LINKAGE FOR THE DIAMOND AND THE PATH WITH FOUR VERTICES

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Abstract. Given graphs G and H, we say G is *H-linked* if for every injective mapping $\ell : V(H) \to V(G)$ we can find a subgraph H' of G that is a subdivision of H, with $\ell(v)$ being the vertex of H' corresponding to each vertex v of H. In this paper we prove two results on H-linkage for 4-vertex graphs H. Goddard showed that 4-connected planar triangulations are 4-ordered, or in other words C_4-linked. We strengthen this by showing that 4-connected planar triangulations are $(K_4 - e)$-linked. X. Yu characterized certain graphs related to P_4-linkage. We use his characterization to show that every 7-connected graph is P_4-linked, and to construct 6-connected graphs that are not P_4-linked.

1. Introduction

A graph is *k-linked* if for any k pairs of vertices $\{u_i, v_i\}$, $1 \le i \le k$, there is a *k-linkage*, namely k internally disjoint paths $\Pi_1, \Pi_2, \ldots, \Pi_k$ such that Π_i joins u_i and v_i. Graph linkage is a very important tool in studying graph minors.

If G and H are graphs, then an *H-subdivision in G* is a subgraph H' of G isomorphic to a subdivision of H. There is an associated map $\ell : V(H) \to V(G)$, where $\ell(v)$ (called a *branch vertex*) is the vertex of H' corresponding to each vertex v of H. We say H' is *consistent with ℓ*. We say G is *H-linked* if for every injection $\ell : V(H) \to V(G)$ there is a consistent H-subdivision.

Properties related to H-subdivisions were first studied by Jung [6] in the 1970s. This idea was recently re-introduced by Kostochka and Yu [9], and independently by Ferrara, Gould, Tansey, and Whalen [2]. Special cases of H-linkage include being k-linked (kK_2-linked), k-connected ($K_{1,k}$-linked, or $(K_2 \cup (k-1)K_1)$-linked), and k-ordered (C_k-linked). Sufficient degree conditions for a graph to be H-linked were extensively studied in [2, 5, 8, 9, 10, 11]. In [13], implications among linkage properties in graphs were studied.

The study of $f(k)$, the minimum t such that t-connected graphs are k-linked, has a long history. After a series of papers by Jung [6], Larman and Mani [12], Mader [14], and Robertson and Seymour [16], the first linear upper bound for f, namely, $f(k) \le 22k$ was proved by

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Bollobás and Thomason [1]. This was improved by Kawarabayashi, Kostochka and Yu [7] to $f(k) \leq 12k$, and Thomas and Wollan [19] showed that $f(k) \leq 10k$.

Jung [6] proved that $f(2) = 6$ and showed that every 4-connected non-planar graph is 2-linked. Later, Seymour [17] and Thomassen [18] independently characterized all non-2-linked graphs. Thomas and Wollan [20] showed that $f(3) \leq 10$, but this bound is not known to be best possible.

In this paper we examine linkage for two small graphs. Let $K_4 - e$ be the graph obtained from K_4 by removing one edge, which can also be described as $K_{1,1,2}$, and is sometimes referred to as the *diamond*. It is clear that a K_4-linked graph is $(K_4 - e)$-linked, a $(K_4 - e)$-linked graph is C_4-linked, a C_4-linked graph is P_4-linked, and a P_4-linked graph is 2-linked. There are examples showing that none of these implications can be reversed. We will investigate $(K_4 - e)$-linkage for planar graphs, and P_4-linkage in general.

Planarity can provide barriers to linkage properties. For example, a 2-connected planar graph with a face of degree 4 or more is not 2-linked (and hence not H-linked for $H = P_4$, C_4, $K_4 - e$ or K_4): there is no 2-linkage for vertices u_1, u_2, v_1, v_2 in order around the face. So, what positive results can be given for H-linkage in planar graphs? One (difficult) approach is to characterize structures that prevent H-linkage; X. Yu [21] did this for K_4-linkage in 4-connected planar graphs. Another approach is to restrict ourselves to graphs without obvious barriers to H-linkage: in particular, to avoid faces of degree 4 or more we may consider triangulations. Goddard [4] showed that 4-connected planar triangulations are C_4-linked. A linkage property somewhat different from the ones we examine in this paper was investigated by Mori [15], who showed that 4-connected planar triangulations are $(3,3)$-*linked*: for all disjoint 3-subsets S_1 and S_2 of vertices, there are vertex-disjoint connected subgraphs H_1 and H_2 with $S_1 \subseteq V(H_1)$ and $S_2 \subseteq V(H_2)$.

Here we strengthen Goddard's result as follows.

Theorem 1.1. *Any 4-connected planar triangulation is $(K_4 - e)$-linked.*

The proof of Theorem 1.1 occupies most of Section 2. We cannot replace '4-connected' here by '3-connected,' '$(K_4 - e)$-linked' by 'K_4-linked,' or 'planar triangulation' by 'planar graph' (even if we increase the connectivity to 5): details are given at the end of Section 2. However, if we just increase the connectivity, then it is known that every 5-connected planar triangulation is K_4-linked. This follows from X. Yu's results; see [21, Cor. 4.3].

Motivated by trying to extend the results of [21] from 4-connected planar graphs to more general settings, X. Yu characterized a family of graphs called obstructions [22, 23, 24]. Let G be a graph, $\{a, b, c\} \subseteq V(G)$, and $\{a', b', c'\} \subseteq V(G)$, such that $\{a, b, c\} \neq \{a', b', c'\}$. $(G, \{a, c\}, \{a', c'\}, (b, b'))$ is an *obstruction* if for every set of three vertex disjoint paths from $\{a, b, c\}$ to $\{a', b', c'\}$ in G, one path is from b to b'.

The problem of characterizing obstructions was posed by Robertson and Seymour (see [22, p. 90]). In [24], Yu stated a characterization of obstructions, and investigated their connectivity.

Theorem 1.2 (X. Yu [24])**.** *Let $(G, \{a, c\}, \{a', c'\}, (b, b'))$ be an obstruction. If $\{a, c\} \neq \{a', c'\}$ and $b \neq b'$, then G is at most 7-connected.* □

In the same paper, Yu constructed a class of obstructions which he claimed were 7-connected. Upon studying these graphs, however, we found that each is only 6-connected. In fact, we will show that there are essentially no 7-connected obstructions.

Theorem 1.3. *Let $(G, \{a, c\}, \{a', c'\}, (b, b'))$ be an obstruction. If $\{a, c\} \neq \{a', c'\}$ and $b \neq b'$, then G is at most 6-connected.*

(Any graph G with a, b, c, a', b', c' chosen so that $\{a, c\} = \{a', c'\}$ or $b = b'$ is an obstruction, so there is no upper bound on the connectivity of these trivial types of obstructions.)

Seymour (see [24, p. 245]) has pointed out a connection between obstructions and the existence of P_4-subdivisions in a graph, which we will state in Section 3. Using this and Theorem 1.3, it follows that every 7-connected graph is P_4-linked. On the other hand, this connection also allows us to modify Yu's construction from [24] to construct instances of 6-connected graphs where a specific P_4-subdivision does not exist.

Theorem 1.4. *Every 7-connected graph is P_4-linked, but there are 6-connected graphs that are not P_4-linked.*

We prove Theorem 1.3 and the first part of Theorem 1.4 in Section 3. The second part of Theorem 1.4 is proved in Section 4. Theorem 1.4 is the first determination of the exact connectivity required for H-linkage for any H with three or more edges, other than for the family of graphs where H-linkage is equivalent to k-connectivity (e.g., $H = K_2 \cup (k-1)K_1$ or $H = K_{1,k}$).

Most notation, definitions and supporting lemmas will be given in the sections where they are needed. However, the following are used in more than one section so we give them here.

We use $K_n(v_1, v_2, \ldots, v_n)$ to denote a labeled complete graph with distinct vertices v_1, v_2, …, v_n. A *near-triangulation* is a plane graph in which all faces are triangles, except perhaps the outer face, which is a cycle. Cutsets in near-triangulations are characterized as follows.

Lemma 1.5. *In a near-triangulation, a minimal cutset induces either a chordless separating cycle, or a chordless path whose ends are vertices on the boundary not joined by an edge of the boundary and which contains no other vertex of the boundary.*

Proof. Add an extra vertex adjacent to all vertices of the outer face to obtain a triangulation, and use the well-known fact that a minimal cutset in a triangulation induces a chordless separating cycle. □

2. $(K_4 - e)$-subdivisions in 4-connected planar triangulations

In this section, we prove Theorem 1.1. Our proof follows the same approach used in [4]. We first introduce some notation and a basic result.

Let f be an edge of graph G. Denote by $G \cdot f$ the graph resulting from G by contracting the edge f to a single vertex and deleting any loops and multiple edges thus formed. More generally, suppose $X \subseteq V(G)$. Then denote by $G \cdot X$ the graph obtained by identifying X to a single vertex and deleting all loops and multiple edges thus formed.

When dealing with a planar triangulation G, we assume it has a fixed embedding in the plane. Given a separating cycle D in G, we may refer to the vertices on the inside or the outside or 'on one side' of D; these terms never include vertices of D itself.

Lemma 2.1 (Goddard [4, Claim 1 and proof of Claim 4])**.** *Suppose G is a 4-connected planar triangulation and $f \in E(G)$. Then*

(i) *$G \cdot f$ is a planar triangulation.*

(ii) *If $G \cdot f$ is not 4-connected then f is in a chordless separating 4-cycle.*

(iii) *If X is the set of vertices on one side of a separating 4-cycle in G, then $G \cdot X$ is also a 4-connected planar triangulation.* □

Proof of Theorem 1.1. By way of contradiction, assume that G is a counterexample with fewest vertices. Represent $K_4 - e$ as $K_4(a_1, a_2, a_3, a_4) - a_2a_4$. We suppose that $B = \{b_1, b_2, b_3, b_4\} \subseteq V(G)$ is such that there is no subdivision of $K_4 - e$ in G with branch vertices b_1, b_2, b_3, b_4, each b_i corresponding to $a_i \in V(K_4 - e)$. When we mention a $(K_4 - e)$-subdivision 'with branch set B' in a graph related to G, we assume this correspondence between B and $V(K_4 - e)$. When a different branch set is used we indicate how this correspondence is modified. An edge of G is *free* if it is not incident with any vertex of B. Two vertices b_i, b_j of B are *pre-adjacent* if $a_ia_j \in E(K_4 - e)$.

Claim 2.2. *Every free edge of G is in a separating 4-cycle.*

Proof. Suppose f is free. If $G \cdot f$ is 4-connected then by Lemma 2.1(i) and minimality of G it has a $(K_4 - e)$-subdivision H' with branch set B. Since the vertex obtained by contracting f is not a branch vertex of H', we can easily modify H' to obtain a desired $(K_4 - e)$-subdivision H in G. Therefore, $G \cdot f$ is not 4-connected and Lemma 2.1(ii) applies. □

Claim 2.3. *Every vertex not in B has degree at least 5.*

Proof. Assume that $u \notin B$ has degree at most 4. Since G is 4-connected, u has degree exactly 4.

We now show that for each $v \in N(u)$, uv is in a separating 4-cycle. Suppose this does not hold for some v. Then $v \in B$ by Claim 2.2, and $G \cdot uv$ is 4-connected by Lemma 2.1(ii). If we contract uv to a vertex x, then there is a $(K_4 - e)$-subdivision H' in $G \cdot uv$ with branch set $B - \{v\} \cup \{x\}$, x playing the role of v. Note that all neighbors of x except one, say v', are neighbors of v. Hence, replacing xv' in H' with vuv' if necessary, we obtain a desired $(K_4 - e)$-subdivision in G, a contradiction. Hence, uv is in a separating 4-cycle.

Suppose the neighbors of u are v_1, v_2, v_3, v_4 in order. The separating 4-cycle for uv_1 must contain v_3 and use another vertex y. The separating 4-cycle for uv_2 and uv_4 must use y as well, by planarity. So G is an octahedron $K_{2,2,2}$, which is K_4-linked, a contradiction. □

Claim 2.4. *Let D be a separating 4-cycle.*

(i) *There is at least one vertex of B on each side of D.*

(ii) *If there is exactly one vertex of B on one side of D, then it is the only vertex of G on that side.*

(iii) *If there are two vertices of B on each side of D, then b_1 and b_3 are on one side of D, and b_2 and b_4 are on the other.*

Proof. By symmetry we may suppose that there are at most two vertices of B inside D. Denote by X the set of all vertices of G inside D. By Lemma 2.1(iii), $G \cdot X$ is a 4-connected planar triangulation.

(i) Suppose that $X \cap B = \emptyset$. Then $|X| \geq 2$, for otherwise the vertex in X has degree four, contradicting Claim 2.3. By the minimality of G, $G \cdot X$ contains a $(K_4 - e)$-subdivision with branch set B which we can then extend to a $(K_4 - e)$-subdivision in G.

(ii) Suppose that $X \cap B = \{x\}$ and $|X| \geq 2$. Contract X into a single vertex x'. Then $G \cdot X$ contains a $(K_4 - e)$-subdivision with branch set $B - \{x\} \cup \{x'\}$, with x' playing the role of x. This can be transformed into a $(K_4 - e)$-subdivision in G.

(iii) If there are two vertices of B on each side of D but (iii) does not hold, then we may assume without loss of generality that $X \cap B = \{b_1, b_2\}$.

Since G is 4-connected, there are four internally disjoint paths Π_1, Π_2, Π_3, Π_4 from b_1 to the four vertices in D. We claim that we may assume that one of these paths contains b_2. For suppose not. Then we may assume that b_2 lies interior to the region bounded by, say, Π_1, Π_2 and an edge of D. There are four paths from b_2 to the boundary of this region, and at least two of these paths must terminate on either Π_1 or Π_2, say on Π_1. Then path Π_1 may be replaced by a new path Π_1' which contains b_2. So, we now assume that b_2 is on Π_1. Let u be the end of Π_1 on D.

Contract X to a single vertex x', and let $B' = B - \{b_1, b_2\} \cup \{x', u\}$, with x' and u playing the roles of b_1 and b_2 respectively. By minimality of G, $G \cdot X$ has a $(K_4 - e)$-subdivision H' with branch set B'. We may assume that $x'u \in E(H')$, and we can transform H' into the required $(K_4 - e)$-subdivision in G using Π_1 and two of Π_2, Π_3, Π_4. $\square$

Let Q be the set of vertices outside of B with degree exactly five. Since G is 4-connected and only the four vertices of B can have degree four, $|Q| \geq 4$ by Euler's formula.

Claim 2.5. *For each vertex $u \in Q$,*

(i) *exactly two neighbors, say x and y, of u are in B;*

(ii) *x and y are not successive neighbors of u in the planar embedding;*

(iii) *x and y are pre-adjacent; and*

(iv) *neither ux nor uy is in a separating 4-cycle.*

Proof. Suppose that $u \in Q$ has neighbors v_1, v_2, v_3, v_4, v_5 in cyclic order.

Suppose that three neighbors of u are in B, and let b be the fourth vertex of B. First assume that three successive neighbors of u, say v_1, v_2, v_3, lie in B. There are four internally disjoint paths from b to $\{v_1, v_2, v_3, u\}$. But then v_1v_2, v_2v_3, v_1uv_3 and the paths from b to v_1, v_2, v_3 give a K_4-subdivision with branch set B, a contradiction. Therefore, no three successive neighbors of u lie in B, so we may assume that $v_1, v_3, v_5 \in B$ and $v_2, v_4 \notin B$. There are four internally disjoint paths from b to $\{v_1, v_3, v_5, u\}$. At least one of v_2 or v_4, say v_2, is used by the path to u. Then v_1v_5, $v_1v_2v_3$, v_3uv_5 and the paths from b to v_1, v_3, v_5 give a K_4-subdivision with B as the branch set, a contradiction. Therefore, at most two neighbors of u are in B.

We now claim that if each of uv_{i-1}, uv_i, uv_{i+1} (subscripts interpreted modulo 5) is in a separating 4-cycle, then at least one of v_{i-1} or v_{i+1} is in B. Without loss of generality suppose that uv_1, uv_2, uv_3 are in separating 4-cycles. The separating 4-cycle for uv_2 must include v_4 or v_5, say v_4, and another vertex v. The separating 4-cycle for uv_3 must also include v, by planarity. But then, since G is 4-connected, v_3 has degree four, and hence $v_3 \in B$ by Claim 2.3, as claimed.

Thus, if three successive neighbors of u are not in B, then the edge from u to each of these neighbors is in a separating 4-cycle by Claim 2.2, and the preceding paragraph yields a contradiction. Hence at most two successive neighbors of u are not in B, and (i) and (ii) follow. Without loss of generality suppose that $v_1, v_3 \in B$ and $v_2, v_4, v_5 \notin B$. If either uv_1 or uv_3 is in a separating 4-cycle then the preceding paragraph again yields a contradiction, so (iv) holds.

If v_1 and v_3 are not pre-adjacent, then upon contracting uv_3 to x', there is a $(K_4 - e)$-subdivision H' in $G \cdot uv_3$ with branch set $B - \{v_3\} \cup \{x'\}$, with x' playing the role of v_3. Since $x'v_1$ is not in H' (because v_1 and x' are not pre-adjacent), we can replace $x'v_5$ by v_3uv_5 and every other edge $x'v$ by v_3v, as necessary, to obtain the desired $(K_4 - e)$-subdivision in G, a contradiction. Hence (iii) holds. □

Claim 2.6. *Two vertices in B have at most one common neighbor in Q.*

Proof. Suppose that two vertices in B, say b and b', have two common neighbors u and v in Q. By Claim 2.5(iii), b and b' are pre-adjacent. Suppose $(bub'v)$ is a separating 4-cycle. Then by Claim 2.4(i) there is exactly one vertex of B on each side of $(bub'v)$. Hence by Claim 2.4(ii) $|V(G)| = 6$ and $\deg u = 4$, a contradiction. So $(bub'v)$ is not a separating 4-cycle, and thus either $uv \in E(G)$ or $bb' \in E(G)$. If $bb' \in E(G)$, vertex u contradicts Claim 2.5(ii). Hence $uv \in E(G)$.

Suppose that in clockwise order the neighbors of u are v, b, u_1, u_2, b', and the neighbors of v are u, b', v_2, v_1, b. Since G is 4-connected, $u_1, u_2 \neq v_1, v_2$, so G contains the cycle $(bu_1u_2b'v_2v_1)$. See Figure 1. By Claim 2.5(i), $u_1, u_2, v_1, v_2 \notin B$. If G contains u_1v_2 then $(ub'v_2u_1)$ is a separating 4-cycle, contradicting Claim 2.5(iv) for ub'. So u_1v_2 and (similarly) u_2v_1 are not edges.

Now uu_1 is a free edge, so by Claim 2.2 there is a separating 4-cycle C containing uu_1. By 4-connectivity, C contains neither ub or uu_2. By Claim 2.5(iv), C does not contain ub' and hence must contain uv. By Claim 2.5(iv), C does not contain vb or vb', and since

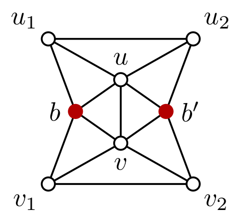

FIGURE 1. Situation in Claim 2.6

$u_1v_2 \notin E(G)$, C does not contain vv_2, so C contains vv_1, and $u_1v_1 \in E(G)$. Similarly, $u_2v_2 \in E(G)$. Since $\deg u_1 \geq 5$, $D = (u_1u_2v_2v_1)$ is a separating 4-cycle. Without loss of generality we may assume that u, v, b, b' are inside D. By Claim 2.4(iii) applied to D, $\{b, b'\} = \{b_1, b_3\}$, and b_2 and b_4 are outside D.

Now suppose one of u_1, u_2, v_1, v_2, say u_1, is in Q. Then by Claim 2.5(i), (ii) and (iv) the neighbors of u_1 in clockwise order are, without loss of generality, u, b, v_1, b_2, u_2. There are four internally disjoint paths from b_4 to D, and the path to u_1 must use b_2 as it is the only neighbor of u_1 outside D. Then the paths from b_4 to v_1 and v_2 joined to v_1b and v_2b' respectively, and the paths b_2u_1b, b_2u_2b' and $buvb'$, give a desired $(K_4 - e)$-subdivision in G, a contradiction.

Therefore none of u_1, u_2, v_1, v_2 is in Q. Since $|Q| \geq 4$, there is $u' \in Q$ outside D. By Claim 2.5(i) u' is adjacent to both b_2 and b_4, but this contradicts Claim 2.5(iii). □

Now consider the subgraph J of G induced by edges of the form ub, $u \in Q$ and $b \in B$. By Claim 2.5(i), each vertex of Q has exactly two neighbors in B, say b_i and b_j with $i < j$, and by Claim 2.6 no other vertex of Q is adjacent to both b_i and b_j. Therefore we can denote this vertex of Q unambiguously as $u_{i,j}$. By Claim 2.5(iii), b_i and b_j are pre-adjacent if $u_{i,j}$ exists. Therefore $|Q| \leq 5$ and J is isomorphic to a subdivision of a $|Q|$-edge subgraph of $K_4 - e$, each edge being subdivided exactly once.

If $u_{i,j}$ exists, then $b_ib_j \notin E(G)$, for otherwise $(b_iu_{i,j}b_j)$ would be a separating triangle by Claim 2.5(ii). If $u_{i,j}$ does not exist, b_ib_j may or may not be an edge.

If $|Q| = 5$, J is the subdivision of $K_4 - e$ that we require. So suppose, then, that $|Q| = 4$. Since G is a plane triangulation, by Claim 2.3 and Euler's formula, it follows that $\deg b_i \leq 4$ for each $b_i \in B$, and hence, since G is 4-connected, all vertices in B have degree exactly four. Modulo automorphisms of $K_4 - e$ there are only two four-edge subgraphs of $K_4 - e$, namely $(K_4 - e) - a_3a_4$ and $(K_4 - e) - a_1a_3$, giving two cases for us to consider.

Case 1. Suppose J is isomorphic to a subdivision of $K_4 - e - a_3a_4$, which we may assume is embedded in the plane as shown at left in Figure 2. To simplify notation we let $u_1 = u_{2,3}$, and $u_i = u_{1,i}$ for $2 \leq i \leq 4$. Since $u_{1,2}, u_{1,3}, u_{1,4}$ all exist, $b_1b_2, b_1b_3, b_1b_4 \notin E(G)$. It suffices to find a b_3b_4-path that is internally disjoint from $B \cup Q$.

Suppose first that u_4 and u_3 are not successive neighbors of b_1. Then the neighbors of b_1 in clockwise order are u_2, u_4, v, u_3, where $v \notin B \cup Q$. Thus, $u_2u_3, u_2u_4, vu_3, vu_4 \in E(G)$. Since

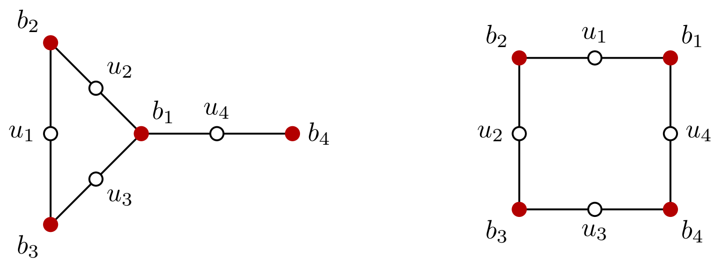

FIGURE 2. The two cases

$u_2b_4, u_4b_2 \notin E(G)$, the fifth neighbor of u_4 is between u_2 and b_4 in clockwise order around u_4, so $vb_4 \in E(G)$. Similarly, $vb_3 \in E(G)$. Thus, b_3vb_4 is the required path.

Now suppose that u_4 and u_3 are successive neighbors of b_1, so that $u_3u_4 \in E(G)$. By Claim 2.5 and since u_4 is not adjacent to b_3, the neighbors of u_3 in clockwise order are b_1, u_4, s, b_3, t, so that $su_4, su_3, sb_3 \in E(G)$. By Claim 2.5(i) for u_3, $s \notin B$. Now by Claim 2.5(ii) for u_4, $sb_4 \in E(G)$. Since $u_{3,4}$ does not exist and $sb_3, sb_4 \in E(G)$, $s \notin Q$. Thus, b_3sb_4 is the required path.

Case 2. Suppose J is isomorphic to a subdivision of $K_4 - e - a_1a_3$, which we may assume is embedded in the plane as shown at right in Figure 2. To simplify notation we let $u_i = u_{i,i+1}$ for $1 \leq i \leq 3$, and $u_4 = u_{1,4}$. It suffices to find a b_1b_3-path that is internally disjoint from $B \cup Q$.

Let H_1 and H_2 be the subgraphs of G consisting of J and everything inside or outside of J, respectively. Then H_i has a b_1b_3-path internally disjoint from $B \cup Q = V(J)$ unless there is a minimal cutset $S \subseteq V(J)$ separating b_1 from b_3 in H_i. Since H_i is a near-triangulation, by Lemma 1.5 such a cutset S must consist of two vertices of J joined by an edge, which can only be u_1u_4, b_2b_4, or u_2u_3. Since G is a counterexample, each H_i must contain one of these edges.

Suppose $b_2b_4 \in E(G)$; without loss of generality assume that $b_2b_4 \in E(H_1)$. Then one of u_1u_4, u_2u_3, say u_1u_4, must be an edge of H_2. Then $(b_2u_1u_4b_4)$ is a separating 4-cycle, contradicting Claim 2.5(iv) for u_1b_2 and u_4b_4. Therefore, $b_2b_4 \notin E(G)$.

Hence, without loss of generality we must have $u_1u_4 \in E(H_1)$ and $u_2u_3 \in E(H_2)$. By 4-connectivity, $(u_4u_1b_1)$ and $(u_3u_2b_3)$ are facial triangles. Let the second facial triangle on u_1u_4 be $(u_1u_4v_1)$ where $v_1 \in V(H_1)$, and let the second facial triangle on u_2u_3 be $(u_2u_3v_3)$, where $v_3 \in V(H_2)$.

Suppose that $v_1 \notin V(J)$. Then by Claim 2.5(ii) applied to u_1 and u_4, $v_1b_2, v_1b_4 \in E(H_1)$. If $v_3 \notin V(J)$, then similarly $v_3b_2, v_3b_4 \in E(H_2)$. Then $(v_1b_2v_3b_4)$ is a separating 4-cycle contradicting Claim 2.4(ii). Therefore $v_3 \in V(J)$. The only possibilities for v_3 are u_1 or u_4; without loss of generality assume that $v_3 = u_1$. Then $(u_1v_1b_4u_3)$ is a separating 4-cycle, contradicting Claim 2.5(iv) for u_3b_4.

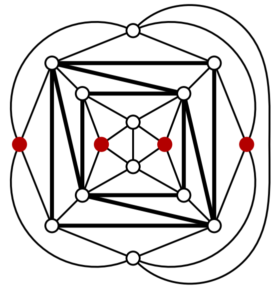

FIGURE 3. Triangulation with no K_4-subdivision

Thus, $v_1 \in V(J)$, and similarly $v_3 \in V(J)$. The only possibilities for v_1 are u_2 or u_3; without loss of generality assume that $v_1 = u_2$. Then $u_1u_2, u_2u_4 \in E(H_1)$. Since $u_2 \in Q$ has degree 5, $u_4b_3 \in E(G)$, contradicting Claim 2.5(i) for u_4.

Thus, the proof of Theorem 1.1 is complete. □

In Theorem 1.1 we cannot replace '4-connected' by '3-connected' because placing b_1 and b_3 on one side of a separating triangle and b_2 and b_4 on the other gives a situation with no $(K_4 - e)$-subdivision. We now sketch constructions (leaving the details to the reader) to show that 'triangulation' cannot be replaced by 'planar graph', and '$(K_4 - e)$-linked' cannot be replaced by 'K_4-linked.'

First, there are 4-connected, or even 5-connected, planar graphs, differing by only one edge from a triangulation, that are not 2-linked, and hence are not $(K_4 - e)$-linked. For example, take a large triangulation with 12 vertices of degree 5 and all other vertices of degree 6 (the dual of a 'fullerene'). Delete an edge between two vertices of degree 6 sufficiently far from every degree 5 vertex. The result can be shown to be 5-connected, and any planar graph on at least four vertices that is not a triangulation is not 2-linked.

Second, as Goddard [4] mentions, there are 4-connected planar triangulations that are not K_4-linked. These can be constructed using X. Yu's characterization [21, Theorem 4.2]. For example, in Figure 3 there are separating 4-cycles (shown with thicker edges) between the inner and outer pairs of solid vertices. In a K_4-subdivision for the solid vertices, the path between the inner solid vertices must use the other two inner vertices, and similarly for the outer vertices. There must be four paths between the inner and outer solid vertices, crossing the separating 4-cycles, but each such path is forced to join both left solid vertices or both right solid vertices, so we cannot get a K_4-subdivision. Infinitely many other examples may be obtained by adding more or fewer separating 4-cycles between the inner and outer solid vertices.

3. MAXIMUM CONNECTIVITY OF OBSTRUCTIONS

In this section we prove that every 7-connected obstruction is an obstruction for a trivial reason. We use this to prove that every 7-connected graph is P_4-linked.

A *separation* in a graph G is a pair of edge-disjoint subgraphs (G_1, G_2) such that $G = G_1 \cup G_2$ and each G_i contains an edge or vertex not in G_{3-i}. If $|V(G_1 \cap G_2)| = k$ then (G_1, G_2) is a *k-separation*. We say that $T \subseteq V(G)$ *separates* $S_1, S_2 \subseteq V(G)$ if S_1 and S_2 are disjoint from T and there is no path from S_1 to S_2 in $G - T$. If H is a subgraph of G, then $N_G(H)$ is $\{v \in V(G) - V(H) \mid vw \in E(G) \text{ for some } w \in V(H)\}$.

We need some definitions and results from [22, 23, 24]. The following definitions of '3-planar' and 'rung' differ slightly from those in [24], but are equivalent. We introduce the idea of the 'foundation' of a 3-planar graph for later reference, and the idea of 'R-equivalence' to describe permitted symmetries for rungs.

Definition 3.1. If G is a graph and $\mathcal{A} = \{A_1, A_2, \ldots, A_k\}$ is a (possibly empty) collection of pairwise disjoint induced subgraphs of G, then we say $(G, \mathcal{A})$ is 3*-planar* if $N_G(A_i) \cap A_j = \emptyset$ for all distinct i and j, $|N_G(A_i)| \leq 3$ for all i, and the graph G' obtained from G by replacing each A_i with a new vertex a_i adjacent to $N_G(A_i)$ is planar.

We call $G - \bigcup_{i=1}^{k} V(A_i)$ the *foundation* of G. If in addition $b_0, b_1, \ldots, b_n$ are (possibly not distinct) vertices of the foundation of G and G' can be embedded in a closed disk with $b_0, b_1, \ldots, b_n$ in cyclic order along its boundary, then we say that $(G, \mathcal{A}, b_0, b_1, \ldots, b_n)$, or just $(G, b_0, b_1, \ldots, b_n)$, is 3*-planar*.

Since a planar graph is at most 5-connected, it is clear that a 3-planar graph whose foundation has at least four vertices is at most 5-connected.

Definition 3.2. Suppose G is a graph, and $\{a, b, c\}$, $\{a', b', c'\}$ are 3-subsets of $V(G)$. Suppose $\{a, b, c\} \neq \{a', b', c'\}$, and G has no 3-separation (G_1, G_2) with $\{a, b, c\} \subseteq V(G_1)$ and $\{a', b', c'\} \subseteq V(G_2)$. Then we call $(G, (a, b, c), (a', b', c'))$ a *rung* if at least one of the following holds:

(1) $b = b'$ or $\{a, c\} = \{a', c'\}$;
(2) $a = a'$ and $(G - a, c, c', b', b)$ is 3-planar;
(3) $\{a, b, c\} \cap \{a', b', c'\} = \emptyset$ and (G, a', b', c', c, b, a) is 3-planar;
(4) $\{a, b, c\} \cap \{a', b', c'\} = \emptyset$, G has a 1-separation (G_1, G_2) such that $\{a, a', b, b'\} \subseteq V(G_1)$, $\{c, c'\} \subseteq V(G_2)$, and (G_1, a, a', b', b) is 3-planar;
(5) $\{a, b, c\} \cap \{a', b', c'\} = \emptyset$, (G, a, a', b', b) is 3-planar, and G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{z, b\}$, $\{a, a', b, b'\} \subseteq V(G_1)$, $\{c, c'\} \subseteq V(G_2)$, and (G_2, c, c', z, b) is 3-planar;
(6) $\{a, b, c\} \cap \{a', b', c'\} = \emptyset$, and there are pairwise edge-disjoint subgraphs G_a, G_c, M of G such that $G = G_a \cup G_c \cup M$, $V(G_a \cap M) = \{u, w\}$, $V(G_c \cap M) = \{p, q\}$, $V(G_a \cap G_c) = \emptyset$, $\{a, a', b'\} \subseteq V(G_a)$, $\{c, c', b\} \subseteq V(G_c)$, (G_a, a, a', b', w, u) is 3-planar, and (G_c, c', c, b, p, q) is 3-planar;
(7) $\{a, b, c\} \cap \{a', b', c'\} = \emptyset$, and there are pairwise edge-disjoint subgraphs G_a, G_c, M of G such that $G = G_a \cup G_c \cup M$, $V(G_a \cap M) = \{b, b', w\}$, $V(G_c \cap M) = \{b, b', p\}$, $V(G_a \cap G_c) = \{b, b'\}$, $\{a, a', b'\} \subseteq V(G_a)$, $\{c, c', b\} \subseteq V(G_c)$, (G_a, a, a', b', w, b) is 3-planar, and (G_c, c', c, b, p, b') is 3-planar.

A structure $(G,(a,b,c),(a',b',c'))$ is said to be *R-equivalent* to itself and to the structures $(G,(a',b',c'),(a,b,c))$, $(G,(c,b,a),(c',b',a'))$ and $(G,(c',b',a'),(c,b,a))$. Anything R-equivalent to a rung is also considered a rung.

If $(G,(a,b,c),(a',b',c'))$ is a rung, then it is not hard to show that $(G,\{a,c\},\{a',c'\},(b,b'))$ is an obstruction [22, Prop. 4.2]. Rungs are 'basic' obstructions which form the building blocks of general obstructions.

Definition 3.3 ([24])**.** Let L be a graph and let $R_1,\dots,R_m$, $m \geq 1$, be edge disjoint subgraphs of L such that

(i) $(R_i,(x_{i-1},v_{i-1},y_{i-1}),(x_i,v_i,y_i))$ is a rung for $1 \leq i \leq m$,

(ii) $V(R_i \cap R_j) = \{x_i,v_i,y_i\} \cap \{x_{j-1},v_{j-1},y_{j-1}\}$ for $1 \leq i < j \leq m$,

(iii) for any $0 \leq i < j \leq m$, if $x_i = x_j$ then $x_k = x_i$ for all $i \leq k \leq j$, if $v_i = v_j$ then $v_k = v_i$ for all $i \leq k \leq j$, and if $y_i = y_j$ then $y_k = y_i$ for all $i \leq k \leq j$.

(iv) $L = (\bigcup_{i=1}^m R_i) + S$, where S consists of edges of L with both endvertices in some $\{x_i,v_i,y_i\}$, $0 \leq i \leq m$.

Then we call $(L,(x_0,v_0,y_0),(x_m,v_m,y_m))$ a *ladder with rungs* $(R_i,(x_{i-1},v_{i-1},y_{i-1}),(x_i,v_i,y_i))$, $i = 1,2,\dots,m$, or simply a *ladder along* $v_0v_1\dots v_m$. Note that [24] has the inequalities in (iii) and (iv) beginning at 1, not 0, but 0 is correct.

Informally, a ladder is obtained from a sequence of rungs by identifying the (a',b',c') vertices of each rung with the (a,b,c) vertices of the next rung. We can also add edges ab, ac, bc, $a'b'$, $a'c'$, $b'c'$ inside any rung. Note that anything R-equivalent to a ladder is also a ladder.

Theorem 3.4 (Yu [24, Theorem 1.3])**.** *Let G be a graph, and let $S = \{a,b,c\}$ and $S' = \{a',b',c'\}$ be 3-subsets of $V(G)$ with $S \neq S'$. Assume that for every $T \subseteq V(G)$ with $|T| \leq 3$, every component of $G - T$ contains a vertex of $S \cup S'$. Then $(G,\{a,c\},\{a',c'\},(b,b'))$ is an obstruction if and only if one of the following statements hold:*

(1) *G has a k-separation (G_1,G_2) with $k \leq 2$, $S \subseteq V(G_1)$, and $S' \subseteq V(G_2)$.*

(2) *G is the edge-disjoint union of a ladder $(L,(a,b,c),(a',b',c'))$ or $(L,(a,b,c),(c',b',a'))$ along $v_0v_1\dots v_m$ (where $b = v_0$ and $b' = v_m$) and a (possibly edgeless) graph J such that $V(J \cap L) = \{w_0,w_1,\dots,w_n\}$ and $(J,w_0,w_1,\dots,w_n)$ is 3-planar, where $w_0,w_1,\dots,w_n$ is the sequence $v_0,v_1,\dots,v_m$ with repetitions removed.* $\square$

In [24], case (2) above is separated into two cases, the situation where J is edgeless being treated as a separate case. Also, in [24] the case of the ladder being $(L,(a,b,c),(c',b',a'))$ is not mentioned, but this needs to be present for the 'only if' part of the theorem to be correct. This is because there is symmetry between a' and c' in the definition of an obstruction, but not in the definition of a rung $(G,(a,b,c),(a',b',c'))$ or ladder $(L,(a,b,c),(a',b',c'))$.

To take the symmetry between a' and c' into account we define $(G,(a,b,c),(a',b',c'))$ to be *X-equivalent* to itself, to $(G,(a,b,c),(c',b',a'))$, and to everything R-equivalent to either of these. Then an *X-rung* is a structure X-equivalent to one of (1)–(7) of Definition 3.2. For every X-rung $(G,(a,b,c),(a',b',c'))$, $(G,\{a,c\},\{a',c'\},(b,b'))$ is an obstruction. We define an *X-ladder* by replacing 'rung' by 'X-rung' in the definition of a ladder. Clearly $(L,(a,b,c),(a',b',c'))$ is an X-ladder if and only if either it is a ladder or $(L,(a,b,c),(c',b',a'))$ is a ladder. Therefore, the condition on L in case (2) of Theorem 3.4 may be restated as '$(L,(a,b,c),(a',b',c'))$ is an X-ladder'.

The following lemma will be used to handle one-rung ladders.

Lemma 3.5. *If $(G,(a,b,c),(a',b',c'))$ is an X-rung not covered by case (1) of Definition 3.2 and $G' = G \cup K_3(a,b,c) \cup K_3(a',b',c') \cup \{bb'\}$, then G' is at most 6-connected.*

Proof. Without loss of generality, assume G is exactly as in one of cases (2)–(7) of Definition 3.2. We examine each case individually.
(2) We can add bc, $b'c'$ and bb' to $G - a$ without destroying 3-planarity, so $G' - a = (G - a) \cup \{bc, b'c', bb'\}$ is 3-planar. Since G is not covered by case (1), b, b', c and c' are distinct, so $G' - a$ has at least four foundation vertices and hence is at most 5-connected. Therefore, G' is at most 6-connected.
(3) We can add the edges of $K_3(a,b,c)$ and $K_3(a',b',c')$ to G without destroying 3-planarity. Thus, $G' - bb'$ is 3-planar with at least six foundation vertices and hence at most 5-connected. Therefore, G' is at most 6-connected.
(4) If $c \neq z$ then $\{a,b,c',z\}$ separates a' and c in G', and if $c = z$ then $\{a',b',c = z\}$ separates a and c' in G'.
(5) $\{a,b,c',z\}$ separates $\{a',b'\} - \{z\}$ and c in G'.
(6) $\{a,b,c',p,q\}$ separates a' and c in G'.
(7) $\{a,b,b',c',p\}$ separates a' and c in G'. □

Now we restate our upper bound on the connectivity of obstructions.

Theorem 1.3. *Let $(G,\{a,c\},\{a',c'\},(b,b'))$ be an obstruction. If $\{a,c\} \neq \{a',c'\}$ and $b \neq b'$, then G is at most 6-connected.*

Proof. Suppose that G is 7-connected. We may assume we have a maximal obstruction, meaning that $(G+e,\{a,c\},\{a',c'\},(b,b'))$ is not an obstruction for any $e \in E(\overline{G})$. Since G is 7-connected, case (1) of Theorem 3.4 does not hold, so case (2) holds. Thus G is the union of an X-ladder $(L,(a,b,c),(a',b',c'))$ and a 3-planar graph $(J,w_0,w_1,\ldots,w_n)$. Let the X-rungs of L be $(R_i,(x_{i-1},v_{i-1},y_{i-1}),(x_i,v_i,y_i))$, $1 \leq i \leq m$, so that $w_0,w_1,\ldots,w_n$ is the sequence $v_0,v_1,\ldots v_m$ with repetitions removed.

Define the *I-rung* R'_i, $1 \leq i \leq m$, to be the subgraph of G induced by $V(R_i)$. By maximality of G and Definition 3.3 (iv), R'_i contains $K_3(x_{i-1},v_{i-1},y_{i-1})$ and $K_3(x_i,v_i,y_i)$ as subgraphs. Also, if $v_{i-1} \neq v_i$ then we may always add the edge $v_{i-1}v_i$ to J, so by maximality $v_{i-1}v_i \in E(R'_i)$ if $v_{i-1} \neq v_i$. In general, R'_1, R'_2, …, R'_m are not pairwise edge-disjoint.

Claim 3.6. *We have $m \geq 2$, and we may assume that every I-rung R'_i, $1 \leq i \leq m$, has one of the following forms:*

(i) $K_4(x_{i-1} = x_i, v_{i-1} = v_i, y_{i-1}, y_i)$, *or* $K_4(x_{i-1}, x_i, v_{i-1} = v_i, y_{i-1} = y_i)$, *or*
(ii) $K_4(x_{i-1} = x_i, v_{i-1}, v_i, y_{i-1} = y_i)$, *or*
(iii) $K_5(x_{i-1}, x_i, v_{i-1} = v_i, y_{i-1}, y_i)$.

Proof. If $m = 1$ (there is only one X-rung) then $V(J) - \{v_0, v_1\}$ is empty, otherwise $\{v_0, v_1\}$ is a cutset. Hence, $G = R'_1$. Since $b \neq b'$ and $\{a, c\} \neq \{a', c'\}$, case (1) of Definition 3.2 does not apply, so R'_1 is at most 6-connected by Lemma 3.5. Therefore, $m \geq 2$.

To satisfy the claim about each I-rung, we may need to swap the labels of x_i and y_i for some values of i, which we can do since swapping x_i and y_i in the X-rung $(R_i, (x_{i-1}, v_{i-1}, y_{i-1}), (x_i, v_i, y_i))$ still yields an X-rung. By processing the X-rungs in order from R_1 up to R_m, each such swap can be done without altering the effects of previous swaps. We describe how to process each X-rung R_i. We use 'R_i is an obstruction' as shorthand for '$(R_i, \{x_{i-1}, y_{i-1}\}, \{x_i, y_i\}, (v_{i-1}, v_i))$ is an obstruction'.

Write $T = \{x_{i-1}, v_{i-1}, y_{i-1}\}$ and $T' = \{x_i, v_i, y_i\}$, which may not be disjoint. If $V(R_i) \neq T \cup T'$, then $T \cup T'$ is a cutset in G of order at most 6, a contradiction. Hence $V(R_i) = T \cup T'$ and $|V(R_i)| \leq 6$. By Definition 3.2, R_i has no 3-separation (G_1, G_2) with $T \subseteq V(G_1)$ and $T' \subseteq V(G_2)$. Therefore, R_i has no 3-cutset S which separates $T - S$ from $T' - S$.

Suppose that $|V(R_i)| = 6$, so that T and T' are disjoint. Let Q be the bipartite graph with vertex set $T \cup T'$ and containing all edges of R_i with one end in T and the other end in T'.

If $u \in T$ has degree 0 or 1 in Q, then u is nonadjacent to two vertices s, t of T'. Then $T \cup T' - \{u, s, t\}$ is a 3-cutset of R_i separating u from s and t, which is not allowed. So every vertex in T, and similarly in T', has degree at least 2 in Q.

Therefore, if Q' is the complement of Q in the $K_{3,3}$ with bipartition (T, T'), it has maximum degree at most 1 and so $Q' \subseteq 3K_2$. Thus, Q contains $K_{3,3} - 3K_2 = C_6$ as a subgraph. But then Q has two disjoint perfect matchings, and in at least one of them v_{i-1} is not matched to v_i, contradicting the fact that R_i is an obstruction.

Now suppose that $|V(R_i)| = 5$. If there is a vertex s of $T - T'$ not adjacent to some vertex t of $T' - T$, then $T \cup T' - \{s, t\}$ is a 3-cutset of R_i separating s and t, which is not allowed. Therefore, R'_i is a K_5. Since $|V(R_i)| = 5$, some vertex in T is equal to some vertex in T'. Since R_i is an obstruction, the equality must be $v_{i-1} = v_i$ and we have (iii) above.

Finally, suppose that $|V(R_i)| = 4$. Not every 4-vertex obstruction is an X-rung; for example, an edgeless graph on vertices x_{i-1}, $x_i = v_{i-1}$, v_i, $y_{i-1} = y_i$ is an obstruction, but not an X-rung. So we must refer to the details of Definition 3.2. $(R_i, (x_{i-1}, v_{i-1}, y_{i-1}), (x_i, v_i, y_i))$ must be X-equivalent to case (1) of Definition 3.2 (or to a degenerate version of case (2), with $b = b'$ or $c = c'$, but then we have case (1) again). Adding any edge between T and T' that is not already present does not violate the conditions of case (1), so by maximality R'_i must be a K_4. By swapping the labels of x_i and y_i if necessary, we can guarantee that R'_i has the form of (i) above (if $b = b'$ in (1)) or (ii) above (if $\{a, c\} = \{a', c'\}$ in (1)). $\square$

Henceforth we assume all I-rungs are as in Claim 3.6. All X-rungs are therefore covered by case (1) of Definition 3.2.

If $v_{i-1} \neq v_i$ then G always contains the edge $v_{i-1}v_i$, which may be in J or in the X-rung R_i. Definition 3.2(1) allows us to assume that it is always in R_i. Thus, L contains the path $W = w_0w_1 \dots w_n$. Also, if $x_i \neq x_{i-1}$ then R_i always contains the edge x_ix_{i-1}. Thus, L contains the path $A = a_0a_1 \dots a_p$, where $a_0, a_1, \dots, a_p$ is the sequence $x_0, x_1, \dots, x_m$ with repetitions removed. Similarly, L contains the path $C = c_0c_1 \dots c_q$, where $c_0, c_1, \dots c_q$ is the sequence $y_0, y_1, \dots, y_m$ with repetitions removed. We have $L = \bigcup_{i=1}^m R'_i$, and $V(L) = V(W) \cup V(A) \cup V(C)$.

From Claim 3.6 and the definition of an X-ladder we observe the following.

(A) Suppose $P, Q \in \{W, A, C\}$ are distinct. If $v \in V(P)$ then the neighbors of v on Q are consecutive.

(B) Every edge of W, A or C belongs to a unique R_i and hence to a unique R'_i.

(C) Suppose $(P, Q) = (A, W), (C, W), (W, A)$ or (W, C). If $uv \in E(P)$, then u and v have exactly one common neighbor on Q, namely the unique vertex of Q in the unique I-rung containing uv. (This does *not* hold with $(P, Q) = (A, C)$ or (C, A).)

Let J' denote the subgraph of G induced by $V(J)$. Then $J' = J \cup W$.

Claim 3.7. *J' may be regarded as a plane graph with outer cycle $(w_0w_1 \dots w_n)$, where $n \geq 2$.*

Proof. Suppose that $n \leq 1$. Then $V(J) = V(W)$, otherwise $V(W)$ is a cutset of order $n + 1 \leq 2$. The only possible edge of J is w_0w_1 when $n = 1$, but this is an edge of L. Therefore J is edgeless. Hence, since $m \geq 2$, $T = \{x_1, v_1, y_1\}$ is a cutset of order 3 separating $V(R_1) - T$ from $V(R_2) - T$ in G. So $n \geq 2$.

Since G is 7-connected, the 3-planar graph J cannot contain any subgraphs H disjoint from L with $|N_J(H)| \leq 3$. So J is in fact planar, and can be embedded in a disk with $w_0, w_1, \dots, w_n$ in cyclic order around the boundary. The edge w_0w_n can always be added to this embedding if it is not already present, so by maximality of G, $w_0w_n \in E(J)$. Moreover, by maximality w_0w_n must be an edge of the outer facial walk, otherwise we can move it into the outer face and add another edge to J. We can also add the edges of W in the outer face to obtain a planar embedding of J' with $(w_0w_1 \dots w_n)$ as outer cycle. $\square$

Claim 3.8. *We may assume that there are i and k so that either (i) w_ia_{k-1}, w_ia_k, $w_ia_{k+1} \in E(L)$, or (ii) w_ia_{k-1}, w_ia_k, $w_{i+1}a_k$, $w_{i+1}a_{k+1} \in E(L)$.*

Proof. For $P = A$ or C, let $n_P(w_i) = |N_L(w_i) \cap V(P)|$. If $n_A(w_i) \geq 3$ or $n_C(w_i) \geq 3$ for some w_i then we may assume (i) by (A). So, assume that $n_A(w_i), n_C(w_i) \leq 2$, so that $d_{L-E(W)}(w_i) \leq 4$, for all w_i. Since G is 7-connected, $d_G(w_i) \geq 7$. Therefore, $d_{J'}(w_i) = d_G(w_i) - d_{L-E(W)}(w_i) \geq 3$ for all w_i.

Let n_3, n_4 and n_5^+ be the number of vertices in W with degree 3, 4, and at least 5 in J', respectively, and let n_{int} be the number of internal vertices of J'. Add a new vertex z and join it to all vertices of the outer cycle $(w_0w_1\dots w_n)$ of J', giving a new planar graph J''. Then $d_{J''}(z) = n_3 + n_4 + n_5^+$; $d_{J''}(w_i) = d_{J'}(w_i) + 1$ for all w_i; and $d_{J''}(v) = d'_J(v) \geq 7$ for all internal vertices v of J', because G is 7-connected.

Since J'' is planar, $6|V(J'')| - 12 \geq 2|E(J'')|$. This implies that $6(n_{\text{int}} + n_3 + n_4 + n_5^+ + 1) - 12 \geq 7n_{\text{int}} + 4n_3 + 5n_4 + 6n_5^+ + (n_3 + n_4 + n_5^+)$, from which $n_3 \geq n_{\text{int}} + n_5^+ + 6 \geq n_5^+ + 6$. Therefore, on W there are two vertices with degree 3 in J' such that no vertices with degree at least 5 in J' lie between them.

Thus, there is $ww' \in E(W)$ with $d_{J'}(w) \leq 3$ and $d_{J'}(w') \leq 4$. Since G is 7-connected, $d_{L-E(W)}(w) \geq 4$ and $d_{L-E(W)}(w') \geq 3$. Since $n_A(w), n_C(w), n_A(w'), n_C(w') \leq 2$, we must have $n_A(w) = n_C(w) = 2$, and without loss of generality $n_A(w') = 2$ and $n_C(w') \geq 1$. Then (ii) follows from (C) and (A). □

If Claim 3.8(i) applies, let $j = i$, and if Claim 3.8(ii) applies, let $j = i + 1$. In either case, w_ia_{k-1}, w_ia_k, w_ja_k, $w_ja_{k+1} \in E(L)$. By (B) there are unique s and t with $a_{k-1}a_k \in E(R_s)$ and $a_ka_{k+1} \in E(R_t)$; clearly $s \leq t$. Then $a_{k-1} = x_{s-1}$, $a_k = x_s = x_{t-1}$, and $a_{k+1} = x_t$. By (C), w_i is the unique vertex of W in R_s, so $w_i = v_{s-1} = v_s$, and w_j is the unique vertex of W in R_t, so $w_j = v_{t-1} = v_t$. Let $y_{s-1} = c_g$ and $y_t = c_h$; clearly $g \leq h$.

Define

$$U_1 = \{a_\alpha \,|\, \alpha < k-1\} \cup \{w_\beta \,|\, \beta < i\} \cup \{c_\gamma \,|\, \gamma < g\},$$
$$U_2 = \{a_k\} \cup \{c_\gamma \,|\, g < \gamma < h\}, \text{ and}$$
$$U_3 = \{a_\alpha \,|\, \alpha > k+1\} \cup \{w_\beta \,|\, \beta > j\} \cup \{c_\gamma \,|\, \gamma > h\}.$$

Because $x_{s-1} = a_{k-1}$, $v_{s-1} = w_i$, and $y_{s-1} = c_g$, if R_r (or R'_r) contains a vertex of U_1 then $r \leq s-1$, and if R_r contains a vertex of U_2 then $r \geq s$. Because $x_t = a_{k+1}$, $v_t = w_j$ and $y_t = c_h$, if R_r contains a vertex of U_2 then $r \leq t$, and if R_r contains a vertex of U_3 then $r \geq t+1$. Therefore there is no R'_r containing both a vertex of U_2 and a vertex of $U_1 \cup U_3$, so there are no edges from U_2 to $U_1 \cup U_3$. There are also no edges from U_2 to $U_0 = V(J) - V(W)$ because U_2 contains no vertex of J. Therefore, $S = V(G) - (U_0 \cup U_1 \cup U_2 \cup U_3) = \{a_{k-1}, a_{k+1}, w_i, w_j, c_g, c_h\}$ separates U_2 from $U_0 \cup U_1 \cup U_3$, which is nonempty because $|V(W)| = n + 1 \geq 3$ and $U_2 \cup S$ contains at most two vertices of W. Therefore, G is not 7-connected, a contradiction which concludes the proof of Theorem 1.3. □

Now we can show that 7-connected graphs are P_4-linked.

Suppose u and v are vertices of G. If $N_G(u) = N_G(v)$, then we say u and v are *nonadjacent twins* of each other in G, and if $N_G[u] = N_G[v]$ we say they are *adjacent twins*. If we add a new vertex w to G adjacent exactly to all vertices of $N_G(v)$ or $N_G[v]$, then we say we have *made a nonadjacent or adjacent twin of* v, respectively.

The following is well known.

Observation 3.9. *Suppose G is k-connected. Let G' be obtained from G by making a nonadjacent (or adjacent) twin of a vertex of G. Then G' is also k-connected.*

Observation 3.10 (Seymour, see [24, p. 245])**.** *Let u, v, w, x be distinct vertices of a graph G.*

(i) *If $vw \in E(G)$ then G has a path through u, v, w, x in that order if and only if G has vertex-disjoint paths from u to v and from w to x.*

(ii) *If $vw \notin E(G)$, construct G' from G by making nonadjacent twins v', w' of v, w respectively. Then G has a path through u, v, w, x in that order if and only if $(G', \{w, w'\}, \{v, v'\}, (u, x))$ is not an obstruction.*

Proof of first part of Theorem 1.4. Suppose G is a 7-connected graph. To show G is P_4-linked we must show there is a path through specified vertices u, v, w, x in that order. If $vw \in E(G)$, then since every 6-connected graph is 2-linked [6], G has vertex-disjoint paths from u to v and from w to x. If $vw \notin E(G)$, make nonadjacent twins v', w' of v and w respectively to obtain a graph G'. By Observation 3.9, G' is 7-connected. By Theorem 1.3, $(G', \{w, w'\}, \{v, v'\}, (u, x))$ is not an obstruction. In either case, G has the desired path by Observation 3.10. □

4. 6-connected graphs without P_4-subdivisions

In this section we prove the second half of Theorem 1.4 by constructing a family of 6-connected graphs that are not P_4-linked. We use Seymour's Observation 3.10, first finding a 6-connected obstruction G and then deriving our example G' from G.

X. Yu [24, pp. 243-245] constructed obstructions that were claimed to be 7-connected; in fact they are only 6-connected. One can derive graphs that are not P_4-linked from these obstructions, but they are only 5-connected. We will modify X. Yu's construction to obtain our examples. Since the crucial issue here is the connectivity of the resulting graphs, we provide a detailed verification that our examples are 6-connected.

We use the terminology and notation of Section 3.

4.1. **Construction of near-triangulation J'.**

We describe a near-triangulation which will play the role of J' in our construction of G.

Let Π_0 be an edge bb'. For each i, $0 \leq i \leq 4$, construct a new path Π_{i+1} so that each vertex of Π_i is adjacent to at least four consecutive vertices on Π_{i+1}, every vertex of Π_{i+1} is adjacent to one or two vertices of Π_i, and the region between Π_i and Π_{i+1} is triangulated. For $i = 5$ we construct Π_6 in the same way, except the first and last vertex of Π_5 each has only one neighbor in Π_6, the first or last vertex of Π_6, respectively. Let J' be the union of the paths Π_1, Π_2, ..., Π_6 and all edges between them. J' is a near-triangulation. Let W^+ be its boundary cycle and write $W^+ = (w_0 w_1 w_2 \dots w_n)$ where $w_0 = b$ and $w_n = b'$. J' may be seen as the subgraph consisting of the solid vertices and thicker edges in Figure 4.

Then from every internal vertex v of J' there are at least seven paths, disjoint except at v, from v to W^+. Suppose v is on Π_i. We may take one path that uses one vertex of each of $\Pi_{i-1}, \Pi_{i-2}, \dots, \Pi_0$; two paths along Π_i from v to each end of Π_i; and four paths that use one vertex of each of $\Pi_{i+1}, \Pi_{i+2}, \dots, \Pi_6$.

Observation 4.1. *If Z is a cycle in J' with at least one vertex inside it, then $|V(Z)| \geq 7$, because the seven paths from the inside vertex to W^+ must intersect Z at distinct points.*

4.2. **Construction of obstruction G and example G'.** Given the vertices $w_0, w_1, \ldots, w_n$ of J' and new vertices $a_0, a_1, \ldots, a_{n-7}$ and $c_0, c_1, \ldots, c_{n-7}$, we form L by taking the union of the following I-rungs (all K_4's as in Claim 3.6(ii) or K_5's as in Claim 3.6(iii)):

$K_4(a_0, w_i, w_{i+1}, c_0)$, $0 \leq i \leq 2$,
$K_5(a_0, a_1, w_3, c_0, c_1)$,
$K_4(a_1, w_i, w_{i+1}, c_1)$, $3 \leq i \leq 4$,
$K_5(a_i, a_{i+1}, w_{i+4}, c_i, c_{i+1})$, $1 \leq i \leq n-9$,
$K_4(a_i, w_{i+3}, w_{i+4}, c_i)$, $2 \leq i \leq n-9$,
$K_4(a_{n-8}, w_i, w_{i+1}, c_{n-8})$, $n-5 \leq i \leq n-6$,
$K_5(a_{n-8}, a_{n-7}, w_{n-3}, c_{n-8}, c_{n-7})$, and
$K_4(a_{n-7}, w_i, w_{i+1}, c_{n-7})$, $n-3 \leq i \leq n-1$.

(So we take three K_4's, then one K_5, then two K_4's, then alternate K_5, K_4, K_5, $\ldots$, K_4, K_5, finishing with two K_4's, then one K_5, then three K_4's.) Then L satisfies the definition of a ladder, and if $G = J' \cup L$, then $(G, \{a_0, c_0\}, \{a_{n-7}, c_{n-7}\}, (w_0, w_n))$ is an obstruction by Theorem 3.4.

Notice that in G, a_i and c_i are adjacent twins for all i, $0 \leq i \leq n-7$.

Now let $G' = G - \{c_0, c_{n-7}\}$. We claim that G' has no P_4-subdivision with branch vertices w_0, a_{n-7}, a_0, w_n in that order along the P_4. By Observation 3.10, there is no P_4-subdivision if and only if when we make a nonadjacent twin a'_{n-7} of a_{n-7} and an nonadjacent twin a'_0 of a_0 we get a graph G'' such that $(G'', \{a_0, a'_0\}, \{a_{n-7}, a'_{n-7}\}, (w_0, w_n))$ is an obstruction. But relabelling a'_0 as c_0 and a'_{n-7} as c_{n-7}, we see that $G'' = G - \{a_0c_0, a_{n-7}c_{n-7}\}$ and since G is an obstruction, G'' is an obstruction. Thus, G' does not have the required P_4-subdivision and so G' is not P_4-linked.

Now we must show that G' is 6-connected.

4.3. **Projections and minimal cutsets.** To examine the connectivity of G and G', we will make use of a simpler graph H. In order to relate cutsets in H to cutsets in G and G', we need the following concepts.

Let G be a graph. By a *minimal* cutset $S \subseteq V(G)$ we mean that no proper subset of S is a cutset.

Suppose H is an induced subgraph of G and we have a map $\pi : V(G) \to V(H)$ such that if $v \in V(H)$ then $\pi(v) = v$, and if $v \notin V(H)$ then $\pi(v)$ is an adjacent twin of v. We call π a *projection* of G onto H. The essential fact we need is the following. We omit the proof, which is not difficult. The word 'minimal' is necessary here.

Lemma 4.2. *Suppose we have a projection π of G onto H. Then S is a minimal cutset of G if and only if $S = \pi^{-1}(T)$ for some minimal cutset T of H.* □

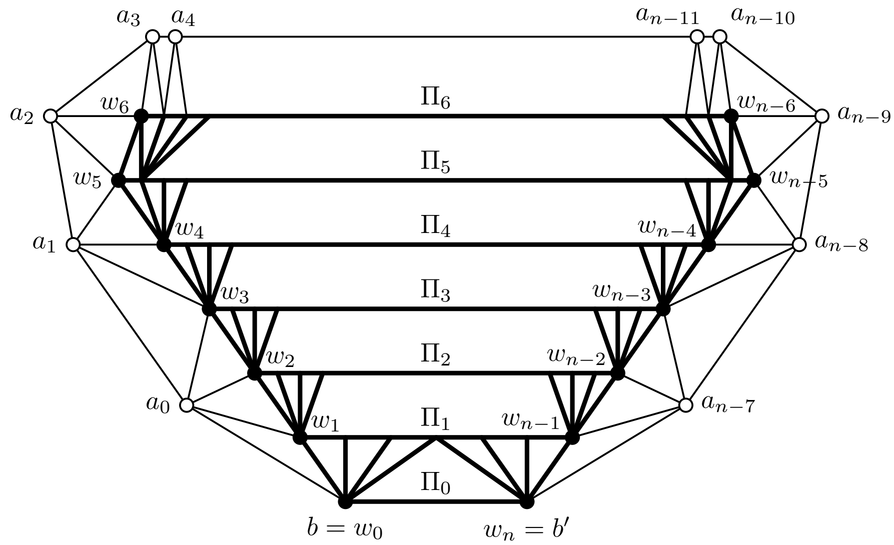

FIGURE 4. Near-triangulation H with subgraph J' (thicker edges)

4.4. **Cutsets in the projection H.** It is convenient to modify slightly our notation from Section 3, and write $A = \{a_0, a_1, \ldots, a_{n-7}\}$ and $C = \{c_0, c_1, \ldots, c_{n-7}\}$ (so these are sets of vertices, not paths). Let $H = G - C$. As shown in Figure 4, H is a near-triangulation with boundary cycle $(w_0 a_0 a_1 a_2 \ldots a_{n-7} w_n)$. Lemma 1.5 and the following observation are useful for investigating cutsets in H.

Observation 4.3. *For $0 \leq i \leq n-8$, a_i and a_{i+1} have a unique common neighbor in H, which is a vertex w_j for some j.*

Since a_i and c_i are adjacent twins in G for each i, we have a projection π from G to H with $\pi(v) = v$ for $v \in V(H)$ and $\pi(c_i) = a_i$ for $0 \leq i \leq n-7$. If π' is the restriction of π to $V(G')$, then π' also gives a projection from G' to H. We will use Lemma 4.2 to examine the cardinality of cutsets in G or G' by looking at weighted minimal cutsets in H. Let ω be a weighting of $V(H)$ so that each vertex a_i in A receives weight 2 (because it will represent both a_i and c_i in G) and each vertex of J' receives weight 1.

Lemma 4.4. *H has no minimal cutset T with $\omega(T) \leq 5$. The only minimal cutsets T with $\omega(T) \leq 6$ are $N_H(a_0)$, $N_H(a_{n-7})$, and $N_H(a_i)$, $2 \leq i \leq n-9$. Thus, H has no minimal cutset T containing a_0 or a_{n-7} with $\omega(T) \leq 6$.*

Proof. Suppose T is a minimal cutset of H with $\omega(T) \leq 6$. Since H is a near-triangulation we may use Lemma 1.5 to analyse the minimal cutsets of T.

First suppose that $|T \cap A| = 0$, i.e., $T \subseteq V(J')$. Since the only vertices of J' on the boundary of H are w_0 and w_n, which are adjacent, T cannot induce a path in H, so T induces a separating cycle in H which is also a cycle in J'. Since the cycle is separating there is a vertex inside it which must be a vertex of J'. Then $\omega(T) = |T| \geq 7$ by Observation 4.1, a contradiction.

Now suppose that $|T \cap A| = 1$, so that $|T - A| \leq 4$; let $T \cap A = \{a_i\}$.

Suppose T induces a chordless separating cycle Z. No 3-cycle incident with any a_i is separating, so $|T| \geq 4$ and T must use nonadjacent $x, y \in N_H(a_i) \cap V(J')$. Since Z is chordless it must avoid the other neighbors of a_i. The only vertices of A with two nonadjacent neighbors in $V(J')$ are a_0, a_1, a_{n-8} and a_{n-7}. However, it is easy to check the pairs of nonadjacent neighbors x, y of each such a_i, and see that there is no vertex $z \notin N_H(a_i)$ with xz, $yz \in E(H)$, and no pair of vertices $x', y' \notin N_H(a_i)$ with xx', yy', $x'y' \in E(H)$.

Thus, T induces a path. The other end must be a vertex in $V(J')$, which must be w_0 or w_n. Suppose it is w_0. Then $a_0, w_n \notin T$. Then T gives a path from w_0 to a_i in $H - \{a_0, w_n\}$, using at most four vertices of J' (including w_0). There is no such path from w_0 to a_{n-7} or a_{n-8}, and if $2 \leq i \leq n-9$ then the path would have to reach Π_5 before reaching a_i and hence would have to use at least six vertices of J'. Therefore $i = 1$, and the only path that will work is $w_0w_1w_2w_3a_1$, giving $T = N_H(a_0)$. Similarly, we get $T = N_H(a_{n-7})$ if the end of the path induced by T is w_n.

Now suppose that $|T \cap A| = 2$, so that $|T - A| \leq 2$; let $T \cap A = \{a_i, a_j\}$, $i < j$. Note that every path between a_i and a_j contains at least one vertex of A or at least two vertices of $V(J')$ unless $j = i + 1$ and the path goes through the unique common neighbor of a_i and a_{i+1} described in Observation 4.3.

Suppose T induces a cycle Z. If $j \neq i + 1$ then the internal vertices of both of the $a_i - a_j$ paths in Z have weight at least two, by the observation above, so $\omega(T) \geq 8$, a contradiction. Thus $j = i + 1$. Now $|T| \geq 4$ because there are no separating 3-cycles using two vertices of A. The common neighbor w_j of a_i and a_{i+1} cannot be a vertex of Z, because at least one of the edges a_iw_j, $a_{i+1}w_j$ would not be an edge of Z and so would be a chord of Z. So Z uses some $x \in N_H(a_i) - \{w_j\}$ and some $y \in N_H(a_{i+1}) - \{w_j\}$. But it can be seen that no such x and y are ever adjacent in H, so T contains at least one additional vertex. But then $\omega(T) \geq 7$, a contradiction.

Thus, T induces a path and so $j \geq i + 2$. Since a_i and a_j with $j \geq i + 2$ have no common neighbor not in A, T must induce a path of the form a_ixya_j where x, y are vertices of J', so xy is an edge of J'. The only edges of J' with ends adjacent to a_i and a_j with $j \geq i + 2$ are edges $w_{i+4}w_{i+5}$ when $j = i + 2$ and $1 \leq i \leq n - 10$, giving $T = N_H(a_{i'})$ with $i' = i + 1$, so that $2 \leq i' \leq n - 9$.

Finally, suppose that $|T \cap A| \geq 3$, so that $|T - A| \leq 0$, i.e., $T \subseteq A$. Then by Lemma 1.5 since T contains at least three vertices of the boundary of H, it must induce a chordless cycle. But the set A induces only a path in H, so this cannot happen.

No minimal cutset that we have found with $\omega(T) = 6$ contains a_0 or a_{n-7}. □

Lemma 4.4 is sufficient, with Lemma 4.2, to show that G is 6-connected, since $|\pi^{-1}(S)| = \omega(S)$ for all $S \subseteq V(H)$. However, we really wish to show that G' is 6-connected. To do this we examine the cutsets of H containing a_0 and a_{n-7} more closely.

Lemma 4.5. *There is no minimal cutset T in H that contains $\{a_0, a_{n-7}\}$ and has $\omega(T) \leq 7$.*

Proof. From Lemma 4.4 there is no such T with $\omega(T) \leq 6$, so we may suppose that T is a minimal cutset containing $\{a_0, a_{n-7}\}$ and with $\omega(T) = 7$.

If T induces a cycle, then, as observed in the proof of Lemma 4.4, the internal vertices of each of the a_0-a_{n-7}-paths in the cycle have weight at least two, so $\omega(T) \geq 8$, a contradiction.

Thus T induces a path whose internal vertices are internal vertices of H. Thus T has the form a_0xzya_{n-7}, where $x \in \{w_1, w_2, w_3\}$ and $y \in \{w_{n-3}, w_{n-2}, w_{n-1}\}$. But no such x has a common neighbor z with any such y, so this cannot happen. □

Proof of second part of Theorem 1.4. Let S be a minimal cutset in G', then by Lemma 4.2 $S = (\pi')^{-1}(T)$ for some minimal cutset of H. Let $t = |T \cap \{a_0, a_{n-7}\}|$, then $|S| = \omega(T) - t$. If $t = 0$ then $\omega(T) \geq 6$ by Lemma 4.4, if $t = 1$ then $\omega(T) \geq 7$ by the last sentence of the statement of Lemma 4.4, and if $t = 2$ then $\omega(T) \geq 8$ by Lemma 4.5. Therefore, $|S| \geq 6$ and hence G' is 6-connected. Thus, we have exhibited a 6-connected graph G' that is not P_4-linked. □

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Material for referees, not to be included in published paper

Appendix A. Proof of Lemma 4.2

We use the notation of Subsection 4.3.

Lemma 4.2. *Suppose we have a projection π of G onto H. Then S is a minimal cutset of G if and only if $S = \pi^{-1}(T)$ for some minimal cutset T of H.*

Proof. Extend π to a map $\widehat{\pi} : V(G) \cup E(G) \to V(H) \cup E(H)$ by $\widehat{\pi}(v) = \pi(v) \in V(H)$ for $v \in V(G)$, and for $uv \in E(G)$, $\widehat{\pi}(uv) = \pi(u)\pi(v) \in E(H)$ if $\pi(u) \neq \pi(v)$, and $\widehat{\pi}(uv) = \pi(u) \in V(H)$ if $\pi(u) = \pi(v)$. Since $\widehat{\pi}$ applies to the vertices and edges of G we can extend it in a natural way to all subgraphs of G. Subgraphs of H can be lifted via $\widehat{\pi}^{-1}$ to subgraphs of G. For $v \in V(H)$, $\widehat{\pi}^{-1}(v)$ is a nonempty clique induced by $\pi^{-1}(v)$, a set of adjacent twins of v in G. For $uv \in E(H)$, $\widehat{\pi}^{-1}(uv)$ is a nonempty complete bipartite graph between the vertices of $\pi^{-1}(u)$ and $\pi^{-1}(v)$.

Now we establish a lemma and make two observations.

Lemma A.1. *Suppose a, b are adjacent twins in G and S is a minimal cutset of G. Then $a \in S \iff b \in S$.*

Proof. Suppose $a \in S$. Since S is minimal, a has neighbors v and w which are in different components of $G - S$. If $v = b$ then v is adjacent to w, contradicting v and w being in different components. So $v \neq b$ and similarly $w \neq b$. Now both v and w are neighbors of b, so if $b \notin S$ then v and w are in the same component of $G - S$, a contradiction. Thus $b \in S$. The reverse implication follows by symmetry. □

Observation A.2. *Since any path in G projects to a (possibly shorter) path in H, and any path in H lifts to a connected subgraph in G, a subgraph K of H is connected if and only if $\widehat{\pi}^{-1}(K)$ is connected.*

Observation A.3. *For any $T \subseteq V(H)$, $\widehat{\pi}^{-1}(H - T) = G - \pi^{-1}(T)$. Therefore, by Observation A.2, T is a cutset of H if and only if $\pi^{-1}(T)$ is a cutset of G.*

Suppose S is a minimal cutset of G. By Lemma A.1, whenever $\pi(u) = \pi(v)$ then u and v are both in or both not in S, so $S = \pi^{-1}(T)$ for some $T \subseteq V(H)$. By Observation A.3, T

is a cutset of H. If a proper subset T' of T is also a cutset of H, then by Observation A.3 $\pi^{-1}(T')$ would be a proper subset of S that is also a cutset of G, a contradiction, so T is minimal.

Now suppose T is a minimal cutset in H. From Observation A.3, $S = \pi^{-1}(T)$ is a cutset in G. Suppose S is not minimal, so there exists a proper subset S' of S that is a minimal cutset. From above $S' = \pi^{-1}(T')$ for some minimal cutset T'. But T' is a proper subset of T, contradicting minimality of T. Therefore S is minimal.

This concludes the proof of Lemma 4.2. □

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