( P_4 \)-linked. On the other hand, this connection also allows us to modify Yu’s construction from [24] to construct instances of 6-connected graphs where a specific \( P_4 \)-subdivision does not exist.

Theorem 1.4. Every 7-connected graph is \( P_4 \)-linked, but there are 6-connected graphs that are not \( P_4 \)-linked.

We prove Theorem 1.3 and the first part of Theorem 1.4 in Section 3. The second part of Theorem 1.4 is proved in Section 4. Theorem 1.4 is the first determination of the exact connectivity required for \( H \)-linkage for any \( H \) with three or more edges, other than for the family of graphs where \( H \)-linkage is equivalent to \( k \)-connectivity (e.g., \( H = K_2 \cup (k - 1)K_1 \) or \( H = K_{1,k} \)).

Most notation, definitions and supporting lemmas will be given in the sections where they are needed. However, the following are used in more than one section so we give them here.

We use \( K_n(v_1, v_2, \ldots, v_n) \) to denote a labeled complete graph with distinct vertices \( v_1, v_2, \ldots, v_n \). A near-triangulation is a plane graph in which all faces are triangles, except perhaps the outer face, which is a cycle. Cutsets in near-triangulations are characterized as follows.

Lemma 1.5. In a near-triangulation, a minimal cutset induces either a chordless separating cycle, or a chordless path whose ends are vertices on the boundary not joined by an edge of the boundary and which contains no other vertex of the boundary.

Proof. Add an extra vertex adjacent to all vertices of the outer face to obtain a triangulation, and use the well-known fact that a minimal cutset in a triangulation induces a chordless separating cycle.

2. \((K_4 - e)\)-Subdivisions in 4-Connected Planar Triangulations

In this section, we prove Theorem 1.1. Our proof follows the same approach used in [4]. We first introduce some notation and a basic result.
Let $f$ be an edge of graph $G$. Denote by $G \cdot f$ the graph resulting from $G$ by contracting the edge $f$ to a single vertex and deleting any loops and multiple edges thus formed. More generally, suppose $X \subseteq V(G)$. Then denote by $G \cdot X$ the graph obtained by identifying $X$ to a single vertex and deleting all loops and multiple edges thus formed.

When dealing with a planar triangulation $G$, we assume it has a fixed embedding in the plane. Given a separating cycle $D$ in $G$, we may refer to the vertices on the inside or the outside or ‘on one side’ of $D$; these terms never include vertices of $D$ itself.

**Lemma 2.1** (Goddard [4, Claim 1 and proof of Claim 4]). Suppose $G$ is a 4-connected planar triangulation and $f \in E(G)$. Then

(i) $G \cdot f$ is a planar triangulation.

(ii) If $G \cdot f$ is not 4-connected then $f$ is in a chordless separating 4-cycle.

(iii) If $X$ is the set of vertices on one side of a separating 4-cycle in $G$, then $G \cdot X$ is also a 4-connected planar triangulation. □

**Proof of Theorem 1.1.** By way of contradiction, assume that $G$ is a counterexample with fewest vertices. Represent $K_4 - e$ as $K_4(a_1, a_2, a_3, a_4) - a_2a_4$. We suppose that $B = \{b_1, b_2, b_3, b_4\} \subseteq V(G)$ is such that there is no subdivision of $K_4 - e$ in $G$ with branch vertices $b_1, b_2, b_3, b_4$, each $b_i$ corresponding to $a_i \in V(K_4 - e)$. When we mention a $(K_4 - e)$-subdivision ‘with branch set $B$’ in a graph related to $G$, we assume this correspondence is modified. An edge of $G$ is free if it is not incident with any vertex of $B$. Two vertices $b_i, b_j$ of $B$ are pre-adjacent if $a_ia_j \in E(K_4 - e)$.

**Claim 2.2.** Every free edge of $G$ is in a separating 4-cycle.

**Proof.** Suppose $f$ is free. If $G \cdot f$ is 4-connected then by Lemma 2.1(i) and minimality of $G$ it has a $(K_4 - e)$-subdivision $H'$ with branch set $B$. Since the vertex obtained by contracting $f$ is not a branch vertex of $H'$, we can easily modify $H'$ to obtain a desired $(K_4 - e)$-subdivision $H$ in $G$. Therefore, $G \cdot f$ is not 4-connected and Lemma 2.1(ii) applies. □

**Claim 2.3.** Every vertex not in $B$ has degree at least 5.

**Proof.** Assume that $u \notin B$ has degree at most 4. Since $G$ is 4-connected, $u$ has degree exactly 4.

We now show that for each $v \in N(u)$, $uv$ is in a separating 4-cycle. Suppose this does not hold for some $v$. Then $v \in B$ by Claim 2.2, and $G \cdot uv$ is 4-connected by Lemma 2.1(ii). If we contract $uv$ to a vertex $x$, then there is a $(K_4 - e)$-subdivision $H'$ in $G \cdot uv$ with branch set $B - \{v\} \cup \{x\}$, $x$ playing the role of $v$. Note that all neighbors of $x$ except one, say $v'$, are neighbors of $v$. Hence, replacing $xv'$ in $H'$ with $uvv'$ if necessary, we obtain a desired $(K_4 - e)$-subdivision in $G$, a contradiction. Hence, $uv$ is in a separating 4-cycle.

Suppose the neighbors of $u$ are $v_1, v_2, v_3, v_4$ in order. The separating 4-cycle for $uv_1$ must contain $v_3$ and use another vertex $y$. The separating 4-cycle for $uv_2$ and $uv_4$ must use $y$ as well, by planarity. So $G$ is an octahedron $K_{2,2,2}$, which is $K_4$-linked, a contradiction. □
Claim 2.4. Let $D$ be a separating 4-cycle.

(i) There is at least one vertex of $B$ on each side of $D$.
(ii) If there is exactly one vertex of $B$ on one side of $D$, then it is the only vertex of $G$ on that side.
(iii) If there are two vertices of $B$ on each side of $D$, then $b_1$ and $b_3$ are on one side of $D$, and $b_2$ and $b_4$ are on the other.

Proof. By symmetry we may suppose that there are at most two vertices of $B$ inside $D$. Denote by $X$ the set of all vertices of $G$ inside $D$. By Lemma 2.1(iii), $G \cdot X$ is a 4-connected planar triangulation.

(i) Suppose that $X \cap B = \emptyset$. Then $|X| \geq 2$, for otherwise the vertex in $X$ has degree four, contradicting Claim 2.3. By the minimality of $G$, $G \cdot X$ contains a $(K_4 - e)$-subdivision with branch set $B$ which we can then extend to a $(K_4 - e)$-subdivision in $G$.

(ii) Suppose that $X \cap B = \{x\}$ and $|X| \geq 2$. Contract $X$ into a single vertex $x'$. Then $G \cdot X$ contains a $(K_4 - e)$-subdivision with branch set $B - \{x\} \cup \{x'\}$, with $x'$ playing the role of $x$. This can be transformed into a $(K_4 - e)$-subdivision in $G$.

(iii) If there are two vertices of $B$ on each side of $D$ but (iii) does not hold, then we may assume without loss of generality that $X \cap B = \{b_1, b_2\}$.

Since $G$ is 4-connected, there are four internally disjoint paths $\Pi_1, \Pi_2, \Pi_3, \Pi_4$ from $b_1$ to the four vertices in $D$. We claim that we may assume that one of these paths contains $b_2$. For suppose not. Then we may assume that $b_2$ lies interior to the region bounded by, say, $\Pi_1, \Pi_2$ and an edge of $D$. There are four paths from $b_2$ to the boundary of this region, and at least two of these paths must terminate on either $\Pi_1$ or $\Pi_2$, say on $\Pi_1$. Then path $\Pi_1$ may be replaced by a new path $\Pi_1'$ which contains $b_2$. So, we now assume that $b_2$ is on $\Pi_1$. Let $u$ be the end of $\Pi_1$ on $D$.

Contract $X$ to a single vertex $x'$, and let $B' = B - \{b_1, b_2\} \cup \{x', u\}$, with $x'$ and $u$ playing the roles of $b_1$ and $b_2$ respectively. By minimality of $G$, $G \cdot X$ has a $(K_4 - e)$-subdivision $H'$ with branch set $B'$. We may assume that $x'u \in E(H')$, and we can transform $H'$ into the required $(K_4 - e)$-subdivision in $G$ using $\Pi_1$ and two of $\Pi_2, \Pi_3, \Pi_4$. \hfill \Box

Let $Q$ be the set of vertices outside of $B$ with degree exactly five. Since $G$ is 4-connected and only the four vertices of $B$ can have degree four, $|Q| \geq 4$ by Euler’s formula.

Claim 2.5. For each vertex $u \in Q$,

(i) exactly two neighbors, say $x$ and $y$, of $u$ are in $B$;
(ii) $x$ and $y$ are not successive neighbors of $u$ in the planar embedding;
(iii) $x$ and $y$ are pre-adjacent; and
(iv) neither $ux$ nor $uy$ is in a separating 4-cycle.

Proof. Suppose that $u \in Q$ has neighbors $v_1, v_2, v_3, v_4, v_5$ in cyclic order.

Suppose that three neighbors of $u$ are in $B$, and let $b$ be the fourth vertex of $B$. First assume that three successive neighbors of $u$, say $v_1, v_2, v_3$, lie in $B$. There are four internally
disjoint paths from $b$ to $\{v_1, v_2, v_3, u\}$. But then $v_1v_2, v_2v_3, v_1uv_3$ and the paths from $b$ to $v_1, v_2, v_3$ give a $K_4$-subdivision with branch set $B$, a contradiction. Therefore, no three successive neighbors of $u$ lie in $B$, so we may assume that $v_1, v_3, v_5 \in B$ and $v_2, v_4 \notin B$. There are four internally disjoint paths from $b$ to $\{v_1, v_3, v_5, u\}$. At least one of $v_2$ or $v_4$, say $v_2$, is used by the path to $u$. Then $v_1v_5, v_1v_2v_3, v_3uv_5$ and the paths from $b$ to $v_1, v_3, v_5$ give a $K_4$-subdivision with $B$ as the branch set, a contradiction. Therefore, at most two neighbors of $u$ are in $B$.

We now claim that if each of $uv_{i-1}, uv_i, uv_{i+1}$ (subscripts interpreted modulo 5) is in a separating 4-cycle, then at least one of $v_{i-1}$ or $v_{i+1}$ is in $B$. Without loss of generality suppose that $uv_1, uv_2, uv_3$ are in separating 4-cycles. The separating 4-cycle for $uv_2$ must include $v_4$ or $v_5$, say $v_4$, and another vertex $v$. The separating 4-cycle for $uv_3$ must also include $v$, by planarity. But then, since $G$ is 4-connected, $v_3$ has degree four, and hence $v_3 \in B$ by Claim 2.3, as claimed.

Thus, if three successive neighbors of $u$ are not in $B$, then the edge from $u$ to each of these neighbors is in a separating 4-cycle by Claim 2.2, and the preceding paragraph yields a contradiction. Hence at most two successive neighbors of $u$ are not in $B$, and (i) and (ii) follow. Without loss of generality suppose that $v_1, v_3 \in B$ and $v_2, v_4, v_5 \notin B$. If either $uv_1$ or $uv_3$ is in a separating 4-cycle then the preceding paragraph again yields a contradiction, so (iv) holds.

If $v_1$ and $v_3$ are not pre-adjacent, then upon contracting $uv_3$ to $x'$, there is a $(K_4 - e)$-subdivision $H'$ in $G \cdot uv_3$ with branch set $B - \{v_3\} \cup \{x'\}$, with $x'$ playing the role of $v_3$. Since $x'v_1$ is not in $H'$ (because $v_1$ and $x'$ are not pre-adjacent), we can replace $x'v_5$ by $v_3uv_5$ and every other edge $x'e$ by $v_3v$, as necessary, to obtain the desired $(K_4 - e)$-subdivision in $G$, a contradiction. Hence (iii) holds. \hfill \□

**Claim 2.6.** Two vertices in $B$ have at most one common neighbor in $Q$.

**Proof.** Suppose that two vertices in $B$, say $b$ and $b'$, have two common neighbors $u$ and $v$ in $Q$. By Claim 2.5(iii), $b$ and $b'$ are pre-adjacent. Suppose $(bu_bv)$ is a separating 4-cycle. Then by Claim 2.4(i) there is exactly one vertex of $B$ on each side of $(bu_bv)$. Hence by Claim 2.4(ii) $|V(G)| = 6$ and $\deg u = 4$, a contradiction. So $(bu_bv)$ is not a separating 4-cycle, and thus either $uv \in E(G)$ or $bb' \in E(G)$. If $bb' \in E(G)$, vertex $u$ contradicts Claim 2.5(ii). Hence $uv \in E(G)$.

Suppose that in clockwise order the neighbors of $u$ are $v, b, u_1, u_2, b'$, and the neighbors of $v$ are $u, b', v_2, v_1, b$. Since $G$ is 4-connected, $u_1, u_2 \neq v_1, v_2$, so $G$ contains the cycle $(bu_1u_2b'v_2v_1)$. See Figure 1. By Claim 2.5(i), $u_1, u_2, v_1, v_2 \notin B$. If $G$ contains $u_1v_2$ then $(ub'v_2u_1)$ is a separating 4-cycle, contradicting Claim 2.5(iv) for $ub'$. So $u_1v_2$ and (similarly) $u_2v_1$ are not edges.

Now $uu_1$ is a free edge, so by Claim 2.2 there is a separating 4-cycle $C$ containing $uu_1$. By 4-connectivity, $C$ contains neither $ub$ or $uu_2$. By Claim 2.5(iv), $C$ does not contain $ub'$ and hence must contain $uv$. By Claim 2.5(iv), $C$ does not contain $vb$ or $vb'$, and since
$u_1v_2 \notin E(G)$, $C$ does not contain $vv_2$, so $C$ contains $vv_1$, and $u_1v_1 \in E(G)$. Similarly, $u_2v_2 \in E(G)$. Since $\deg u_1 \geq 5$, $D = (u_1u_2v_2v_1)$ is a separating 4-cycle. Without loss of generality we may assume that $u, v, b, b'$ are inside $D$. By Claim 2.4(iii) applied to $D$, $\{b, b'\} = \{b_1, b_3\}$, and $b_2$ and $b_4$ are outside $D$.

Now suppose one of $u_1, u_2, v_1, v_2$, say $u_1$, is in $Q$. Then by Claim 2.5(i), (ii) and (iv) the neighbors of $u_1$ in clockwise order are, without loss of generality, $u, b, v_1, b_2, u_2$. There are four internally disjoint paths from $b_4$ to $D$, and the path to $u_1$ must use $b_2$ as it is the only neighbor of $u_1$ outside $D$. Then the paths from $b_4$ to $v_1$ and $v_2$ joined to $v_1b$ and $v_2b'$ respectively, and the paths $b_2u_1b, b_2u_2b'$ and $buvb'$, give a desired $(K_4 - e)$-subdivision in $G$, a contradiction.

Therefore none of $u_1, u_2, v_1, v_2$ is in $Q$. Since $|Q| \geq 4$, there is $u' \in Q$ outside $D$. By Claim 2.5(i) $u'$ is adjacent to both $b_2$ and $b_4$, but this contradicts Claim 2.5(iii).

Now consider the subgraph $J$ of $G$ induced by edges of the form $ub, u \in Q$ and $b \in B$. By Claim 2.5(i), each vertex of $Q$ has exactly two neighbors in $B$, say $b_i$ and $b_j$ with $i < j$, and by Claim 2.6 no other vertex of $Q$ is adjacent to both $b_i$ and $b_j$. Therefore we can denote this vertex of $Q$ unambiguously as $u_{i,j}$. By Claim 2.5(iii), $b_i$ and $b_j$ are pre-adjacent if $u_{i,j}$ exists. Therefore $|Q| \leq 5$ and $J$ is isomorphic to a subdivision of a $|Q|$-edge subgraph of $K_4 - e$, each edge being subdivided exactly once.

If $u_{i,j}$ exists, then $b_ib_j \notin E(G)$, for otherwise $(b_iu_{i,j}b_j)$ would be a separating triangle by Claim 2.5(ii). If $u_{i,j}$ does not exist, $b_ib_j$ may or may not be an edge.

If $|Q| = 5$, $J$ is the subdivision of $K_4 - e$ that we require. So suppose, then, that $|Q| = 4$. Since $G$ is a plane triangulation, by Claim 2.3 and Euler’s formula, it follows that $\deg b_i \leq 4$ for each $b_i \in B$, and hence, since $G$ is 4-connected, all vertices in $B$ have degree exactly four. Modulo automorphisms of $K_4 - e$ there are only two four-edge subgraphs of $K_4 - e$, namely $(K_4 - e) - a_3a_4$ and $(K_4 - e) - a_1a_3$, giving two cases for us to consider.

*Case 1.* Suppose $J$ is isomorphic to a subdivision of $K_4 - e - a_3a_4$, which we may assume is embedded in the plane as shown at left in Figure 2. To simplify notation we let $u_1 = u_{2,3}$, and $u_i = u_{i,i}$ for $2 \leq i \leq 4$. Since $u_{1,2}, u_{1,3}, u_{1,4}$ all exist, $b_1b_2, b_1b_3, b_1b_4 \notin E(G)$. It suffices to find a $b_3b_4$-path that is internally disjoint from $B \cup Q$.

Suppose first that $u_4$ and $u_3$ are not successive neighbors of $b_1$. Then the neighbors of $b_2$ in clockwise order are $u_2, u_4, v, u_3$, where $v \notin B \cup Q$. Thus, $u_2u_3, u_2u_4, vu_3, vu_4 \in E(G).$ Since

![Figure 1. Situation in Claim 2.6](image-url)
Figure 2. The two cases

$u_2b_4, u_4b_2 \notin E(G)$, the fifth neighbor of $u_4$ is between $u_2$ and $b_4$ in clockwise order around $u_4$, so $vb_4 \in E(G)$. Similarly, $vb_3 \in E(G)$. Thus, $b_3vb_4$ is the required path.

Now suppose that $u_4$ and $u_3$ are successive neighbors of $b_1$, so that $u_3u_4 \in E(G)$. By Claim 2.5 and since $u_4$ is not adjacent to $b_3$, the neighbors of $u_3$ in clockwise order are $b_1, u_4, s, b_3, t$, so that $su_4, su_3, sb_3 \in E(G)$. By Claim 2.5(i) for $u_3$, $s \notin B$. Now by Claim 2.5(ii) for $u_4$, $sb_4 \in E(G)$. Since $u_3, u_4$ does not exist and $sb_3, sb_4 \in E(G)$, $s \notin Q$. Thus, $b_3sb_4$ is the required path.

Case 2. Suppose $J$ is isomorphic to a subdivision of $K_4 - e - a_1a_3$, which we may assume is embedded in the plane as shown at right in Figure 2. To simplify notation we let $u_i = u_{i-1} + 1$ for $1 \leq i \leq 3$, and $u_4 = u_{3,4}$. It suffices to find a $b_1b_3$-path that is internally disjoint from $B \cup Q$.

Let $H_1$ and $H_2$ be the subgraphs of $G$ consisting of $J$ and everything inside or outside of $J$, respectively. Then $H_i$ has a $b_1b_3$-path internally disjoint from $B \cup Q = V(J)$ unless there is a minimal cutset $S \subseteq V(J)$ separating $b_1$ from $b_3$ in $H_i$. Since $H_i$ is a near-triangulation, by Lemma 1.5 such a cutset $S$ must consist of two vertices of $J$ joined by an edge, which can only be $u_1u_4$, $b_2b_4$, or $u_2u_3$. Since $G$ is a counterexample, each $H_i$ must contain one of these edges.

Suppose $b_2b_4 \in E(G)$; without loss of generality assume that $b_2b_4 \in E(H_1)$. Then one of $u_1u_4, u_2u_3$, say $u_1u_4$, must be an edge of $H_2$. Then $(b_2u_1u_4b_4)$ is a separating 4-cycle, contradicting Claim 2.5(iv) for $u_1b_2$ and $u_4b_4$. Therefore, $b_2b_4 \notin E(G)$.

Hence, without loss of generality we must have $u_1u_4 \in E(H_1)$ and $u_2u_3 \in E(H_2)$. By 4-connectivity, $(u_4u_1b_1)$ and $(u_3u_2b_3)$ are facial triangles. Let the second facial triangle on $u_1u_4$ be $(u_1u_4v_1)$ where $v_1 \in V(H_1)$, and let the second facial triangle on $u_2u_3$ be $(u_2u_3v_3)$, where $v_3 \in V(H_2)$.

Suppose that $v_1 \notin V(J)$. Then by Claim 2.5(ii) applied to $u_1$ and $u_4$, $v_1b_2, v_1b_4 \in E(H_1)$. If $v_3 \notin V(J)$, then similarly $v_3b_2, v_3b_4 \in E(H_2)$. Then $(v_1b_2v_3b_4)$ is a separating 4-cycle contradicting Claim 2.4(ii). Therefore $v_3 \in V(J)$. The only possibilities for $v_3$ are $u_1$ or $u_4$; without loss of generality assume that $v_3 = u_1$. Then $(u_1v_1b_4u_3)$ is a separating 4-cycle, contradicting Claim 2.5(iv) for $u_3b_4$. 


Thus, \( v_1 \in V(J) \), and similarly \( v_3 \in V(J) \). The only possibilities for \( v_1 \) are \( u_2 \) or \( u_3 \); without loss of generality assume that \( v_1 = u_2 \). Then \( u_1u_2, u_2u_4 \in E(H_1) \). Since \( u_2 \in Q \) has degree 5, \( u_4b_3 \in E(G) \), contradicting Claim 2.5(i) for \( u_4 \).

Thus, the proof of Theorem 1.1 is complete. \( \square \)

In Theorem 1.1 we cannot replace ‘4-connected’ by ‘3-connected’ because placing \( b_1 \) and \( b_3 \) on one side of a separating triangle and \( b_2 \) and \( b_4 \) on the other gives a situation with no \((K_4 - e)\)-subdivision. We now sketch constructions (leaving the details to the reader) to show that ‘triangulation’ cannot be replaced by ‘planar graph’, and ‘\((K_4 - e)\)-linked’ cannot be replaced by ‘\(K_4\)-linked’.

First, there are 4-connected, or even 5-connected, planar graphs, differing by only one edge from a triangulation, that are not \(2\)-linked, and hence are not \((K_4 - e)\)-linked. For example, take a large triangulation with 12 vertices of degree 5 and all other vertices of degree 6 (the dual of a ‘ fullerene’). Delete an edge between two vertices of degree 6 sufficiently far from every degree 5 vertex. The result can be shown to be \(5\)-connected, and any planar graph on at least four vertices that is not a triangulation is not \(2\)-linked.

Second, as Goddard [4] mentions, there are 4-connected planar triangulations that are not \(K_4\)-linked. These can be constructed using X. Yu’s characterization [21, Theorem 4.2]. For example, in Figure 3 there are separating 4-cycles (shown with thicker edges) between the inner and outer pairs of solid vertices. In a \(K_4\)-subdivision for the solid vertices, the path between the inner solid vertices must use the other two inner vertices, and similarly for the outer vertices. There must be four paths between the inner and outer solid vertices, crossing the separating 4-cycles, but each such path is forced to join both left solid vertices or both right solid vertices, so we cannot get a \(K_4\)-subdivision. Infinitely many other examples may be obtained by adding more or fewer separating 4-cycles between the inner and outer solid vertices.

3. Maximum connectivity of obstructions

In this section we prove that every 7-connected obstruction is an obstruction for a trivial reason. We use this to prove that every 7-connected graph is \(P_4\)-linked.
A separation in a graph $G$ is a pair of edge-disjoint subgraphs $(G_1, G_2)$ such that $G = G_1 \cup G_2$ and each $G_i$ contains an edge or vertex not in $G_{3-i}$. If $|V(G_1 \cap G_2)| = k$ then $(G_1, G_2)$ is a $k$-separation. We say that $T \subseteq V(G)$ separates $S_1, S_2 \subseteq V(G)$ if $S_1$ and $S_2$ are disjoint from $T$ and there is no path from $S_1$ to $S_2$ in $G - T$. If $H$ is a subgraph of $G$, then $N_G(H)$ is $\{v \in V(G) - V(H) \mid vw \in E(G) \text{ for some } w \in V(H)\}$.

We need some definitions and results from [22, 23, 24]. The following definitions of ‘3-planar’ and ‘rung’ differ slightly from those in [24], but are equivalent. We introduce the idea of the ‘foundation’ of a 3-planar graph for later reference, and the idea of ‘$R$-equivalence’ to describe permitted symmetries for rungs.

**Definition 3.1.** If $G$ is a graph and $\mathcal{A} = \{A_1, A_2, \ldots, A_k\}$ is a (possibly empty) collection of pairwise disjoint induced subgraphs of $G$, then we say $(G, \mathcal{A})$ is 3-planar if $N_G(A_i) \cap A_j = \emptyset$ for all distinct $i$ and $j$, $|N_G(A_i)| \leq 3$ for all $i$, and the graph $G'$ obtained from $G$ by replacing each $A_i$ with a new vertex $a_i$ adjacent to $N_G(A_i)$ is planar.

We call $G - \bigcup_{i=1}^k V(A_i)$ the foundation of $G$. If in addition $b_0, b_1, \ldots, b_n$ are (possibly not distinct) vertices of the foundation of $G$ and $G'$ can be embedded in a closed disk with $b_0, b_1, \ldots, b_n$ in cyclic order along its boundary, then we say that $(G, \mathcal{A}, b_0, b_1, \ldots, b_n)$, or just $(G, b_0, b_1, \ldots, b_n)$, is 3-planar.

Since a planar graph is at most 5-connected, it is clear that a 3-planar graph whose foundation has at least four vertices is at most 5-connected.

**Definition 3.2.** Suppose $G$ is a graph, and $\{a, b, c\}, \{a', b', c'\}$ are 3-subsets of $V(G)$. Suppose $\{a, b, c\} \neq \{a', b', c'\}$, and $G$ has no 3-separation $(G_1, G_2)$ with $\{a, b, c\} \subseteq V(G_1)$ and $\{a', b', c'\} \subseteq V(G_2)$. Then we call $(G, (a, b, c), (a', b', c'))$ a rung if at least one of the following holds:

1. $b = b'$ or $\{a, c\} = \{a', c'\}$;
2. $a = a'$ and $(G - a, c, c', b', b)$ is 3-planar;
3. $\{a, b, c\} \cap \{a', b', c'\} = \emptyset$ and $(G, a', b', c', c, b, a)$ is 3-planar;
4. $\{a, b, c\} \cap \{a', b', c'\} = \emptyset$, $G$ has a 1-separation $(G_1, G_2)$ such that $\{a, a', b, b'\} \subseteq V(G_1)$, $\{c, c'\} \subseteq V(G_2)$, and $(G_1, a, a', b', b)$ is 3-planar;
5. $\{a, b, c\} \cap \{a', b', c'\} = \emptyset$, $(G, a, a', b', b)$ is 3-planar, and $G$ has a separation $(G_1, G_2)$ such that $V(G_1 \cap G_2) = \{z, b\}$, $\{a, a', b, b'\} \subseteq V(G_1)$, $\{c, c'\} \subseteq V(G_2)$, and $(G_2, c, c', z, b)$ is 3-planar;
6. $\{a, b, c\} \cap \{a', b', c'\} = \emptyset$, and there are pairwise edge-disjoint subgraphs $G_a, G_c, M$ of $G$ such that $G = G_a \cup G_c \cup M$, $V(G_a \cap M) = \{u, w\}$, $V(G_c \cap M) = \{p, q\}$, $V(G_a \cap G_c) = \emptyset$, $\{a, a', b'\} \subseteq V(G_a)$, $\{c, c', b\} \subseteq V(G_c)$, $(G_a, a, a', b', w, u)$ is 3-planar, and $(G_c, c', c, b, p, q)$ is 3-planar;
7. $\{a, b, c\} \cap \{a', b', c'\} = \emptyset$, and there are pairwise edge-disjoint subgraphs $G_a, G_c, M$ of $G$ such that $G = G_a \cup G_c \cup M$, $V(G_a \cap M) = \{b, b'\}$, $V(G_c \cap M) = \{b, b', p\}$, $V(G_a \cap G_c) = \{b, b'\}$, $\{a, a', b'\} \subseteq V(G_a)$, $\{c, c', b\} \subseteq V(G_c)$, $(G_a, a, a', b', w, b)$ is 3-planar, and $(G_c, c', c, b, p, b')$ is 3-planar.
A structure \((G, (a, b, c), (a', b', c'))\) is said to be \(R\)-equivalent to itself and to the structures \((G, (a', b', c'), (a, b, c))\), \((G, (c, b, a), (c', b, a'))\) and \((G, (c', b', a'), (c, b, a))\). Anything \(R\)-equivalent to a rung is also considered a rung.

If \((G, (a, b, c), (a', b', c'))\) is a rung, then it is not hard to show that \((G, \{a, c\}, \{a', c'\}, (b, b'))\) is an obstruction [22, Prop. 4.2]. Rungs are ‘basic’ obstructions which form the building blocks of general obstructions.

**Definition 3.3** ([24]). Let \(L\) be a graph and let \(R_1, \ldots, R_m, m \geq 1\), be edge disjoint subgraphs of \(L\) such that

(i) \((R_i, (x_{i-1}, v_{i-1}, y_{i-1}), (x_i, v_i, y_i))\) is a rung for \(1 \leq i \leq m\),

(ii) \(V(R_i \cap R_j) = \{x_i, v_i, y_i\} \cap \{x_{j-1}, v_{j-1}, y_{j-1}\}\) for \(1 \leq i < j \leq m\),

(iii) for any \(0 \leq i < j \leq m\), if \(x_i = x_{i-1}\) then \(x_k = x_i\) for all \(i \leq k \leq j\), if \(v_i = v_j\) then \(v_k = v_i\) for all \(i \leq k \leq j\), and if \(y_i = y_j\) then \(y_k = y_i\) for all \(i \leq k \leq j\).

(iv) \(L = (\bigcup_{i=1}^m R_i) + S\), where \(S\) consists of edges of \(L\) with both endvertices in some \(\{x_i, v_i, y_i\}\), \(0 \leq i \leq m\).

Then we call \((L, (x_0, v_0, y_0), (x_m, v_m, y_m))\) a **ladder with rungs** \((R_i, (x_{i-1}, v_{i-1}, y_{i-1}), (x_i, v_i, y_i)), i = 1, 2, \ldots, m\), or simply a ladder along \(v_0v_1 \ldots v_m\). Note that [24] has the inequalities in (iii) and (iv) beginning at 1, not 0, but 0 is correct.

Informally, a ladder is obtained from a sequence of rungs by identifying the \((a', b', c')\) vertices of each rung with the \((a, b, c)\) vertices of the next rung. We can also add edges \(ab, ac, bc, a'b', a'c', b'c'\) inside any rung. Note that anything \(R\)-equivalent to a ladder is also a ladder.

**Theorem 3.4** (Yu [24, Theorem 1.3]). Let \(G\) be a graph, and let \(S = \{a, b, c\}\) and \(S' = \{a', b', c'\}\) be 3-subsets of \(V(G)\) with \(S \neq S'\). Assume that for every \(T \subseteq V(G)\) with \(|T| \leq 3\), every component of \(G - T\) contains a vertex of \(S \cup S'\). Then \((G, \{a, c\}, \{a', c'\}, (b, b'))\) is an obstruction if and only if one of the following statements hold:

1. \(G\) has a \(k\)-separation \((G_1, G_2)\) with \(k \leq 2\), \(S \subseteq V(G_1)\), and \(S' \subseteq V(G_2)\).
2. \(G\) is the edge-disjoint union of a ladder \((L, (a, b, c), (a', b', c'))\) or \((L, (a, b, c), (c', b', a'))\) along \(v_0v_1 \ldots v_m\) (where \(b = v_0\) and \(b' = v_m\)) and a (possibly edgeless) graph \(J\) such that \(V(J \cap L) = \{w_0, w_1, \ldots, w_n\}\) and \((J, w_0, w_1, \ldots, w_n)\) is 3-planar, where \(w_0, w_1, \ldots, w_n\) is the sequence \(v_0, v_1, \ldots, v_m\) with repetitions removed. \(\square\)

In [24], case (2) above is separated into two cases, the situation where \(J\) is edgeless being treated as a separate case. Also, in [24] the case of the ladder being \((L, (a, b, c), (c', b', a'))\) is not mentioned, but this needs to be present for the ‘only if’ part of the theorem to be correct. This is because there is symmetry between \(a'\) and \(c'\) in the definition of an obstruction, but not in the definition of a rung \((G, (a, b, c), (a', b', c'))\) or ladder \((L, (a, b, c), (a', b', c'))\).

To take the symmetry between \(a'\) and \(c'\) into account we define \((G, (a, b, c), (a', b', c'))\) to be \(X\)-equivalent to itself, to \((G, (a, b, c), (c', b', a'))\), and to everything \(R\)-equivalent to...
either of these. Then an X-rung is a structure X-equivalent to one of (1)–(7) of Definition 3.2. For every X-rung \((G, (a, b, c), (a', b', c'))\), \((G, \{a, c\}, \{a', c'\}, (b, b'))\) is an obstruction. We define an X-ladder by replacing ‘rung’ by ‘X-rung’ in the definition of a ladder. Clearly \((L, (a, b, c), (a', b', c'))\) is an X-ladder if and only if either it is a ladder or \((L, (a, b, c), (c', b', a'))\) is a ladder. Therefore, the condition on \(L\) in case (2) of Theorem 3.4 may be restated as ‘\((L, (a, b, c), (a', b', c'))\) is an X-ladder’.

The following lemma will be used to handle one-rung ladders.

**Lemma 3.5.** If \((G, (a, b, c), (a', b', c'))\) is an X-rung not covered by case (1) of Definition 3.2 and \(G' = G \cup K_3(a, b, c) \cup K_3(a', b', c') \cup \{bb'\}\), then \(G'\) is at most 6-connected.

**Proof.** Without loss of generality, assume \(G\) is exactly as in one of cases (2)–(7) of Definition 3.2. We examine each case individually.

(2) We can add \(bc, b'c'\) and \(bb'\) to \(G - a\) without destroying 3-planarity, so \(G' - a = (G - a) \cup \{bc, b'c', bb'\}\) is 3-planar. Since \(G\) is not covered by case (1), \(b, b', c,\) and \(c'\) are distinct, so \(G' - a\) has at least four foundation vertices and hence is at most 5-connected. Therefore, \(G'\) is at most 6-connected.

(3) We can add the edges of \(K_3(a, b, c)\) and \(K_3(a', b', c')\) to \(G\) without destroying 3-planarity. Thus, \(G' - bb'\) is 3-planar with at least six foundation vertices and hence at most 5-connected. Therefore, \(G'\) is at most 6-connected.

(4) If \(c \neq z\) then \(\{a, b, c', z\}\) separates \(a'\) and \(c\) in \(G'\), and if \(c = z\) then \(\{a', b', c = z\}\) separates \(a\) and \(c'\) in \(G'\).

(5) \(\{a, b, c', z\}\) separates \(\{a', b'\} \setminus \{z\}\) and \(c\) in \(G'\).

(6) \(\{a, b, c', p, q\}\) separates \(a'\) and \(c\) in \(G'\).

(7) \(\{a, b, b', c', p\}\) separates \(a'\) and \(c\) in \(G'\). \(\Box\)

Now we restate our upper bound on the connectivity of obstructions.

**Theorem 1.3.** Let \((G, \{a, c\}, \{a', c'\}, (b, b'))\) be an obstruction. If \(\{a, c\} \neq \{a', c'\}\) and \(b \neq b'\), then \(G\) is at most 6-connected.

**Proof.** Suppose that \(G\) is 7-connected. We may assume we have a maximal obstruction, meaning that \((G + e, \{a, c\}, \{a', c'\}, (b, b'))\) is not an obstruction for any \(e \in E(G)\). Since \(G\) is 7-connected, case (1) of Theorem 3.4 does not hold, so case (2) holds. Thus \(G\) is the union of an X-ladder \((L, (a, b, c), (a', b', c'))\) and a 3-planar graph \((J, w_0, w_1, \ldots, w_n)\). Let the X-rungs of \(L\) be \((R_i, (x_{i-1}, v_{i-1}, y_{i-1}), (x_i, v_i, y_i))\), \(1 \leq i \leq m\), so that \(w_0, w_1, \ldots, w_n\) is the sequence \(v_0, v_1, \ldots, v_m\) with repetitions removed.

Define the I-rung \(R'_i\), \(1 \leq i \leq m\), to be the subgraph of \(G\) induced by \(V(R_i)\). By maximality of \(G\) and Definition 3.3 (iv), \(R'_i\) contains \(K_3(x_{i-1}, v_{i-1}, y_{i-1})\) and \(K_3(x_i, v_i, y_i)\) as subgraphs. Also, if \(v_{i-1} \neq v_i\) then we may always add the edge \(v_{i-1}v_i\) to \(J\), so by maximality \(v_{i-1}v_i \in E(R'_i)\) if \(v_{i-1} \neq v_i\). In general, \(R'_1, R'_2, \ldots, R'_m\) are not pairwise edge-disjoint.

**Claim 3.6.** We have \(m \geq 2\), and we may assume that every I-rung \(R'_i\), \(1 \leq i \leq m\), has one of the following forms:
(i) $K_4(x_{i-1} = x_i, v_{i-1} = v_i, y_{i-1}, y_i)$, or $K_4(x_{i-1}, x_i, v_{i-1} = v_i, y_{i-1} = y_i)$, or

(ii) $K_4(x_{i-1} = x_i, v_{i-1}, v_i, y_{i-1} = y_i)$, or

(iii) $K_5(x_{i-1}, x_i, v_{i-1} = v_i, y_{i-1} = y_i)$.

**Proof.** If $m = 1$ (there is only one X-rung) then $V(J) - \{v_0, v_1\}$ is empty, otherwise $\{v_0, v_1\}$ is a cutset. Hence, $G = R_i'$. Since $b \neq b'$ and $\{a, c\} \neq \{a', c'\}$, case (1) of Definition 3.2 does not apply, so $R_i'$ is at most 6-connected by Lemma 3.5. Therefore, $m \geq 2$.

To satisfy the claim about each I-rung, we may need to swap the labels of $x_i$ and $y_i$ for some values of $i$, which we can do since swapping $x_i$ and $y_i$ in the X-rung $(R_i, (x_{i-1}, v_{i-1}, y_{i-1}), (x_i, v_i, y_i))$ still yields an X-rung. By processing the X-rungs in order from $R_1$ up to $R_m$, each such swap can be done without altering the effects of previous swaps. We describe how to process each X-rung $R_i$. We use ‘$R_i$ is an obstruction’ as shorthand for ‘$(R_i, \{x_{i-1}, y_{i-1}\}, \{x_i, y_i\}, (v_{i-1}, v_i))$ is an obstruction’.

Write $T = \{x_{i-1}, v_{i-1}, y_{i-1}\}$ and $T' = \{x_i, v_i, y_i\}$, which may not be disjoint. If $V(R_i) \neq T \cup T'$, then $T \cup T'$ is a cutset in $G$ of order at most 6, a contradiction. Hence $V(R_i) = T \cup T'$ and $|V(R_i)| \leq 6$. By Definition 3.2, $R_i$ has no 3-separation $(G_1, G_2)$ with $T \subseteq V(G_1)$ and $T' \subseteq V(G_2)$. Therefore, $R_i$ has no 3-cutset $S$ which separates $T - S$ from $T' - S$.

Suppose that $|V(R_i)| = 6$, so that $T$ and $T'$ are disjoint. Let $Q$ be the bipartite graph with vertex set $T \cup T'$ and containing all edges of $R_i$ with one end in $T$ and the other end in $T'$.

If $u \in T$ has degree 0 or 1 in $Q$, then $u$ is nonadjacent to two vertices $s, t$ of $T'$. Then $T \cup T' - \{u, s, t\}$ is a 3-cutset of $R_i$ separating $u$ from $s$ and $t$, which is not allowed. So every vertex in $T$, and similarly in $T'$, has degree at least 2 in $Q$.

Therefore, if $Q'$ is the complement of $Q$ in the $K_{3,3}$ with bipartition $(T, T')$, it has maximum degree at most 1 and so $Q' \subseteq 3K_2$. Thus, $Q$ contains $K_{3,3} - 3K_2 = C_6$ as a subgraph. But then $Q$ has two disjoint perfect matchings, and in at least one of them $v_{i-1}$ is not matched to $v_i$, contradicting the fact that $R_i$ is an obstruction.

Now suppose that $|V(R_i)| = 5$. If there is a vertex $s$ of $T - T'$ not adjacent to some vertex $t$ of $T' - T$, then $T \cup T' - \{s, t\}$ is a 3-cutset of $R_i$ separating $s$ and $t$, which is not allowed. Therefore, $R_i$ is a $K_5$. Since $|V(R_i)| = 5$, some vertex in $T$ is equal to some vertex in $T'$. Since $R_i$ is an obstruction, the equality must be $v_{i-1} = v_i$ and we have (iii) above.

Finally, suppose that $|V(R_i)| = 4$. Not every 4-vertex obstruction is an X-rung; for example, an edgeless graph on vertices $x_{i-1}$, $x_i = v_{i-1}$, $v_i$, $y_{i-1} = y_i$ is an obstruction, but not an X-rung. So we must refer to the details of Definition 3.2. $(R_i, (x_{i-1}, v_{i-1}, y_{i-1}), (x_i, v_i, y_i))$ must be X-equivalent to case (1) of Definition 3.2 (or to a degenerate version of case (2), with $b = b'$ or $c = c'$, but then we have case (1) again). Adding any edge between $T$ and $T'$ that is not already present does not violate the conditions of case (1), so by maximality $R_i$ must be a $K_4$. By swapping the labels of $x_i$ and $y_i$ if necessary, we can guarantee that $R_i$ has the form of (i) above (if $b = b'$ in (1)) or (ii) above (if $\{a, c\} = \{a', c'\}$ in (1)).
Henceforth we assume all I-rungs are as in Claim 3.6. All X-rungs are therefore covered by case (1) of Definition 3.2.

If $v_{i-1} \neq v_i$ then $G$ always contains the edge $v_{i-1}v_i$, which may be in $J$ or in the X-rung $R_i$. Definition 3.2(1) allows us to assume that it is always in $R_i$. Thus, $L$ contains the path $W = w_0w_1 \ldots w_n$. Also, if $x_i \neq x_{i-1}$ then $R_i$ always contains the edge $x_ix_{i-1}$. Thus, $L$ contains the path $A = a_0a_1 \ldots a_p$, where $a_0, a_1, \ldots, a_p$ is the sequence $x_0, x_1, \ldots, x_m$ with repetitions removed. Similarly, $L$ contains the path $C = c_0c_1 \ldots c_q$, where $c_0, c_1, \ldots, c_q$ is the sequence $y_0, y_1, \ldots, y_m$ with repetitions removed. We have $L = \bigcup_{i=1}^m R'_i$, and $V(L) = V(W) \cup V(A) \cup V(C)$.

From Claim 3.6 and the definition of an X-ladder we observe the following.

(A) Suppose $P, Q \in \{W, A, C\}$ are distinct. If $v \in V(P)$ then the neighbors of $v$ on $Q$ are consecutive.

(B) Every edge of $W, A$ or $C$ belongs to a unique $R_i$ and hence to a unique $R'_i$.

(C) Suppose $(P, Q) = (A, W), (C, W), (W, A)$ or $(W, C)$. If $uv \in E(P)$, then $u$ and $v$ have exactly one common neighbor on $Q$, namely the unique vertex of $Q$ in the unique I-rung containing $uv$. (This does not hold with $(P, Q) = (A, C)$ or $(C, A)$.)

Let $J'$ denote the subgraph of $G$ induced by $V(J)$. Then $J' = J \cup W$.

Claim 3.7. $J'$ may be regarded as a plane graph with outer cycle $(w_0w_1 \ldots w_n)$, where $n \geq 2$.

Proof. Suppose that $n \leq 1$. Then $V(J) = V(W)$, otherwise $V(W)$ is a cutset of order $n + 1 \leq 2$. The only possible edge of $J$ is $w_0w_1$ when $n = 1$, but this is an edge of $L$. Therefore $J$ is edgeless. Hence, since $m \geq 2$, $T = \{x_1, v_1, y_1\}$ is a cutset of order $3$ separating $V(R_1) - T$ from $V(R_2) - T$ in $G$. So $n \geq 2$.

Since $G$ is 7-connected, the 3-planar graph $J$ cannot contain any subgraphs $H$ disjoint from $L$ with $|N_J(H)| \leq 3$. So $J$ is in fact planar, and can be embedded in a disk with $w_0, w_1, \ldots, w_n$ in cyclic order around the boundary. The edge $w_0w_n$ can always be added to this embedding if it is not already present, so by maximality of $G$, $w_0w_n \in E(J)$. Moreover, by maximality $w_0w_n$ must be an edge of the outer facial walk, otherwise we can move it into the outer face and add another edge to $J$. We can also add the edges of $W$ in the outer face to obtain a planar embedding of $J'$ with $(w_0w_1 \ldots w_n)$ as outer cycle.

Claim 3.8. We may assume that there are $i$ and $k$ so that either (i) $w_ia_{k-1}, w_ia_k, w_ia_{k+1} \in E(L)$, or (ii) $w_ia_{k-1}, w_ia_k, w_{i+1}a_k, w_{i+1}a_{k+1} \in E(L)$.

Proof. For $P = A$ or $C$, let $n_P(w_i) = |N_L(w_i) \cap V(P)|$. If $n_A(w_i) \geq 3$ or $n_C(w_i) \geq 3$ for some $w_i$ then we may assume (i) by (A). So, assume that $n_A(w_i), n_C(w_i) \leq 2$, so that $d_{L - E(W)}(w_i) \leq 4$, for all $w_i$. Since $G$ is 7-connected, $d_G(w_i) \geq 7$. Therefore, $d_{J'}(w_i) = d_G(w_i) - d_{L - E(W)}(w_i) \geq 3$ for all $w_i$.

Let $n_3, n_4$ and $n_5^+$ be the number of vertices in $W$ with degree 3, 4, and at least 5 in $J'$, respectively, and let $n_{\text{int}}$ be the number of internal vertices of $J'$. Add a new vertex $z$ and
join it to all vertices of the outer cycle \((w_0w_1\ldots w_n)\) of \(J'\), giving a new planar graph \(J''\). Then \(d_{J''}(z) = n_3 + n_4 + n_6^+; d_{J''}(w_i) = d_{J'}(w_i) + 1\) for all \(w_i\); and \(d_{J''}(v) = d_{J'}(v) \geq 7\) for all internal vertices \(v\) of \(J'\), because \(G\) is 7-connected.

Since \(J''\) is planar, \(6|V(J'')| - 12 \geq 2|E(J'')|\). This implies that \(6(n_{int} + n_3 + n_4 + n_6^+ + 1) - 12 \geq 7n_{int} + 4n_3 + 5n_4 + 6n_6^+ + (n_3 + n_4 + n_6^+)\), from which \(n_3 \geq n_{int} + n_6^+ + 6 \geq n_5^+ + 6\). Therefore, on \(W\) there are two vertices with degree 3 in \(J'\) such that no vertices with degree at least 5 in \(J'\) lie between them.

Thus, there is \(ww' \in E(W)\) with \(d_{J'}(w) \leq 3\) and \(d_{J'}(w') \leq 4\). Since \(G\) is 7-connected, \(d_{L-E(W)}(w) \geq 4\) and \(d_{L-E(W)}(w') \geq 3\). Since \(n_A(w), n_C(w), n_A(w'), n_C(w') \leq 2\), we must have \(n_A(w) = n_C(w) = 2\), and without loss of generality \(n_A(w') = 2\) and \(n_C(w') \geq 1\). Then (ii) follows from (C) and (A).

If Claim 3.8(i) applies, let \(j = i\), and if Claim 3.8(ii) applies, let \(j = i + 1\). In either case, \(w_iak_i, w_ia_k, w_ja_k, w_jak_{i+1} \in E(L)\). By (B) there are unique \(s\) and \(t\) with \(a_{k-1}a_k \in E(R_s)\) and \(a_ka_{k+1} \in E(R_t)\); clearly \(s \leq t\). Then \(a_{k-1} = x_{s-1}, a_k = x_s = x_{t-1}\), and \(a_{k+1} = x_t\). By (C), \(w_i\) is the unique vertex of \(W\) in \(R_s\), so \(w_i = v_{s-1} = v_s\), and \(w_j\) is the unique vertex of \(W\) in \(R_t\), so \(w_j = v_{t-1} = v_t\). Let \(y_{s-1} = h_g\) and \(y_t = c_h\); clearly \(g \leq h\).

Define
\[
U_1 = \{a_\alpha | \alpha < k - 1\} \cup \{w_\beta | \beta < i\} \cup \{c_\gamma | \gamma < g\},
\]
\[
U_2 = \{a_k\} \cup \{c_\gamma | g < \gamma < h\},
\]
\[
U_3 = \{a_\alpha | \alpha > k + 1\} \cup \{w_\beta | \beta > j\} \cup \{c_\gamma | \gamma > h\}.
\]

Because \(x_{s-1} = a_{k-1}, v_{s-1} = w_i\), and \(y_{s-1} = c_{g}\), if \(R_r\) (or \(R'_r\)) contains a vertex of \(U_1\) then \(r \leq s - 1\), and if \(R_r\) contains a vertex of \(U_2\) then \(r \geq s\). Because \(x_t = a_{k+1}, v_t = w_j\), and \(y_t = c_h\), if \(R_r\) contains a vertex of \(U_2\) then \(r \leq t\), and if \(R_r\) contains a vertex of \(U_3\) then \(r \geq t + 1\). Therefore there is no \(R'_r\) containing both a vertex of \(U_2\) and a vertex of \(U_1 \cup U_3\), so there are no edges from \(U_2\) to \(U_1 \cup U_3\). There are also no edges from \(U_2\) to \(U_1 \cup U_3\) (since \(U_2\) contains no vertex of \(J\)). Therefore, \(S = V(G) - (U_0 \cup U_1 \cup U_2 \cup U_3) = \{a_{k-1}, a_{k+1}, w_i, w_j, c_g, c_h\}\) separates \(U_2\) from \(U_0 \cup U_1 \cup U_3\), which is nonempty because \(|V(W)| = n + 1 \geq 3\) and \(U_2 \cup S\) contains at most two vertices of \(W\). Therefore, \(G\) is not 7-connected, a contradiction which concludes the proof of Theorem 1.3.

Now we can show that 7-connected graphs are \(P_4\)-linked.

Suppose \(u\) and \(v\) are vertices of \(G\). If \(N_G(u) = N_G(v)\), then we say \(u\) and \(v\) are nonadjacent twins of each other in \(G\), and if \(N_G[u] = N_G[v]\) we say they are adjacent twins. If we add a new vertex \(w\) to \(G\) adjacent exactly to all vertices of \(N_G(v)\) or \(N_G[v]\), then we say we have made a nonadjacent or adjacent twin of \(v\), respectively.

The following is well known.

**Observation 3.9.** Suppose \(G\) is \(k\)-connected. Let \(G'\) be obtained from \(G\) by making a nonadjacent (or adjacent) twin of a vertex of \(G\). Then \(G'\) is also \(k\)-connected.
Observation 3.10 (Seymour, see [24, p. 245]). Let \( u, v, w, x \) be distinct vertices of a graph \( G \).

(i) If \( vw \in E(G) \) then \( G \) has a path through \( u, v, w, x \) in that order if and only if \( G \) has vertex-disjoint paths from \( u \) to \( v \) and from \( w \) to \( x \).

(ii) If \( vw \notin E(G) \), construct \( G' \) from \( G \) by making nonadjacent twins \( v', w' \) of \( v, w \) respectively. Then \( G \) has a path through \( u, v, w, x \) in that order if and only if \((G', \{w, w'\}, \{v, v'\}, (u, x))\) is not an obstruction.

Proof of first part of Theorem 1.4. Suppose \( G \) is a 7-connected graph. To show \( G \) is \( P_4 \)-linked we must show there is a path through specified vertices \( u, v, w, x \) in that order. If \( vw \in E(G) \), then since every 6-connected graph is 2-linked [6], \( G \) has vertex-disjoint paths from \( u \) to \( v \) and from \( w \) to \( x \). If \( vw \notin E(G) \), make nonadjacent twins \( v', w' \) of \( v, w \) respectively to obtain a graph \( G' \). By Observation 3.9, \( G' \) is 7-connected. By Theorem 1.3, \((G', \{w, w'\}, \{v, v'\}, (u, x))\) is not an obstruction. In either case, \( G \) has the desired path by Observation 3.10. \( \square \)

4. 6-CONNECTED GRAPHS WITHOUT \( P_4 \)-SUBDIVISIONS

In this section we prove the second half of Theorem 1.4 by constructing a family of 6-connected graphs that are not \( P_4 \)-linked. We use Seymour’s Observation 3.10, first finding a 6-connected obstruction \( G \) and then deriving our example \( G' \) from \( G \).

X. Yu [24, pp. 243-245] constructed obstructions that were claimed to be 7-connected; in fact they are only 6-connected. One can derive graphs that are not \( P_4 \)-linked from these obstructions, but they are only 5-connected. We will modify X. Yu’s construction to obtain our examples. Since the crucial issue here is the connectivity of the resulting graphs, we provide a detailed verification that our examples are 6-connected.

We use the terminology and notation of Section 3.

4.1. Construction of near-triangulation \( J' \). We describe a near-triangulation which will play the role of \( J' \) in our construction of \( G \).

Let \( \Pi_0 \) be an edge \( bb' \). For each \( i, \) \( 0 \leq i \leq 4 \), construct a new path \( \Pi_{i+1} \) so that each vertex of \( \Pi_i \) is adjacent to at least four consecutive vertices on \( \Pi_{i+1} \), every vertex of \( \Pi_{i+1} \) is adjacent to one or two vertices of \( \Pi_i \), and the region between \( \Pi_i \) and \( \Pi_{i+1} \) is triangulated. For \( i = 5 \) we construct \( \Pi_6 \) in the same way, except the first and last vertex of \( \Pi_5 \) each has only one neighbor in \( \Pi_6 \), the first or last vertex of \( \Pi_6 \), respectively. Let \( J' \) be the union of the paths \( \Pi_1, \Pi_2, \ldots, \Pi_6 \) and all edges between them. \( J' \) is a near-triangulation. Let \( W^+ \) be its boundary cycle and write \( W^+ = (w_0w_1w_2\ldots w_n) \) where \( w_0 = b \) and \( w_n = b' \). \( J' \) may be seen as the subgraph consisting of the solid vertices and thicker edges in Figure 4.

Then from every internal vertex \( v \) of \( J' \) there are at least seven paths, disjoint except at \( v \), from \( v \) to \( W^+ \). Suppose \( v \) is on \( \Pi_i \). We may take one path that uses one vertex of each of \( \Pi_{i-1}, \Pi_{i-2}, \ldots, \Pi_0 \); two paths along \( \Pi_i \) from \( v \) to each end of \( \Pi_i \); and four paths that use one vertex of each of \( \Pi_{i+1}, \Pi_{i+2}, \ldots, \Pi_6 \).
**Observation 4.1.** If $Z$ is a cycle in $J'$ with at least one vertex inside it, then $|V(Z)| \geq 7$, because the seven paths from the inside vertex to $W^+$ must intersect $Z$ at distinct points.

4.2. **Construction of obstruction $G$ and example $G'$.** Given the vertices $w_0, w_1, \ldots, w_n$ of $J'$ and new vertices $a_0, a_1, \ldots, a_{n-7}$ and $c_0, c_1, \ldots, c_{n-7}$, we form $L$ by taking the union of the following $I$-rungs (all $K_4$’s as in Claim 3.6(ii) or $K_5$’s as in Claim 3.6(iii)):

- $K_4(a_0, w_i, w_{i+1}, c_0), 0 \leq i \leq 2$,
- $K_5(a_0, a_1, w_3, c_0, c_1)$,
- $K_4(a_1, w_i, w_{i+1}, c_1), 3 \leq i \leq 4$,
- $K_5(a_1, a_{i+1}, w_{i+4}, c_i, c_{i+1}), 1 \leq i \leq n - 9$,
- $K_4(a_i, w_{i+3}, w_{i+4}, c_i), 2 \leq i \leq n - 9$,
- $K_4(a_{n-8}, w_i, w_{i+1}, c_{n-8}), n - 5 \leq i \leq n - 6$,
- $K_5(a_{n-8}, a_{n-7}, w_{n-3}, c_{n-8}, c_{n-7})$ and
- $K_4(a_{n-7}, w_i, w_{i+1}, c_{n-7}), n - 3 \leq i \leq n - 1$.

(So we take three $K_4$’s, then one $K_5$, then two $K_4$’s, then alternate $K_5, K_4, K_5, \ldots, K_4, K_5$, finishing with two $K_4$’s, then one $K_5$, then three $K_4$’s.) Then $L$ satisfies the definition of a ladder, and if $G = J' \cup L$, then $(G, \{a_0, c_0\}, \{a_{n-7}, c_{n-7}\}, (w_0, w_n))$ is an obstruction by Theorem 3.4.

Notice that in $G$, $a_i$ and $c_i$ are adjacent twins for all $i$, $0 \leq i \leq n - 7$.

Now let $G' = G - \{c_0, c_{n-7}\}$. We claim that $G'$ has no $P_4$-subdivision with branch vertices $w_0, a_{n-7}, a_0, w_n$ in that order along the $P_4$. By Observation 3.10, there is no $P_4$-subdivision if and only if when we make a nonadjacent twin $a'_0$ of $a_0$ we get a graph $G''$ such that $(G'', \{a_0, a'_0\}, \{a_{n-7}, a'_0\}, (w_0, w_n))$ is an obstruction. But relabelling $a'_0$ as $c_0$ and $a'_n$ as $c_{n-7}$, we see that $G'' = G - \{a_0c_0, a_{n-7}c_{n-7}\}$ and since $G$ is an obstruction, $G''$ is an obstruction. Thus, $G'$ does not have the required $P_4$-subdivision and so $G'$ is not $P_4$-linked.

Now we must show that $G'$ is 6-connected.

4.3. **Projections and minimal cutsets.** To examine the connectivity of $G$ and $G'$, we will make use of a simpler graph $H$. In order to relate cutsets in $H$ to cutsets in $G$ and $G'$, we need the following concepts.

Let $G$ be a graph. By a minimal cutset $S \subseteq V(G)$ we mean that no proper subset of $S$ is a cutset.

Suppose $H$ is an induced subgraph of $G$ and we have a map $\pi : V(G) \to V(H)$ such that if $v \in V(H)$ then $\pi(v) = v$, and if $v \notin V(H)$ then $\pi(v)$ is an adjacent twin of $v$. We call $\pi$ a projection of $G$ onto $H$. The essential fact we need is the following. We omit the proof, which is not difficult. The word ‘minimal’ is necessary here.

**Lemma 4.2.** Suppose we have a projection $\pi$ of $G$ onto $H$. Then $S$ is a minimal cutset of $G$ if and only if $S = \pi^{-1}(T)$ for some minimal cutset $T$ of $H$.  

\[\square\]
4.4. Cutsets in the projection $H$. It is convenient to modify slightly our notation from Section 3, and write $A = \{a_0, a_1, \ldots, a_{n-7}\}$ and $C = \{c_0, c_1, \ldots, c_{n-7}\}$ (so these are sets of vertices, not paths). Let $H = G - C$. As shown in Figure 4, $H$ is a near-triangulation with boundary cycle $(w_0a_0a_1a_2\ldots a_{n-7}w_n)$. Lemma 1.5 and the following observation are useful for investigating cutsets in $H$.

Observation 4.3. For $0 \leq i \leq n-8$, $a_i$ and $a_{i+1}$ have a unique common neighbor in $H$, which is a vertex $w_j$ for some $j$.

Since $a_i$ and $c_i$ are adjacent twins in $G$ for each $i$, we have a projection $\pi$ from $G$ to $H$ with $\pi(v) = v$ for $v \in V(H)$ and $\pi(c_i) = a_i$ for $0 \leq i \leq n-7$. If $\pi'$ is the restriction of $\pi$ to $V(G')$, then $\pi'$ also gives a projection from $G'$ to $H$. We will use Lemma 4.2 to examine the cardinality of cutsets in $G$ or $G'$ by looking at weighted minimal cutsets in $H$. Let $\omega$ be a weighting of $V(H)$ so that each vertex $a_i$ in $A$ receives weight 2 (because it will represent both $a_i$ and $c_i$ in $G$) and each vertex of $J'$ receives weight 1.

Lemma 4.4. $H$ has no minimal cutset $T$ with $\omega(T) \leq 5$. The only minimal cutsets $T$ with $\omega(T) \leq 6$ are $N_H(a_0)$, $N_H(a_{n-7})$, and $N_H(a_i)$, $2 \leq i \leq n-9$. Thus, $H$ has no minimal cutset $T$ containing $a_0$ or $a_{n-7}$ with $\omega(T) \leq 6$.

Proof. Suppose $T$ is a minimal cutset of $H$ with $\omega(T) \leq 6$. Since $H$ is a near-triangulation we may use Lemma 1.5 to analyse the minimal cutsets of $T$.

First suppose that $|T \cap A| = 0$, i.e., $T \subseteq V(J')$. Since the only vertices of $J'$ on the boundary of $H$ are $w_0$ and $w_n$, which are adjacent, $T$ cannot induce a path in $H$, so $T$ induces a separating cycle in $H$ which is also a cycle in $J'$. Since the cycle is separating there is a vertex inside it which must be a vertex of $J'$. Then $\omega(T) = |T| \geq 7$ by Observation 4.1, a contradiction.
Now suppose that \(|T \cap A| = 1\), so that \(|T - A| \leq 4\); let \(T \cap A = \{a_i\}\).

Suppose \(T\) induces a chordless separating cycle \(Z\). No 3-cycle incident with any \(a_i\) is separating, so \(|T| \geq 4\) and \(T\) must use nonadjacent \(x, y \in N_H(a_i) \cap V(J')\). Since \(Z\) is chordless it must avoid the other neighbors of \(a_i\). The only vertices of \(A\) with two nonadjacent neighbors in \(V(J')\) are \(a_0, a_1, a_{n-8}\) and \(a_{n-7}\). However, it is easy to check the pairs of nonadjacent neighbors \(x, y\) of each such \(a_i\), and see that there is no vertex \(z \notin N_H(a_i)\) with \(xz, yz \in E(H)\), and no pair of vertices \(x', y' \notin N_H(a_i)\) with \(xx', yy', x'y' \in E(H)\).

Thus, \(T\) induces a path. The internal vertices of \(V(J')\), which must be \(w_0\) or \(w_n\). Suppose it is \(w_0\). Then \(a_0, w_n \notin T\). Then \(T\) gives a path from \(w_0\) to \(a_i\) in \(H - \{a_0, w_n\}\), using at most four vertices of \(J'\) (including \(w_0\)). There is no such path from \(w_0\) to \(a_{n-7}\) or \(a_{n-8}\), and if \(2 \leq i \leq n - 9\) then the path would have to reach \(\Pi_5\) before reaching \(a_i\) and hence would have to use at least six vertices of \(J'\). Therefore \(i = 1\), and the only path that will work is \(w_0w_1w_2w_3a_1\), giving \(T = N_H(a_0)\). Similarly, we get \(T = N_H(a_{n-7})\) if the end of the path induced by \(T\) is \(w_n\).

Now suppose that \(|T \cap A| = 2\), so that \(|T - A| \leq 2\); let \(T \cap A = \{a_i, a_j\}, i < j\). Note that every path between \(a_i\) and \(a_j\) contains at least one vertex of \(A\) or at least two vertices of \(V(J')\) unless \(j = i + 1\) and the path goes through the unique common neighbor of \(a_i\) and \(a_{i+1}\) described in Observation 4.3.

Suppose \(T\) induces a cycle \(Z\). If \(j \neq i + 1\) then the internal vertices of both of the \(a_i - a_j\) paths in \(Z\) have weight at least two, by the observation above, so \(\omega(T) \geq 8\), a contradiction. Thus \(j = i + 1\). Now \(|T| \geq 4\) because there are no separating 3-cycles using two vertices of \(A\). The common neighbor \(w_j\) of \(a_i\) and \(a_{i+1}\) cannot be a vertex of \(Z\), because at least one of the edges \(a_iw_j, a_{i+1}w_j\) would not be an edge of \(Z\) and so would be a chord of \(Z\). So \(Z\) uses some \(x \in N_H(a_i) - \{w_j\}\) and some \(y \in N_H(a_{i+1}) - \{w_j\}\). But it can be seen that no such \(x\) and \(y\) are ever adjacent in \(H\), so \(T\) contains at least one additional vertex. But then \(\omega(T) \geq 7\), a contradiction.

Thus, \(T\) induces a path and so \(j \geq i + 2\). Since \(a_i\) and \(a_j\) with \(j \geq i + 2\) have no common neighbor not in \(A\), \(T\) must induce a path of the form \(a_ixya_j\) where \(x, y\) are vertices of \(J'\), so \(xy\) is an edge of \(J'\). The only edges of \(J'\) with ends adjacent to \(a_i\) and \(a_j\) with \(j \geq i + 2\) are edges \(w_{i+4}w_{i+5}\) when \(j = i + 2\) and \(1 \leq i \leq n - 10\), giving \(T = N_H(a_i)\) with \(i' = i + 1\), so that \(2 \leq i' \leq n - 9\).

Finally, suppose that \(|T \cap A| \geq 3\), so that \(|T - A| \leq 0\), i.e., \(T \subseteq A\). Then by Lemma 1.5 since \(T\) contains at least three vertices of the boundary of \(H\), it must induce a chordless cycle. But the set \(A\) induces only a path in \(H\), so this cannot happen.

No minimal cutset that we have found with \(\omega(T) = 6\) contains \(a_0\) or \(a_{n-7}\).

Lemma 4.4 is sufficient, with Lemma 4.2, to show that \(G\) is 6-connected, since \(|\pi^{-1}(S)| = \omega(S)\) for all \(S \subseteq V(H)\). However, we really wish to show that \(G'\) is 6-connected. To do this we examine the cutsets of \(H\) containing \(a_0\) and \(a_{n-7}\) more closely.

**Lemma 4.5.** There is no minimal cutset \(T\) in \(H\) that contains \(\{a_0, a_{n-7}\}\) and has \(\omega(T) \leq 7\).
Proof. From Lemma 4.4 there is no such \( T \) with \( \omega(T) \leq 6 \), so we may suppose that \( T \) is a minimal cutset containing \( \{a_0, a_{n-7}\} \) and with \( \omega(T) = 7 \).

If \( T \) induces a cycle, then, as observed in the proof of Lemma 4.4, the internal vertices of each of the \( a_0-a_{n-7} \)-paths in the cycle have weight at least two, so \( \omega(T) \geq 8 \), a contradiction.

Thus \( T \) induces a path whose internal vertices are internal vertices of \( H \). Thus \( T \) has the form \( a_0xzya_{n-7} \), where \( x \in \{w_1, w_2, w_3\} \) and \( y \in \{w_{n-3}, w_{n-2}, w_{n-1}\} \). But no such \( x \) has a common neighbor \( z \) with any such \( y \), so this cannot happen.

Proof of second part of Theorem 1.4. Let \( S \) be a minimal cutset in \( G' \), then by Lemma 4.2 \( S = (\pi')^{-1}(T) \) for some minimal cutset of \( H \). Let \( t = |T \cap \{a_0, a_{n-7}\}| \), then \( |S| = \omega(T) - t \).

If \( t = 0 \) then \( \omega(T) \geq 6 \) by Lemma 4.4, if \( t = 1 \) then \( \omega(T) \geq 7 \) by the last sentence of the statement of Lemma 4.4, and if \( t = 2 \) then \( \omega(T) \geq 8 \) by Lemma 4.5. Therefore, \( |S| \geq 6 \) and hence \( G' \) is 6-connected. Thus, we have exhibited a 6-connected graph \( G' \) that is not \( P_4 \)-linked. \( \square \)

References

Lemma 4.2. Suppose we have a projection \( \pi \) of \( G \) onto \( H \). Then \( S \) is a minimal cutset of \( G \) if and only if \( S = \pi^{-1}(T) \) for some minimal cutset \( T \) of \( H \).

Proof. Extend \( \pi \) to a map \( \hat{\pi} : V(G) \cup E(G) \to V(H) \cup E(H) \) by \( \hat{\pi}(v) = \pi(v) \in V(H) \) for \( v \in V(G) \), and for \( uv \in E(G) \), \( \hat{\pi}(uv) = \pi(u)\pi(v) \in E(H) \) if \( \pi(u) \neq \pi(v) \), and \( \hat{\pi}(uv) = \pi(u) \in V(H) \) if \( \pi(u) = \pi(v) \). Since \( \hat{\pi} \) applies to the vertices and edges of \( G \) we can extend it in a natural way to all subgraphs of \( G \). Subgraphs of \( H \) can be lifted via \( \hat{\pi}^{-1} \) to subgraphs of \( G \). For \( v \in V(H) \), \( \hat{\pi}^{-1}(v) \) is a nonempty clique induced by \( \pi^{-1}(v) \), a set of adjacent twins of \( v \) in \( G \). For \( uv \in E(H) \), \( \hat{\pi}^{-1}(uv) \) is a nonempty complete bipartite graph between the vertices of \( \pi^{-1}(u) \) and \( \pi^{-1}(v) \).

Now we establish a lemma and make two observations.

Lemma A.1. Suppose \( a, b \) are adjacent twins in \( G \) and \( S \) is a minimal cutset of \( G \). Then \( a \in S \iff b \in S \).

Proof. Suppose \( a \in S \). Since \( S \) is minimal, \( a \) has neighbors \( v \) and \( w \) which are in different components of \( G - S \). If \( v = b \) then \( v \) is adjacent to \( w \), contradicting \( v \) and \( w \) being in different components. So \( v \neq b \) and similarly \( w \neq b \). Now both \( v \) and \( w \) are neighbors of \( b \), so if \( b \notin S \) then \( v \) and \( w \) are in the same component of \( G - S \), a contradiction. Thus \( b \in S \). The reverse implication follows by symmetry. \( \square \)

Observation A.2. Since any path in \( G \) projects to a (possibly shorter) path in \( H \), and any path in \( H \) lifts to a connected subgraph in \( G \), a subgraph \( K \) of \( H \) is connected if and only if \( \hat{\pi}^{-1}(K) \) is connected.

Observation A.3. For any \( T \subseteq V(H) \), \( \hat{\pi}^{-1}(H - T) = G - \pi^{-1}(T) \). Therefore, by Observation A.2, \( T \) is a cutset of \( H \) if and only if \( \pi^{-1}(T) \) is a cutset of \( G \).

Suppose \( S \) is a minimal cutset of \( G \). By Lemma A.1, whenever \( \pi(u) = \pi(v) \) then \( u \) and \( v \) are both in or both not in \( S \), so \( S = \pi^{-1}(T) \) for some \( T \subseteq V(H) \). By Observation A.3, \( T \)
is a cutset of \( H \). If a proper subset \( T' \) of \( T \) is also a cutset of \( H \), then by Observation A.3 \( \pi^{-1}(T') \) would be a proper subset of \( S \) that is also a cutset of \( G \), a contradiction, so \( T \) is minimal.

Now suppose \( T \) is a minimal cutset in \( H \). From Observation A.3, \( S = \pi^{-1}(T) \) is a cutset in \( G \). Suppose \( S \) is not minimal, so there exists a proper subset \( S' \) of \( S \) that is a minimal cutset. From above \( S' = \pi^{-1}(T') \) for some minimal cutset \( T' \). But \( T' \) is a proper subset of \( T \), contradicting minimality of \( T \). Therefore \( S \) is minimal.

This concludes the proof of Lemma 4.2. □

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