The Chvátal-Erdős Condition for Prism-Hamiltonicity

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Abstract

The prism over a graph G is the cartesian product $G \square K_2$. It is known that the property of having a Hamiltonian prism (prism-Hamiltonicity) is stronger than that of having a 2-walk (spanning closed walk using every vertex at most twice) and weaker than that of having a Hamilton path. For a graph G, it is known that $\alpha(G) \leq 2\kappa(G)$, where $\alpha(G)$ is the independence number and $\kappa(G)$ is the connectivity, imples existence of a 2-walk in G, and the bound is sharp. West asked for a bound on $\alpha(G)$ in terms of $\kappa(G)$ guaranteeing prism-Hamiltonicity. In this paper we answer this question and prove that $\alpha(G) \leq 2\kappa(G)$ implies the stronger condition, prism-Hamiltonicity of G.

1 Introduction

In this paper, we consider only simple, finite, and undirected graphs. Let G be a graph. By $\kappa(G)$ and $\alpha(G)$ we mean the connectivity and independence number of G, respectively. The *prism* over a graph G is the cartesian product $G \square K_2$. If $G \square K_2$ is Hamiltonian, we say that G is *prism-Hamiltonian*. A *t-tree* of G is a spanning tree of G with maximum degree at most t. A *t-walk* of G is a spanning closed walk that visits every vertex at most t times.

Kaiser et al. [8] showed that the property of having a Hamiltonian prism is stronger than that of having a 2-walk and weaker than that of having a Hamilton path, i.e.,

$$\text{Hamilton path} \Rightarrow \text{prism-Hamiltonian} \Rightarrow \text{2-walk},$$

and there are examples in [8] showing that none of these implications can be reversed. It is of interest to determine whether or not a graph fits in between the properties of having a Hamilton path and having a 2-walk. In particular, which graphs are prism-Hamiltonian even though they may not have a Hamilton path?

Chvátal and Erdős proved the following theorem.

Theorem 1.1 (Chvátal and Erdős [5]). *Let G be a graph with at least three vertices. If $\alpha(G) \le \kappa(G)$, then G is Hamiltonian.*

Suppose G is a graph with $|V(G)| \ge 2$ and $\alpha(G) \le \kappa(G) + 1$. By adding a new vertex v adjacent to all vertices of G, we construct G' which satisfies the hypothesis of Theorem 1.1. Hence G' is Hamiltonian, so that $G = G' - v$ has a Hamilton path. This also holds if $|V(G)| = 1$, giving the following corollary.

Corollary 1.2. *Let G be a graph. If $\alpha(G) \le \kappa(G) + 1$, then G has a Hamilton path.*

Moreover, it is known that $\alpha(G) \le 2\kappa(G)$ implies existence of a 2-walk for G [7].

Problem 1.3 (West [9]). Given k, what is the largest value of a such that if G is a graph with $\kappa(G) = k$ and $\alpha(G) = a$, then the prism over G is Hamiltonian?

For $a > k$, the complete bipartite graph $K_{k,a}$ is k-connected and has independence number a. When $a > 2k$, the prism over $K_{k,a}$ is not Hamiltonian, since deleting the $2k$ vertices of degree $a + 1$ leaves a components. Hence the answer to this problem is at most $2k$.

The following theorem is our answer to this question.

Theorem 1.4. *Let G be a graph with at least two vertices. If $\alpha(G) \le 2\kappa(G)$, then G is prism-Hamiltonian.*

This theorem shows that the Chvátal-Erdős condition sufficient for being prism-Hamiltonian is the same as for the weaker property of having a 2-walk.

Here we list the results that we need in our proofs.

Theorem 1.5 (Bondy and Lovász [2]). *Let S be a set of k vertices in a k-connected graph G, where $k \ge 3$. Then there exists an even cycle in G through every vertex of S.*

Theorem 1.6 (Jackson and Wormald [7]). *The existence of a t-tree implies the existence of a t-walk, and the existence of a t-walk implies the existence of a $(t + 1)$-tree.*

Theorem 1.7 (Batagelj and Pisanski [1]). *Let T be a tree with maximum degree $\Delta(T) \ge 2$. Then $T \Box C_t$ is Hamiltonian if and only if $\Delta(T) \le t$.*

A *spanning cactus* in a graph G is a spanning connected subgraph of maximum degree 3 that is the union of vertex-disjoint cycles $C_1, C_2, \ldots, C_s$ and vertex-disjoint paths $P_1, P_2, \ldots, P_t$ such that the graph has no cycles other than $C_1, C_2, \ldots, C_s$. The cactus is said to be *even* if all of its cycles are even, that is, if the cactus is a bipartite graph.

Lemma 1.8 (Čada et al. [3]). *If G contains a spanning even cactus, then G is prism-Hamiltonian.*

2 Proof of Theorem 1.4

Recall that Theorem 1.4 states that if G is a connected graph then $\alpha(G) \leq 2\kappa(G)$ implies prism-Hamiltonicity of G.

Let $P = a_1a_2 \dots a_n$ be a path with n vertices. By $P[a_i, a_j]$ and $P(a_i, a_j)$ for $1 \leq i < j \leq n$ we mean the paths $a_i a_{i+1} \dots a_j$ and $a_{i+1}a_{i+2} \dots a_{j-1}$, respectively. Similarly, we can define $P[a_i, a_j)$ and $P(a_i, a_j]$.

Proof of Theorem 1.4. If $\alpha(G) \leq \kappa(G) + 1$ then, by Corollary 1.2, G has a Hamilton path, and hence is prism-Hamiltonian by Lemma 1.8. So we may assume that $\kappa(G) + 2 \leq \alpha(G) \leq 2\kappa(G)$. Thus, $\kappa(G) \geq 2$.

We break the proof into two cases, $\kappa(G) = 2$ and $\kappa(G) \geq 3$. Somewhat surprisingly, we have to work harder in the first case; in the second case Bondy and Lovász's Theorem 1.5 does a significant amount of the work.

Case 1. Suppose that $\kappa(G) = 2$. Since $\kappa(G) + 2 = 4 \leq \alpha(G) \leq 2\kappa(G) = 4$, we have $\alpha(G) = 4$. By adding two adjacent vertices (a complete graph on two vertices, K_2) to G that are adjacent to all vertices of G, we obtain a new graph, say G'. Then $\kappa(G') = \alpha(G') = 4$. Therefore by Theorem 1.1 G' is Hamiltonian. Removing these two new vertices implies that G has a Hamilton path or two vertex-disjoint paths P_1 and P_2 that cover all vertices of G. In the former case G is prism-Hamiltonian, so we assume the latter case. Let u_1 and u_2 be the end vertices of P_1 and v_1 and let v_2 be the end vertices of P_2.

Claim 1. Each of P_1 and P_2 contains more than one vertex; otherwise, G is prism-Hamiltonian.

Proof. Suppose $u_1 = u_2 = u$. Since G is 2-connected, there are two edges from u to P_2, say ub_1 and ub_2. If b_1 or b_2 belongs to $\{v_1, v_2\}$, then G has a Hamilton path, and hence is prism-Hamiltonian. Now suppose b_1 is the neighbor of u closest to v_1 in P_2. Since G is 2-connected, there exists an edge $xy \in E(G)$ such that $x \in V(P_2[v_1, b_1))$ and $y \in V(P_2(b_1, v_2])$. One of the cycles $P_1[y, b_2] \cup b_2ub_1 \cup P_2[b_1, x] \cup xy$, $P_2[b_1, b_2] \cup b_2ub_1$ or $P_2[x, y] \cup xy$ is an even cycle and the even cycle together with remaining two path segments of P_2 form a spanning even cactus, and hence G is prism-Hamiltonian. □

Suppose $u_1 \neq u_2$ and $v_1 \neq v_2$. Since G is 2-connected, there are distinct vertices $a_1, a_2 \in V(P_1)$ and $b_1, b_2 \in V(P_2)$ such that $a_1a_2, b_1b_2 \in E(G)$. We may assume that u_1, a_1, a_2, u_2 occur in that order on P_1, and v_1, b_1, b_2, v_2 occur in that order on P_2.

Claim 2. The orders of the paths $P_1[a_1, a_2]$ and $P_2[b_1, b_2]$ have different parity; otherwise, G is prism-Hamiltonian.

Proof. Suppose the orders of the paths $P_1[a_1, a_2]$ and $P_2[b_1, b_2]$ have same parity. Then $P_1[a_1, a_2] \cup a_2b_2 \cup P_2[b_2, b_1] \cup b_1a_1$ is an even cycle. This cycle together with remaining path segments of P_1 and P_2 form a spanning even cactus, i.e., the even cycle together with $P_1 - P_1[a_1, a_2]$ and $P_2 - P_2[b_1, b_2]$. Therefore G is prism-Hamiltonian. □

Claim 3. If $P_2[x, y] \cap P_2[b_1, b_2]$ has at least one edge and $xy \in E(G) \setminus E(P_2)$ for $x, y \in V(P_2)$, then $P_2[x, y] \cup yx$ is an even cycle; otherwise, G is prism-Hamiltonian.

Proof. Suppose $P_2[x, y] \cup yx$ is an odd cycle. By Claim 2, $P_1[a_1, a_2] \cup a_2b_2 \cup P_2[b_2, b_1] \cup b_1a_1$ is an odd cycle. Then combining these two odd cycles form an even cycle which yields to a spanning even cactus. (The same statement holds for edges between two vertices of P_1.) □

Claim 4. The set $\{u_1, u_2, v_1, v_2\}$ is an independent set; otherwise, G is prism-Hamiltonian.

Proof. By contradiction suppose $\{u_1, u_2, v_1, v_2\}$ is not an independent set. If $u_iv_j \in E(G)$ for $i, j \in \{1, 2\}$ then G contains a Hamilton path and hence it is prism-Hamiltonian.

Thus, we may assume that $u_1u_2 \in E(G)$, i.e., $P_1 \cup u_1u_2$ is a cycle. By Claim 3, $P_1 \cup u_1u_2$ is an even cycle. We can assume that b_1 is the closest neighbor of a vertex of P_1 on P_2 to v_1. Then, by 2-connectedness and since b_1 is the closest vertex to v_1 adjacent to a vertex of P_1, there is an edge xy such that $x \in V(P_2[v_1, b_1))$ and $y \in V(P_2(b_1, v_2])$. Then by Claim 3, $P_2[y, x] \cup xy$ is an even cycle. Therefore $P_1 \cup u_1u_2$ and $P_2[x, y] \cup yx$ are even cycles and together with the edge a_1b_1 and remaining path segments of P_2 form a spanning even cactus. □

Claim 5. There is no edge xy with $x \in V(P_1) - \{a_1, a_2\}$ and $y \in V(P_2) - \{b_1, b_2\}$; otherwise, G is prism-Hamiltonian.

Proof. Suppose $xy \in E(G)$ for $x \in V(P_1)$ and $y \in V(P_2)$. By Claim 2, $P_1[a_1, a_2] \cup a_2b_2 \cup P_2[b_2, b_1] \cup b_1a_1$ is an odd cycle. Then one of the cycles $Z_1 = P_1[x, a_1] \cup a_1b_1 \cup P_2[b_1, y] \cup yx$ or $Z_2 = P_1[x, a_2] \cup a_2b_2 \cup P_2[b_2, y] \cup yx$ is even and together with remaining path segments of P_1 and P_2 forms a spanning even cactus. □

Claim 6. There is no edge xy such that either (i) $x \in \{u_1, u_2\}$ and $y \in V(P_2) - \{b_1, b_2\}$ or (ii) $x \in V(P_1) - \{a_1, a_2\}$ and $y \in \{v_1, v_2\}$; otherwise, G is prism-Hamiltonian.

Proof. Without loss of generality suppose that (i) holds with $x = u_1$. If $x \neq a_1$ then the result follows by Claim 5, so suppose that $x = u_1 = a_1$. The proof of Claim 5 fails when $x = a_1$ because if we need to construct a spanning even cactus from the cycle Z_1 then we would have to attach two path segments of P_1 at $x = a_1$, creating a degree 4 vertex, which is not allowed. However, since $x = u_1 = a_1$ here one of these path segments is trivial (just the single vertex u_1) so this does not create a problem now, and we may proceed as in the proof of Claim 5. □

Now we may suppose that $\{u_1, u_2, v_1, v_2\}$ is an independent set. By Claim 2, the paths $P_1[a_1, a_2]$ and $P_2[b_1, b_2]$ have different parity. Without loss of generality we can assume that $P_2[b_1, b_2]$ has an odd number of vertices, and therefore there is a vertex $x \in V(P_2(b_1, b_2))$. Since $\alpha(G) = 4$, and $S = \{u_1, u_2, v_1, v_2\}$ is an independent set, x is adjacent to some vertex in S. By Claim 6, we may assume that x is adjacent to neither u_1 nor u_2. Without loss of generality we may assume that x is adjacent to v_1. Then by Claim 3, the cycle $P_2[x, v_1] \cup v_1x$ is even. If $a_1 = u_1$ then we have a spanning even cactus using the cycle $P_2[x, v_1] \cup v_1x$ and paths $P_2[x, v_2]$ and $b_1u_1 \cup P_1$, so we may assume that $a_1 \neq u_1$. By 2-connectedness there is an edge yz such that $y \in V(P_1[u_1, a_1))$ and $z \in V(P_2) \cup V(P_1(a_1, u_2])$. So we have the following cases.

Case 1.1. If $z \in V(P_1(a_1, v_2])$, by Claim 3 we may assume that $yz \cup P_1[z, y]$ is an even cycle. Then the cycles $v_1x \cup P_2[x, v_1]$ and $yz \cup P_1[z, y]$ together with the edge a_1b_1 and remaining path segments of P_1 and P_2 form a spanning even cactus.

Case 1.2. Suppose $z \in V(P_2)$. By Claim 5 we can assume that $z = b_1$ or $z = b_2$ which lead us to the following cases.

Case 1.2.1. Suppose $z = b_2$. Then we can assume that the cycle $yb_2a_2 \cup P_1[a_2, y]$ is even; otherwise, the cycle $P_2[b_1, b_2] \cup b_2y \cup P_1[y, a_1] \cup a_1b_1$ is even and yields a spanning even cactus. Therefore the even cycles $yb_2a_2 \cup P_1[a_2, y]$ and $v_1x \cup P_2[x, v_1]$ together with the edge a_1b_1 and remaining path segments of P_1 and P_2 form a spanning even cactus.

Case 1.2.2. Suppose $z = b_1$. Then for the same reason as above we can assume that the cycle $yb_1a_1 \cup P_1[a_1, y]$ is even. Therefore there is a vertex $c \in V(P_1(y, a_1))$. We can assume that $ca_1 \in E(P_1)$. Since $\alpha(G) = 4$, and $S = \{u_1, u_2, v_1, v_2\}$ is an independent set, c is adjacent to some vertex in S. By Claim 6 we may assume that c is adjacent to neither v_1 nor v_2. If $u_2c \in E(G)$ then by Claim 3, $P_1[c, u_2] \cup u_2c$ is an even cycle and together with $P_2[v_1, x] \cup xv_1$ it yields a spanning even cactus. Hence we may assume that $u_1c \in E(G)$.

If $u_1c \notin E(P_1)$, then we may assume that $P_1[c, u_1] \cup u_1c$ is an odd cycle; otherwise, together with $P_2[v_1, x] \cup xv_1$ it yields a spanning even cactus. If $P_1[c, u_1] \cup u_1c$ is odd, then $P_1[c, a_2] \cup a_2b_2 \cup P_2[b_2, b_1] \cup b_1y \cup P_1[y, u_1] \cup u_1c$ is an even cycle and together with remaining path segments of P_1 and P_2 forms a spanning even cactus. Therefore we may assume that $u_1c \in E(P_1)$, which implies $y = u_1$. Then $P_2[v_1, x] \cup xv_1$ together with paths $b_1u_1 \cup P_1$ and $P_2[x, v_2]$ forms a spanning even cactus.

Case 2. Suppose that $k = \kappa(G) \geq 3$. Let $\alpha = \alpha(G)$ and let $t = \alpha - k \geq 2$. Let G' be the graph G together with a K_t and all edges from these new t vertices to $V(G)$. Then $\alpha(G') = \alpha(G) \leq \kappa(G') = \kappa(G) + t$, hence by Theorem 1.1 G' is Hamiltonian. By removing these t new vertices, we can cover all the vertices of G by $r \leq t$ vertex-disjoint paths, $P_1, P_2, \ldots, P_r$. Let $v_1, \ldots, v_r$ be one of the end vertex of each of these r paths. By Theorem 1.5 there is an even cycle, say C, passing through $v_1, \ldots, v_r$. Now we put a direction on each of these r paths starting from v_i, $1 \leq i \leq r$. Our goal is attaching some paths to C to form a spanning even cactus.

Suppose C intersects P_i at $w_1^i = v_i$, w_2^i, …, $w_{k_i}^i$, in that order along P_i. Let $x_{k_i}^i$ be the end of P_i other than v_i, and for $1 \leq j \leq k_i - 1$ let x_j^i be the vertex immediately before w_{j+1}^i on P_i. Then we add the paths $P_i[w_j^i, x_j^i]$, $1 \leq i \leq r$, $1 \leq j \leq k_i$, to C. This process will form a spanning even cactus. Hence, G is prism-Hamiltonian. □

3 Conclusion

It is known [7, Theorem 5.3] that $\alpha(G) \leq t\kappa(G)$ implies Hamiltonicity of $G[K_t]$ (the lexicographic product of G and K_t). As an extension of Theorems 1.1 and 1.4 we can ask whether $\alpha(G) \leq t\kappa(G)$ implies Hamiltonicity of $G \square K_t$ when $t \geq 3$. We can prove the following slightly weaker result.

Proposition 3.1. *Let G be a graph, and $t \geq 3$ an integer. If $\alpha(G) \leq (t-1)\kappa(G)$ then $G \square C_t$, and hence $G \square K_t$, is Hamiltonian.*

Proof. We know that $\alpha(G) \leq (t-1)\kappa$ implies existence of a $(t-1)$-walk in G. By Theorem 1.6 existence of a $(t-1)$-walk implies the existence of a t-tree and hence, by Theorem 1.7, Hamiltonicity of $G \square C_t$. □

We assume the reader is familiar with the idea of toughness, introduced by Chvátal [4], who conjectured that for some fixed t every t-tough graph is Hamiltonian. For $k \geq 3$ we know that $(1/(k-2))$-tough graphs have a k-tree and hence a k-walk [7, 10], and 4-tough graphs have a 2-walk [6]. Kaiser et al. [8, Conjecture 4] make the natural conjecture that for some fixed t all t-tough graphs are prism-Hamiltonian, and show that t must be at least 9/8.

While it appears very difficult to show that some constant toughness implies Hamiltonicity or even prism-Hamiltonicity, Chvátal-Erdős conditions combined with some simple observations suffice to show that $\Omega(\sqrt{n})$-tough graphs have these properties. As far as we can tell, no one has noted this before. Suppose G is a non-complete n-vertex t-tough graph; let $\alpha = \alpha(G)$ and $\kappa = \kappa(G)$. By [4, Propositions 1.3 and 1.4], $\kappa \geq 2t$ and $t \leq (n-\alpha)/\alpha$, or $n/(t+1) \geq \alpha$. Using these, we obtain the following.

Proposition 3.2. *Suppose $t > 0$, $n \geq 3$, and G is a t-tough n-vertex graph.*

(i) If $2t(t+1) \geq n$ (e.g., if $t \geq \sqrt{n/2}$), then G is Hamiltonian.

(ii) If $4t(t+1) \geq n$ (e.g., if $t \geq \sqrt{n}/2$), then G is prism-Hamiltonian.

Proof. We may assume G is non-complete. If $p \geq 0$ and $2pt(t+1) \geq n$ then $p\kappa \geq 2pt \geq n/(t+1) \geq \alpha$. Applying Theorem 1.1 when $p = 1$ and Theorem 1.4 when $p = 2$ gives the result. □

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