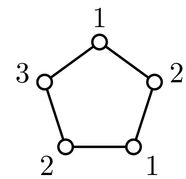
COLOURINGS

Reading: B&M 14.1,7 (vert. col.); 10.1-3 & 11.1-2 (planar).

- *k-colouring*: $c : V(G) \to S$, $|S| = k$ (often $S = \{1, 2, \ldots, k\}$).
- *proper* colouring: no two adjacent vertices get same colour. Often 'proper' implicit when we talk about colourings.
- *k-colourable*: has proper k-colouring.
- *chromatic number* $\chi(G)$: smallest k for which G is k-colourable; *k-chromatic* means $\chi(G) = k$.

Example:
vertices $\leftrightarrow$ meetings
edges $\leftrightarrow$ conflicts
colouring $\leftrightarrow$ schedule



C_5 has proper 3-colouring, no proper 2-colouring, so $\chi = 3$.

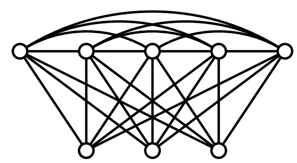
Assume: All graphs simple for vertex-colourings. If loop, no proper colouring. Parallel edges make no difference.

Inequalities: (X1) Each colour forms an independent set, so $\chi(G) \geq \lceil n/\alpha(G) \rceil$.

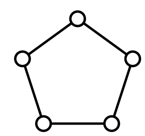
(X2) $\omega(G)$ = size of largest clique (complete subgraph): $\chi(G) \geq \omega(G)$.

Both of these can be tight, or not.

$K_5 \vee \overline{K_3}$:
$\chi = 6 > \lceil n/\alpha \rceil = \lceil 8/3 \rceil = 3$
$\chi = 6 = \omega$



C_5:
$\chi = 3 = \lceil n/\alpha \rceil = \lceil 5/2 \rceil$
$\chi = 3 > \omega = 2$



Greedy colouring: Colours $1, 2, \ldots$

given ordering $v_1, v_2, \ldots, v_n$ of vertices
for $i = 1$ to n {
  $c(v_i)$ = smallest colour not already used on a neighbour of v_i
}

Usually uses more than χ colours. Finding χ is NP-hard.

(X3) $c(v) \leq$ (# previously coloured neighbors of v) + 1 $\forall$ v, so uses $\leq \Delta + 1$ colours. Hence $\chi \leq \Delta + 1$. Sometimes tight, almost always not.

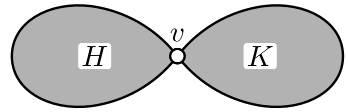
Brooks' Theorem: If G is connected and not an odd cycle or complete then $\chi(G) \leq \Delta(G)$.

Sketch of proof: Won't provide full details. Idea is to order vertices carefully, use greedy colouring. Let $\Delta = \Delta(G)$.

(1) $\Delta \leq 2$. G is path with ≥ 2 edges or even cycle so $\chi = \Delta = 2$.

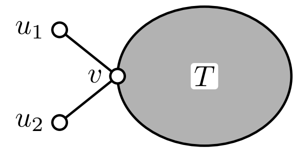
(2) G is not regular. Choose v with $d(v) < \Delta$. Grow spanning tree T from root v (Local TCM), adding vertices $v = v_1, v_2, \ldots, v_n$. Apply greedy colouring in reverse order $v_n, v_{n-1}, \ldots, v_1$.
- When colour $u \neq v$, has parent in T that comes later. So $c(u) \leq d(u) \leq \Delta$.
- And $c(v) \leq d(v) + 1 \leq \Delta$.

(3) G is k-regular, $k \geq 3$, with cutvertex v. Δ-colour H, K by (2) since not regular. Permute colours on K so $c_H(v) = c_K(v)$ and combine c_H, c_K.



(4) G is k-regular, $k \geq 3$, and 2-connected.
Will find spanning tree as shown, then:

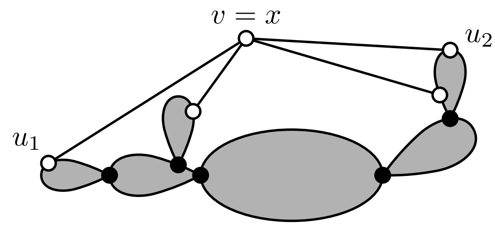
$c(u_1) = c(u_2) = 1$,
colour T towards v as in (2),
$c(v) \leq d(v)$ since u_1, u_2 same colour.



To find tree: Choose any x; since $G \neq K_n$, $d(x) = k \leq n-2$.

If $\kappa(G-x) \geq 2$ take $u_1 = x$ and any path $u_1 v u_2$; build T from v in $G - \{u_1 = x, u_2\}$.

If $\kappa(G-x) = 1$ then x has a neighbour in each leaf block of $G - x$; take u_1, u_2 as neighbours in two such blocks and build tree from $v = x$ in $G - \{u_1, u_2\}$. Use $d(v) = k \geq 3$. ∎



Chromatic polynomials

Assume: all graphs simple again back to vertex-colouring.

Idea: as one approach to 4CT, Birkhoff thought of counting k-colourings of a graph, idea developed by Whitney and Tutte.

◦ $P(G,k) = \#$ of proper k-colourings of G with colours $1, 2, \ldots, k$.

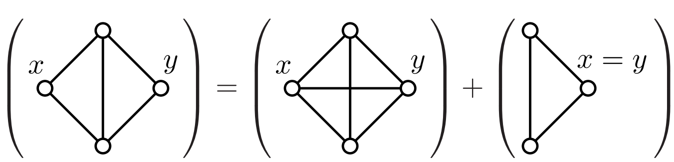
Example:

$P(P_3, 1) = 0$,
$P(P_3, 2) = 2$,
$P(P_3, 3) = 12$.

◦ $P(K_n, k) = k(k-1)(k-2)\ldots(k-n+1) = P(k,n) = {}^kP_n = (k)_n$.
◦ $P(\overline{K_n}, k) = k^n$.
◦ $P(T,k) = k(k-1)^n$ for any tree T – colour outward from arbitrary root. Hence $P(P_3,k) = k(k-1)^2$.
◦ If G_1, G_2 vertex-disjoint, $P(G_1 \cup G_2, k) = P(G_1,k)P(G_2,k)$.

Expansion formula: If $xy \notin E(G)$, $\quad P(G,k) = P(G+xy,k) + P(G_{x=y},k) \quad$ (E).
First term is colourings where $c(x) \neq c(y)$, second term colourings where $c(x) = c(y)$.
$G_{x=y} = G/\{x,y\}$: identify x, y into single vertex.

Example: Represent $P(G,k)$ by just (G):



$= (K_4) + (K_3) = k(k-1)(k-2)(k-3) + k(k-1)(k-2)$
$= k(k-1)(k-2)(k-3+1) = k(k-1)(k-2)^2$
$= k^4 - 5k^3 + 8k^2 - 4k$.

Observation X5: By repeated use of (E) can express any $P(G,k)$ as sum of $P(K_t,k)$'s. Therefore $P(G,k)$ is a polynomial in k, *chromatic polynomial* of G. Can also show has degree n, coefficients alternate in sign, and is monic (leading coefficient is 1).

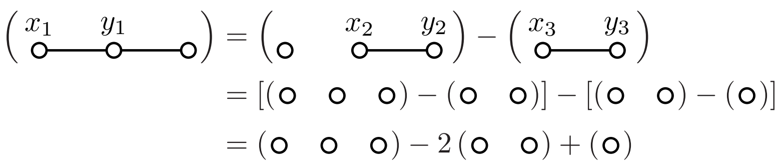
However, computing $P(G,k)$ is NP-hard, since if can find it, can determine $\chi(G)$ = first k with $P(G,k) > 0$. Formula (E) gives exponential time algorithm.

Can turn formula (E) around.

Deletion-contraction formula: Suppose $xy \in E(G)$. Let $H = G \backslash xy$ using $\backslash$ for deletion rather than usual $-$, standard when talking about deletion and contraction together, then

$P(H,k) = P(H+xy,k) + P(H_{x=y},k)$,
i.e., $P(G\backslash xy) = P(G,k) + P(G/xy)$,
so that $P(G,k) = P(G\backslash xy,k) - P(G/xy,k)$ (DC).

Example: (P_3) again:

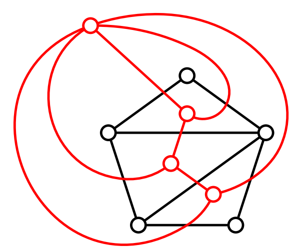


$= k^3 - 2k^2 + k = k(k-1)^2$ as before.

Using (DC) can reduce any graph to edgeless graphs and compute $P(G,k)$ that way, gives another proof that it's a polynomial. Typically use (E) for dense graphs and (DC) for sparse graphs.

Planar graphs and the Four Colour Problem

Recall: graph planar if can be drawn in plane without edge crossings. Not all graphs planar, e.g., can show K_5, $K_{3,3}$ not planar.



Graph G
Dual G^*

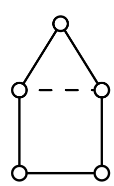
Four Colour Problem (Francis Guthrie, 1852)**:** Can we colour maps (plane graphs) so that faces different colours if share an edge? Transform to vertex-colouring by *duality*.

Before discussing colourings need another fundamental result.

Euler's formula: Let G be a *plane graph* (specific crossing-free drawing of planar graph) with r *faces* (regions determined by graph, including outside). If G is connected then $n - m + r = 2$.

Proof: By induction on m for fixed n. The smallest m is $m = n-1$ when G is a tree; then $r = 1$ and the result holds.

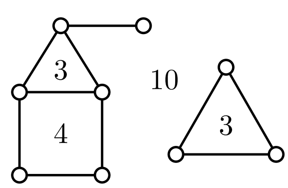
So suppose result true for graphs with fewer edges than G, which is not a tree. So G has an edge e not a cutedge. Let $G' = G - e$, then $n' = n$, $m' = m - 1$ and since e is not a cutedge, $r' = r - 1$. G' is connected, so by induction $2 = n' - m' + r' = n - (m-1) + (r-1) = n - m + r$. ∎



$n = 5,\ m = 6,\ r = 3$
$\rightarrow$
$n' = 5,\ m' = 5,\ r' = 2$

∘ *degree* $d(f)$ of face f = total length of all boundary walks.
∘ $F(G)$ = set of faces of G.

Face Degree-Sum Formula: $\sum_{f \in F(G)} d(f) = 2m$. Every face has two sides (as well as two ends!) Or apply Degree-Sum Formula to dual.



Putting Euler's formula and Face Degree-Sum Formula together gives very useful result (for colourings and other things).

Theorem. Let G be a simple planar graph with $n \geq 3$. Then $m \leq 3n - 6$.

Proof: First suppose G is connected and has plane embedding. Because G is simple and $n \geq 3$, $d(f) \geq 3$ for all $f \in F(G)$. So

$$2m = \sum_{f \in F(G)} d(f) \geq 3|F(G)| = 3r \qquad \text{so} \qquad r \leq 2m/3.$$

By Euler's Formula,

$$2 = n - m + r \leq n - m + 2m/3 = n - m/3 \qquad \text{or equivalently} \qquad m \leq 3n - 6.$$

If G is disconnected add edges to get connected simple planar G', then $m \leq m' \leq 3n' - 6 = 3n - 6$. ∎

Corollary. K_5 is not planar. ($m = 10 \not\leq 3n - 6 = 9$.)

Corollary. (a) The average degree of a simple planar graph is less than 6. True if $n \leq 2$, and if $n \geq 3$ then, by regular Degree-Sum Formula, $d(G) = 2m/n \leq 2(3n-6)/n < 6$.

(b) Thus, a planar graph G must have a vertex of degree at most 5.

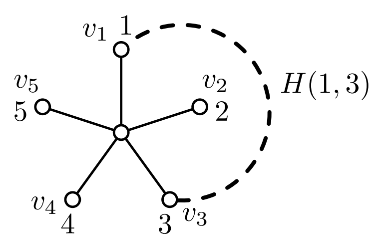
Now can say something about colourings.

Observation: Every planar graph is 6-colourable. Remove v of degree ≤ 5, 6-colour $G - v$ by induction, add v back. Or can think of repeatedly removing vertices of degree 5, then applying greedy colouring in reverse order of removal.

Five Colour Theorem. Every planar graph G is 5-colourable.

Proof: By induction on n. Use plane drawing of G.
- If $n \leq 5$ result is true.
- Suppose $n \geq 6$ and result is true for graphs with fewer vertices than G. Let v have degree ≤ 5 in G and 5-colour $G' = G - v$ by induction.

If ≤ 4 colours used on $N(v)$ can colour v too. So $d(v) = 5$ and each v_i is different colour i. Let $H(i, j)$ be subgraph induced by vertices of colours i and j. Kempe chain.



If v_1, v_3 in different components of $H(1,3)$, switch $1 \leftrightarrow 3$ in v_3's component, colour v with 3.

So v_1, v_3 in same component of $H(1,3)$. But then v_2 and v_4 must be in different components of $H(2,4)$, since components of $H(1,3)$ and $H(2,4)$ can't cross. Switch $2 \leftrightarrow 4$ in v_4's component, colour v with 4. ∎

Four Colour Theorem (Appel and Haken, 1976)**:** Every planar graph G is 4-colourable. Proof complicated, computer checking of hundreds of cases.