1. (presented by Richard Anstee)

The following conjecture, due to Alspach (1988), is well-known, but remains unsolved.

**Conjecture:** Let $G$ be a $2k$-regular simple graph decomposed into $k$ 2-factors $F_1, F_2, \ldots, F_k$. Then there is a matching $M \subseteq E(G)$ such that $|M \cap F_i| = 1$, for all $i = 1, \ldots, k$.

**Known results:** If one relaxes the condition $|M \cap F_i| = 1$ to $|M \cap F_i| \leq 1$ in the above conjecture, one can find such a matching $M$ with $|M| > k - k^{2/3}$.

A weaker conjecture is the following.

**Conjecture:** Let $G$ be a $2k$-regular simple graph. Then there is a decomposition of $G$ into $k$ 2-factors $F_1, F_2, \ldots, F_k$ and a matching $M \subseteq E(G)$ with $|M \cap F_i| = 1$, for all $i = 1, \ldots, k$.

2. (presented by Mark Ellingham)

Since this is a graph factors workshop, I will define $k$-walks and $k$-trails in a factor-based way. A $k$-trail in a graph is a connected even $[2, 2k]$-factor and a $k$-walk is a connected even weighted $[2, 2k]$-factor, where weighted means that an edge can be used multiple times.

Chvátal conjectured in 1973 that sufficiently tough graphs have a hamilton cycle. Since a hamilton cycle is a 1-walk and also a 1-trail, we can weaken Chvátal’s conjecture to ask if, for a given $k \geq 2$, sufficiently tough graphs have a $k$-walk or a $k$-trail.

**Known results:** For $k$-walks, the answer is known. Jackson and Wormald used a result of Win to show that $(1/(k - 2))$-tough graphs have a $k$-walk for $k \geq 3$, and Ellingham and Zha showed that 4-tough graphs have a 2-walk.

However, for $k$-trails nothing is known.

**Problem:** So my question is whether, for a given $k \geq 2$, sufficiently tough graphs have a $k$-trail.

Mike Jacobson commented that it may be worth asking more generally if sufficiently tough graphs have a spanning eulerian subgraph, without any restrictions on the degree, since $k$-trails may alternatively be thought of as spanning eulerian subgraphs of maximum degree at most $2k$. Now that I have thought about this, I realized that the answer to this is known. If a graph is 2-tough, it is 4-connected and hence 4-edge-connected, and
4-edge-connected graphs are known to have a spanning eulerian subgraph.

3. (presented by Ron Gould)
   Suppose graph $G$ is Hamiltonian. What minimum degree condition is sufficient to imply that $G$ contains a 2-factor with exactly two cycles?

**Known results:** $\delta(G) = 4$ fails. To see this, construct a graph by stringing together in cyclic fashion a collection of copies of $K_5 - e$'s alternating with single edges, where the single edges attach to the $K_5 - e$'s at the endvertices of the missing edges $e$.

   The best known result, due to Faudree, Gould, Jacobson, Lesniak and Saito, is that $\delta(G) = 5n/12$ suffices.

4. (presented by Mikio Kano)
   The linear arboricity $\lambda_a(G)$ of a graph $G$ is the minimum number $r$ such that $E(G)$ is decomposed into $r$ linear forests $F_1 \cup F_2 \cup \cdots \cup F_r$, where a linear forest is just a forest all components of which are paths.

**Conjecture:** (Akiyama, Exoo and Harary) The linear arboricity of any $r$-regular simple graph is $\lceil \frac{r+1}{2} \rceil$.

   This conjecture is true for some values of $r$ (e.g., for all $r \leq 6$) and has been shown to hold for almost all $r$-regular graphs.

**Problem 1:** Find a counterexample to the conjecture for some odd integer $r \geq 7$.

Let $G$ be a graph and let $f : V(G) \rightarrow \{1, 3, 5, \ldots \}$. A subgraph $H$ of $G$ is called a $(1, f)$-odd subgraph if $\deg_H(x) \in \{1, 3, \ldots, f(x)\}$ for all $x \in V(H)$. A $(1, f)$-odd subgraph is said to be **maximum** if $G$ has no $(1, f)$-odd $H'$ such that $|H'| > |H|$. Many results on matchings given in Chapter 3 of the book “Matching Theory” by Lovász and Plummer can be extended to $(1, f)$-subgraphs.

**Problem 2:** Can we extend some of the results in Chapters 4,5,7 of this book to $(1, f)$-odd subgraphs?

5. (due to Atsushi Kaneko; presented by Keiko Kotani)
   Let $G$ be a simple graph. If $G$ has a connected spanning subgraph $F$ such that
   \[
   \deg_F(v) \begin{cases} 
   \geq 2 & \text{if } v = x; \\
   = 2 & \text{if } v \neq x,
   \end{cases}
   \]
   for every vertex $x \in V(G)$, then $G$ has a Hamilton cycle.

6. (presented by Katsuhiro Ota)
   Let $H$ be a graph and $f : V(H) \rightarrow \text{integers}$. The total excess of $H$ from $f$ is defined as: $te(H, f) = \sum_{v \in V(H)} \max \{d_H(v) - f(v), 0\}$.
For a constant function \( f = c \), write \( te(H,c) \) for \( te(H,f) \).

**Problem 0:** Given \( g, f : V(G) \rightarrow \text{integers} \) and \( k \), a positive integer, find a necessary and sufficient condition for the existence of a \( (g, \infty) \)-factor \( H \) with \( te(H,f) \leq k \).

**Conjecture 1:** Let \( G \) be a 3-connected graph on a surface of Euler genus \( k \geq 3 \). Then \( G \) has a spanning tree \( T \) with \( te(T,3) \leq 2k - 5 \).

(Here the *Euler genus* is defined to be twice the ordinary genus for an orientable surface and to be the crosscap number for a non-orientable surface.)

**Known results:**

(a) If \( k \leq 2 \), then there exists a spanning tree \( T \) with \( te(T,3) = 0 \). [Gao-Richter, 1994]

(b) Conjecture 1 is true for graphs with “high” representativity. [Kawarabayashi, Nakamoto and Ota, 2003].

(c) The value \( 2k - 5 \) in the statement of the above conjecture would be best possible.

**Conjecture 2:** Let \( G \) be a 3-connected graph on a surface of Euler genus \( k \geq 2 \). Then \( G \) has a 2-connected spanning subgraph \( H \) with \( te(H,6) \leq 6k - 12 \).

**Known results:**

(a) The value \( 6k - 12 \) in the above conjecture would be best possible.

(b) If \( k \leq 2 \), then there exists a 2-connected spanning subgraph \( H \) with \( te(H,6) = 0 \). [Gao 1995]

(c) Conjecture 2 is true for graphs with “high” representativity. [Kawarabayashi, Nakamoto and Ota, 2003].

7. (presented by Mike Plummer)

**Problem:** What is the computational complexity of the following problem: “Given a graph \( G \), find the maximum value of \( k \) for which the graph \( G \) is \( k \)-extendable”?

8. (presented by Roger Yu)

A graph \( G \) on at least \( 2k + 2 \) vertices is said to be \( k \)-extendable if every matching of size \( k \) extends to (i.e., is a subset of) a perfect matching in \( G \).

A graph \( G \) is said to be maximal \( k \)-extendable if \( G \) is \( k \)-extendable, but if any missing edge \( e \) is added to \( G \), the resulting graph \( G + e \) is not \( k \)-extendable.

**Problem:** Characterize maximal \( k \)-extendable graphs for \( k \geq 2 \).

**Known results:** If either (a) \( k = 1 \), (b) \( G \) is bipartite, or (c) if \( k \) is relatively large (i.e., \( k \geq (|V(G)| + 1)/3 \)), such a characterization is known.

9. (presented by Anthony Hilton)

A list assignment to a graph \( G \) is a map \( L : V(G) \rightarrow P(C) \), where \( C \) is a set of “colours” and \( P(C) \) is the collection of all subsets of \( C \). A \( k \)-set colouring of \( G \) is a map \( \phi : V(G) \rightarrow Q(C) \), where \( Q(C) \) is the collection of all \( k \)-subsets of \( C \), such that, if \( v_1 \) and \( v_2 \) are any two adjacent vertices, then \( \phi(v_1) \cap \phi(v_2) = \emptyset \). For a \( k \)-colouring from \( L \), we require that \( \phi(v) \subseteq L(v) \), for all \( v \in V(G) \).
We say that a pair \((G, L)\), where \(L\) is a list assignment to \(G\), satisfies the \(k\)th Hall condition, if, for each subgraph \(H\) of \(G\),
\[
\sum_{\sigma \in C} \alpha(\sigma, L, H) \geq k|V(H)|,
\]
where \(\alpha(\sigma, L, H)\) is the independence number of the subgraph of \(H\) induced by the vertices of \(H\) which have \(\sigma\) in their lists.

It is easy to see that Hall’s \(k\)-condition is a necessary condition for the existence of a \(k\)-set colouring of \(G\) from \(L\). If \(k = 1\) and \(G\) is a complete graph, then Hall’s 1-condition is also sufficient (this is equivalent to Hall’s Theorem). However, in general Hall’s condition is not sufficient.

We define the \(k\)-th Hall number of \(G\), \(h^{(k)}(G)\), to be the least integer \(j\) such that if \(G\) and \(L\) satisfy the \(k\)-th Hall condition, and if \(|L(v)| \geq j\) for all \(v \in V(G)\), then \(G\) must have a \(k\)-set colouring from \(L\).

The question we have not so far been able to resolve is:

**Problem:** Does
\[
\lim_{k \to \infty} \frac{h^{(k)}(G)}{k}
\]
exist?

The answer for similar parameters such as the \(k\)-th chromatic number and the \(k\)-th choice number is “yes”; this means that we can define the fractional chromatic number and the fractional choice number to be
\[
\chi_f = \lim_{k \to \infty} \frac{\chi^{(k)}(G)}{k}
\]
and
\[
ch_f = \lim_{k \to \infty} \frac{ch^{(k)}(G)}{k}
\]
and, remarkably, \(\chi_f(G) = ch_f(G)\). The answer to our problem is “yes” if
\[
\chi_f(G) > \frac{|V(G)|}{\alpha(G)},
\]
where \(\alpha(G)\) is the independence number of \(G\); in this case \(\lim_{k \to \infty} h^{(k)}(G)/k = \chi_f(G)\).