COLOURINGS


- **k-colouring**: \(c : V(G) \to S, \vert S \vert = k\) (often \(S = \{1, 2, \ldots , k\}\)).
- **proper colouring**: no two adjacent vertices get same colour. Often ‘proper’ implicit when we talk about colourings.
- **k-colourable**: has proper \(k\)-colouring.
- **chromatic number** \(\chi(G)\): smallest \(k\) for which \(G\) is \(k\)-colourable; \(k\)-chromatic means \(\chi(G) = k\).

**Example:**
vertices ↔ meetings
edges ↔ conflicts
colouring ↔ schedule
\(C_5\) has proper 3-colouring, no proper 2-colouring, so \(\chi = 3\).

**Assume**: All graphs simple for vertex-colourings. If loop, no proper colouring. Parallel edges make no difference.

**Inequalities**: (X1) Each colour forms an independent set, so \(\chi(G) \geq \lceil n/\alpha(G) \rceil\).

(X2) \(\omega(G)\) = size of largest clique (complete subgraph): \(\chi(G) \geq \omega(G)\).

Both of these can be tight, or not.

**Greedy colouring**: Colours 1, 2, \ldots, given ordering \(v_1, v_2, \ldots , v_n\) of vertices
for \(i = 1\) to \(n\) { \(c(v_i) = \) smallest colour not already used on a neighbour of \(v_i\) }

Usually uses more than \(\chi\) colours. Finding \(\chi\) is NP-hard.

(X3) \(c(v) \leq (\# \text{ previously coloured neighbors of } v) + 1 \forall v\), so uses \(\leq \Delta + 1\) colours. Hence \(\chi \leq \Delta + 1\). Sometimes tight, almost always not.

**Brooks’ Theorem**: If \(G\) is connected and not an odd cycle or complete then \(\chi(G) \leq \Delta(G)\).

**Sketch of proof**: Won’t provide full details. Idea is to order vertices carefully, use greedy colouring. Let \(\Delta = \Delta(G)\).

1. \(\Delta \leq 2\). \(G\) is path with \(\geq 2\) edges or even cycle so \(\chi = \Delta = 2\).

2. \(G\) is not regular. Choose \(v\) with \(d(v) < \Delta\). Grow spanning tree \(T\) from root \(v\) (Local TCM), adding vertices \(v = v_1, v_2, \ldots , v_n\). Apply greedy colouring in reverse order \(v_n, v_{n-1}, \ldots , v_1\).
   - When colour \(u \neq v\), has parent in \(T\) that comes later. So \(c(u) \leq d(u) \leq \Delta\).
   - And \(c(v) \leq d(v) + 1 \leq \Delta\).

3. \(G\) is \(k\)-regular, \(k \geq 3\), with cutvertex \(v\). \(\Delta\)-colour \(H, K\) by (2) since not regular. Permute colours on \(K\) so \(c_H(v) = c_K(v)\) and combine \(c_H, c_K\).
(4) $G$ is $k$-regular, $k \geq 3$, and 2-connected.  
Will find spanning tree as shown, then:
\[ c(u_1) = c(u_2) = 1, \]

\[ \text{colour } T \text{ towards } v \text{ as in (2),} \]

\[ c(v) \leq d(v) \text{ since } u_1, u_2 \text{ same colour.} \]

To find tree: Choose any $x$; since $G \neq K_n$, $d(x) = k \leq n - 2$.

If $\kappa(G - x) \geq 2$ take $u_1 = x$ and any path $u_1 u_2$; build $T$ from $v$ in $G - \{u_1 = x, u_2\}$.

If $\kappa(G - x) = 1$ then $x$ has a neighbour in each leaf block of $G - x$; take $u_1, u_2$ as neighbours in two such blocks and build tree from $v = x$ in $G - \{u_1, u_2\}$. Use $d(v) = k \geq 3$. 

\[ 1 \]

**Chromatic polynomials**

**Assume:** all graphs simple again back to vertex-colouring.

Idea: as one approach to 4CT, Birkhoff thought of counting $k$-colourings of a graph, idea developed by Whitney and Tutte.

\[ \circ \; P(G, k) = \# \text{ of proper } k\text{-colourings of } G \text{ with colours } 1, 2, \ldots, k. \]

**Example:**

\[ P(P_3, 1) = 0, \]

\[ P(P_3, 2) = 2, \]

\[ P(P_3, 3) = 12. \]

\[ \circ \; P(K_n, k) = k(k - 1)(k - 2) \ldots (k - n + 1) = P(k, n) = k^\text{P}_n = (k)_n. \]

\[ \circ \; P(\overline{K_n}, k) = k^n. \]

\[ \circ \; P(T, k) = k(k - 1)^n \text{ for any tree } T \text{ – colour outward from arbitrary root. Hence } P(P_3, k) = k(k - 1)^2. \]

\[ \circ \; \text{If } G_1, G_2 \text{ vertex-disjoint, } P(G_1 \cup G_2, k) = P(G_1, k)P(G_2, k). \]

**Expansion formula:** If $xy \notin E(G)$, 

\[ P(G, k) = P(G + xy, k) + P(G_{x=y}, k) \quad \text{(E).} \]

First term is colourings where $c(x) \neq c(y)$, second term colourings where $c(x) = c(y)$.

$G_{x=y} = G/\{x, y\}$: identify $x, y$ into single vertex.

**Example:** Represent $P(G, k)$ by just (G):

\[
\begin{pmatrix}
   x & y \\
   \end{pmatrix} = 
\begin{pmatrix}
   x & y \\
   \end{pmatrix} + 
\begin{pmatrix}
   x = y \\
   \end{pmatrix} \\
   = (K_4) + (K_3) = k(k - 1)(k - 2)(k - 3) + k(k - 1)(k - 2) \\
   = k(k - 1)(k - 2)(k - 3 + 1) = k(k - 1)(k - 2)^2 \\
   = k^4 - 5k^3 + 8k^2 - 4k.
\]

**Observation X5:** By repeated use of (E) can express any $P(G, k)$ as sum of $P(K_t, k)$’s. Therefore $P(G, k)$ is a polynomial in $k$, chromatic polynomial of $G$. Can also show has degree $n$, coefficients alternate in sign, and is monic (leading coefficient is 1).
However, computing $P(G,k)$ is NP-hard, since if can find it, can determine $\chi(G) = \text{first } k$ with $P(G,k) > 0$. Formula (E) gives exponential time algorithm.

Can turn formula (E) around.

**Deletion-contraction formula:** Suppose $xy \in E(G)$. Let $H = G \setminus xy$ using \ for deletion rather than usual $-$, standard when talking about deletion and contraction together, then

$$P(H,k) = P(H + xy,k) + P(H_{x=y},k),$$

i.e., $P(G \setminus xy) = P(G,k) + P(G/xy)$, so that $P(G,k) = P(G \setminus xy,k) - P(G/xy,k)$ (DC).

**Example:** $(P_3)$ again:

$$\begin{pmatrix} x_1 & y_1 \\ \circ & \circ \end{pmatrix} - \begin{pmatrix} x_2 & y_2 \\ \circ & \circ \end{pmatrix} = \begin{pmatrix} x_3 & y_3 \\ \circ & \circ \end{pmatrix}$$

$$= [((\circ & \circ & \circ) - (\circ & \circ)) - ((\circ & \circ) - (\circ))]$$

$$= (\circ & \circ & \circ) - 2(\circ & \circ) + (\circ)$$

$$= k^3 - 2k^2 + k = k(k-1)^2$$

as before.

Using (DC) can reduce any graph to edgeless graphs and compute $P(G,k)$ that way, gives another proof that it’s a polynomial. Typically use (E) for dense graphs and (DC) for sparse graphs.

**Planar graphs and the Four Colour Problem**

Recall: graph planar if can be drawn in plane without edge crossings. Not all graphs planar, e.g., can show $K_5$, $K_{3,3}$ not planar.

**Four Colour Problem** (Francis Guthrie, 1852): Can we colour maps (plane graphs) so that faces different colours if share an edge? Transform to vertex-colouring by duality.

Before discussing colourings need another fundamental result.

**Euler’s formula:** Let $G$ be a plane graph (specific crossing-free drawing of planar graph) with $r$ faces (regions determined by graph, including outside). If $G$ is connected then $n - m + r = 2$.

**Proof:** By induction on $m$ for fixed $n$. The smallest $m$ is $m = n - 1$ when $G$ is a tree; then $r = 1$ and the result holds.

So suppose result true for graphs with fewer edges than $G$, which is not a tree. So $G$ has an edge $e$ not a cutedge. Let $G' = G - e$, then $n' = n$, $m' = m - 1$ and since $e$ is not a cutedge, $r' = r - 1$. $G'$ is connected, so by induction $2 = n' - m' + r' = n - (m - 1) + (r - 1) = n - m + r$.

- degree $d(f)$ of face $f =$ total length of all boundary walks.
- $F(G) =$ set of faces of $G$.

**Face Degree-Sum Formula:** $\sum_{f \in F(G)} d(f) = 2m$. Every face has two sides (as well as two ends!) Or apply Degree-Sum Formula to dual.

Putting Euler’s formula and Face Degree-Sum Formula together gives very useful result (for colourings and other things).
**Theorem.** Let $G$ be a simple planar graph with $n \geq 3$. Then $m \leq 3n - 6$.

**Proof:** First suppose $G$ is connected and has plane embedding. Because $G$ is simple and $n \geq 3$, $d(f) \geq 3$ for all $f \in F(G)$. So

$$2m = \sum_{f \in F(G)} d(f) \geq 3|F(G)| = 3r \quad \text{so} \quad r \leq 2m/3.$$ 

By Euler’s Formula,

$$2 = n - m + r \leq n - m + 2m/3 = n - m/3 \quad \text{or equivalently} \quad m \leq 3n - 6.$$ 

If $G$ is disconnected add edges to get connected simple planar $G'$, then $m \leq m' \leq 3n' - 6 = 3n - 6$. 

**Corollary.** $K_5$ is not planar. ($m = 10 \not\leq 3n - 6 = 9$.)

**Corollary.** (a) The average degree of a simple planar graph is less than 6. True if $n \leq 2$, and if $n \geq 3$ then, by regular Degree-Sum Formula, $d(G) = 2m/n \leq 2(3n - 6)/n < 6$. 

(b) Thus, a planar graph $G$ must have a vertex of degree at most 5.

Now can say something about colourings.

**Observation:** Every planar graph is 6-colourable. Remove $v$ of degree $\leq 5$, 6-colour $G - v$ by induction, add $v$ back. Or can think of repeatedly removing vertices of degree 5, then applying greedy colouring in reverse order of removal.

**Five Colour Theorem.** Every planar graph $G$ is 5-colourable.

**Proof:** By induction on $n$. Use plane drawing of $G$.

• If $n \leq 5$ result is true.

• Suppose $n \geq 6$ and result is true for graphs with fewer vertices than $G$. Let $v$ have degree $\leq 5$ in $G$ and 5-colour $G' = G - v$ by induction.

If $\leq 4$ colours used on $N(v)$ can colour $v$ too. So $d(v) = 5$ and each $v_i$ is different colour $i$. Let $H(i,j)$ be subgraph induced by vertices of colours $i$ and $j$. Kempe chain.

If $v_1, v_3$ in different components of $H(1,3)$, switch $1 \leftrightarrow 3$ in $v_3$’s component, colour $v$ with 3. So $v_1, v_3$ in same component of $H(1,3)$. But then $v_2$ and $v_4$ must be in different components of $H(2,4)$, since components of $H(1,3)$ and $H(2,4)$ can’t cross. Switch $2 \leftrightarrow 4$ in $v_4$’s component, colour $v$ with 4. 

**Four Colour Theorem** (Appel and Haken, 1976): Every planar graph $G$ is 4-colourable. Proof complicated, computer checking of hundreds of cases.