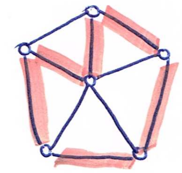
HAMILTON CYCLES

Reading: 18.1, 3.

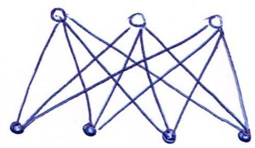
Recall: *hamilton* path or cycle: spanning.
hamiltonian graph: has hamilton cycle.
traceable graph: has hamilton path.

$c(G)$: number of components of G.

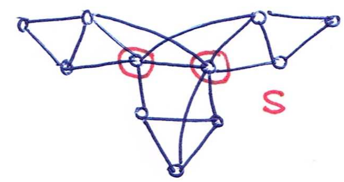


Necessary condition

Examples: $K_{3,4}$: not hamiltonian, cannot alternate $X - Y - X - Y \dots$.

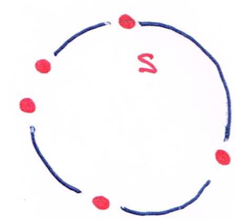


G not hamiltonian: $c(C - S) \le 2$, $C - S$ spanning subgraph of $G - S$ with $c(G - S) = 3$, impossible. More general version of bipartite problem.



Toughness condition: If G has a hamilton cycle then $c(G - S) \le |S| \; \forall \; S \subseteq V(G)$, $S \ne \emptyset$.

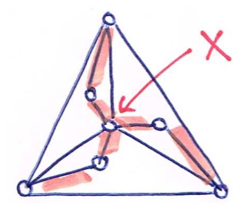
G is t-*tough* if $c(G - S) \le |S|/t \; \forall \; S \subseteq V(G)$ with $c(G - S) \ge 2$. So hamiltonian $\Rightarrow$ 1-tough. Book calls 1-tough just 'tough'.



Notes: (1) 1-tough $\Rightarrow$ 2-connected (no cutvertex).

(2) High connectivity cannot guarantee hamiltonian. $K_{k,k+1}$ is k-connected but not 1-tough so not hamiltonian.
But 1-tough $\not\Rightarrow$ hamiltonian (example from Chvatal, 1973).

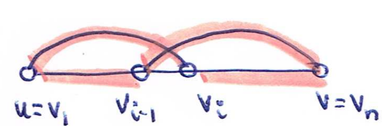
Toughness Conjecture (Chvátal, 1973): Sufficiently tough graphs are hamiltonian. (At present know required toughness would have to be at least 9/4.)



Degree-based sufficient conditions

Dirac's Theorem: If G is a simple graph with $\delta \ge n/2$, $n \ge 3$, then G is hamiltonian.

Proof: Suppose not. Let G be a maximal n-vertex graph satisfying the condition that is not hamiltonian. Then G is not complete, so there are two nonadjacent vertices u, v. By maximality of G, $G + uv$ is hamiltonian, with hamilton cycle C which must include uv. Then $P = C - uv = v_1 v_2 v_3 \dots v_n$ ($v_1 = u$, $v_n = v$) is a hamilton path in G. Let
$S = \{i \mid u \sim v_i\} \subseteq [2, n-1]$;
$T = \{i \mid v \sim v_{i-1}\} \subseteq [3, n]$.
Then $S \cup T \subseteq [2, n]$ so $|S \cup T| \le n - 1$. But $|S| + |T| = d(u) + d(v) = n$, so $S \cap T \ne \emptyset$. If $i \in S \cap T$ then G has the hamilton cycle C' shown, which is a contradiction. ∎

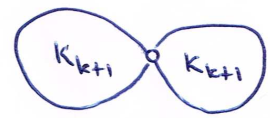


Ore observed that this can be strengthened.

Ore's Theorem: Suppose G is an n-vertex simple graph, $n \geq 3$, and $d(u) + d(v) \geq n$ for all distinct nonadjacent u, v. Then G is hamiltonian.

Proof: Same. Need conditional *for all* nonadjacent u, v so can use maximality argument.

Note: (1) Theorems of Ore and Dirac are sharp: Take graph with vertex separating G into two K_{k+1}'s: $n = 2k + 1$ and
$\delta = k = (n-1)/2$;
$d(u) + d(v) = k + k = 2k = n - 1 \;\forall$ nonadjacent u, v.
But no hamiltonian cycle since not 2-connected.



(2) Observe: G has hamilton path $\Leftrightarrow G \vee K_1$ has hamilton cycle. Can deduce

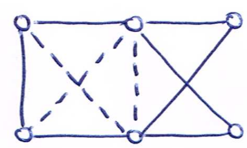
$$\left.\begin{array}{r} d(u) + d(v) \geq n - 1 \;\forall \text{ distinct nonadj } u, v \\ \delta \geq (n-1)/2 \end{array}\right\} \Rightarrow G \text{ has hamilton path (is traceable)}$$

In fact, can make an even more general statement.

Lemma H1: Let G be an n-vertex simple graph with distinct nonadjacent vertices u, v. If $d(u) + d(v) \geq n$ then G is hamiltonian $\Leftrightarrow G + uv$ is hamiltonian.
Proof: ($\Rightarrow$) Obvious. ($\Leftarrow$) Almost same. Just start with $G + uv$ being hamiltonian.

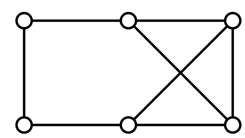
addable edge uv: $u \not\sim v$, $d(u) + d(v) \geq n$. My terminology, not standard.

Bondy-Chvátal closure: Given G, repeatedly add addable edges until reach graph G^c with no more addable edges. Can show G^c is unique: *Bondy-Chvátal closure of* G. G hamiltonian $\Leftrightarrow G^c$ hamiltonian. Theorems of Dirac and Ore just cases where G^c is complete.



Example where BCC complete although Dirac, Ore don't work.

Leads to other complicated degree-based conditions; see book.

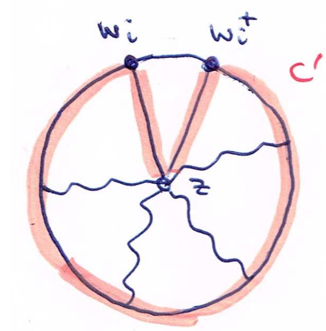


Sufficient condition: connectivity and independence

Saw before that connectivity by itself cannot guarantee hamiltonian. But if connectivity high relative to size of independent sets, then get hamiltonian.

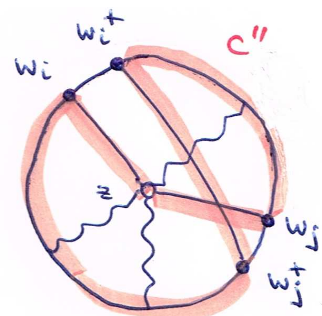
Recall $\alpha(G)$: size of maximum independent set.

Chvátal-Erdős Theorem: If G is loopless, $n \geq 3$ and $\kappa(G) \geq \alpha(G)$ then G is hamiltonian.



Proof: If G is supercomplete this is obvious, so suppose G is not supercomplete. Then we may assume G is simple (since parallel edges do not affect hamiltonicity, and since there are nonadjacent vertices κ is determined by vertex cutsets) and $n \geq \kappa + 2$.

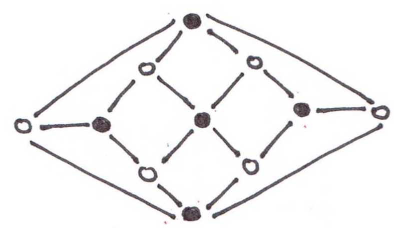
Let C be a longest cycle in G, and assume G is not hamiltonian, so there exists $z \notin V(C)$. We know $|V(C)| \geq \kappa$ (since any κ vertices lie on a cycle). For each $x \in V(C)$ let x^+ be the vertex immediately following x on C (taking a fixed orientation of C). There are κ paths from v to C, vertex-disjoint except at v, ending at $w_1, w_2, \ldots, w_\kappa$. If $z \sim w_i^+$ then we have a longer cycle C' as shown. If $w_i^+ \sim w_j^+$ then we have a longer cycle C'' as shown (even if $w_i^+ w_j^+ \in E(C)$). Hence $\{z, w_1^+, w_2^+, \ldots, w_\kappa^+\}$ is an independent set of size $\kappa + 1$, a contradiction. Hence C must be a hamilton cycle. ∎



Connectivity and planarity

Recall: *planar* graph: can be drawn in plane without crossings.

Theorem (Tutte, 1956): If G is a 4-connected planar graph then G is hamiltonian.



Note: Not all 3-connected planar graphs are hamiltonian, e.g. Herschel graph (Fig. 18.1(b) in book). Even if restrict to cubic (3-regular) graphs, not hamiltonian; rules out approach to 4 Colour Theorem.