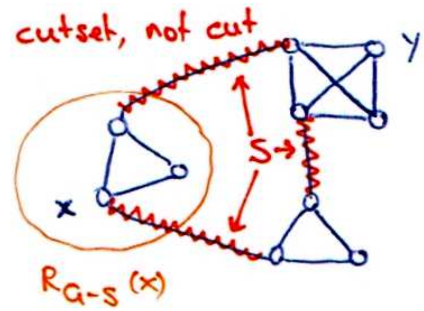
CONNECTIVITY

Reading: 9.1-3

Edge connectivity

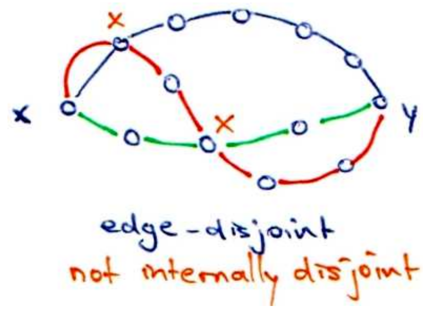


Think of trying to get message from x to y, adversary trying to stop you by taking out edges of graph.

- ◦ *edge cutset:* $S \subseteq E(G)$ so $G - S$ is disconnected.
- ◦ *edge cut:* $S \subseteq E(G)$ so that there exists $X \subseteq V(G)$, $X \neq \emptyset, V(G)$, with $S = \delta X$. Not quite the same thing.

(K1) Any edge cut is an edge cutset, but not vice versa. See picture.

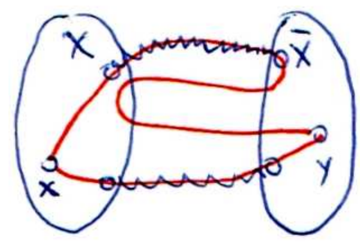
(K2) Any edge cutset S contains an edge cut δX, where X is vertex set of any component of $G - S$.



Follows that every *minimal* edge cutset is an edge cut.

- ◦ *xy-edge cutset* S where $G - S$ has no xy-path; $s'(x, y)$ = minimum size of an xy-edge cutset.
- ◦ *xy-edge cut*: $S = \delta X$ with $x \in X$, $y \in \overline{X}$; $c'(x, y)$ = minimum size of an xy-edge cut.

(K3) Like (K1) and (K2), every xy-edge cutset contains an xy-edge cut, so $s'(x, y) \geq c'(x, y)$, and every xy-edge cut is an xy-edge cutset, so $s'(x, y) \leq c'(x, y)$. Thus, $s'(x, y) = c'(x, y)$.



(K4) Let $\mathcal{P}$ be a collection of *edge-disjoint* xy-paths, and δX an xy-edge cut. Each path in $\mathcal{P}$ must contain a distinct edge of δX, so $|\mathcal{P}| \leq |\delta X|$.

- ◦ $p'(x, y)$ = maximum number of *edge-disjoint* xy-paths.

(K5) Thus $p'(x, y) \leq c'(x, y)$. And if we can find $\mathcal{P}$ and δX with $|\mathcal{P}| = |\delta X|$, $\mathcal{P}$ must be maximum and δX must be minimum, and then $p'(x, y) = c'(x, y)$.

Edge Version of Menger's Theorem: If x, y are distinct vertices of a graph G, then $p'(x, y) = c'(x, y) = s'(x, y)$.

Proof: We just need to find $\mathcal{P}$ and δX as in (K5).

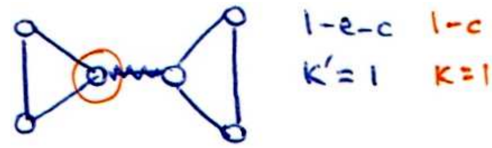
Take the associated digraph D of G (replace each edge by two oppositely directed arcs). Give each $a \in A(D)$ capacity $c(a) = 1$. Let f be an integer-valued maximum xy-flow in D and $\delta_D^+ X$ the associated minimum xy-cut. By the MFMC Theorem, $\operatorname{val} f = c(\delta_D^+ X) = |\delta_D^+ X|$.

By flow decomposition we may assume f is acyclic, and break f into an integer linear combination of flows of value 1 along directed xy-paths. If $\mathcal{P}'$ is the collection of paths then $\operatorname{val} f = |P'|$, $\operatorname{supp} f = \cup_{p \in \mathcal{P}'} A(P)$ is acyclic, and each arc is in at most one path. Since $\operatorname{supp} f$ is acyclic, no two paths in $\mathcal{P}'$ use opposite arcs. Important for saying corresponding paths in G are edge-disjoint.

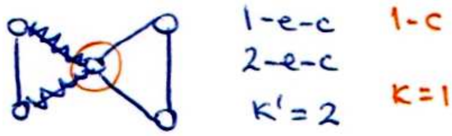
Thus, G has a corresponding collection $\mathcal{P}$ of edge-disjoint xy-paths, where $|\mathcal{P}| = |\mathcal{P}'| = \operatorname{val} f = |\delta_D^+ X| = |\delta X|$. ∎

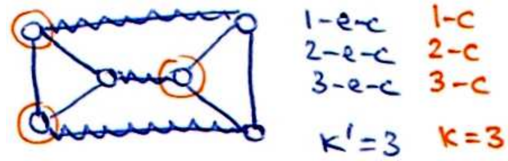
Notes: (1) Also directed version, proof even simpler.
(2) Can use max flow algorithm to find $p'/c'/s'(x, y)$, min cut, max set of edge-disjoint paths.

- G is *k-edge-connected* if $G - S$ is connected for all $S \subseteq E(G)$ with $|S| < k$. Equivalent to $s'/c'/p'(x, y) \geq k \ \forall$ distinct $x, y \in V(G)$.



- *edge-connectivity* $\kappa'(G)$ is the largest k for which G is k-edge-connected. I.e., $\kappa'(G)$ is the minimum size of any edge cutset. Thus $\kappa'(G) = \min\{s'/c'/p'(x, y) \mid x, y \in V(G), x \neq y\}$. Observe that $\kappa'(G) \leq \delta(G)$.





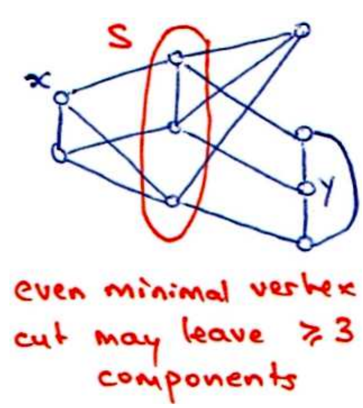
Note: Edge-connectivity of a one-vertex graph is conventionally taken as 1. Could argue for 0, ∞.

B&M define κ' using p', but definition using cutsets more standard. Equivalent by edge version of Menger's Thm.

Global Edge Version of Menger's Theorem: G with $n \geq 2$ is k-edge-connected if and only if there are k edge-disjoint paths between any pair of distinct vertices.

(Vertex) connectivity

Again trying to get message from x to y, adversary can now take out VERTICES of graph. Want to define connectivity $\kappa(G)$ and relate to existence of paths. This time no cut/cutset distinction, but another issue.



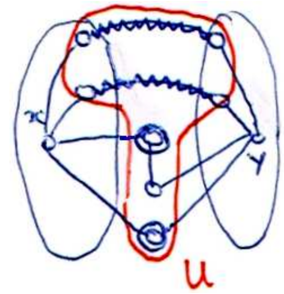
- *vertex cut(set)*: $S \subseteq V(G)$ so that $G - S$ is disconnected;
  xy-vertex cut(set) S: $G - S$ has x, y but no xy-path (so $x, y \notin S$);
  $c^{\mathrm{v}}(x, y) =$ minimum size of an xy-vertex cut.

Issue: If graph *supercomplete* (any two distinct vertices adjacent), no vertex cuts. If x, y adjacent, no xy-vertex cut. Various ways to handle this. (1) Define connectivity using paths, not cuts (what B&M do). (2) Treat supercomplete graphs/adjacent vertices as special cases (very common approach). (3) Expand definition of cut (my approach, nonstandard).

For adjacent vertices, does adversary give up? No, prefers vertices, but will target edges as well if necessary. Nonstandard, but works nicely.

- a *unit* in a graph is either a vertex or an edge;
  unit cutset: $U \subseteq V(G) \cup E(G)$ so $G - U$ is disconnected;
  xy-unit cutset: $U \subseteq V(G) \cup E(G)$ so $G - U$ has x, y but no xy-path;
  $c(x, y) =$ minimum size of xy-unit cutset.

(K6) If x and y are nonadjacent, any xy-unit cutset can be replaced by an xy-vertex cutset of the same or smaller size. Replace each edge by one of its ends not equal to x or y. Hence $c(x, y) = c^{\mathrm{v}}(x, y)$ when x and y are nonadjacent.



- *internally disjoint* xy-paths: nothing in common except x and y (no common internal vertices or edges, i.e. no common internal units).

(K7) Let $\mathcal{P}$ be a collection of internally disjoint xy-paths, and U an xy-unit cutset. Each path in $\mathcal{P}$ must contain a distinct element of U, so $|\mathcal{P}| \le |U|$.

- $p(x, y)$ = maximum number of internally disjoint xy-paths.

(K8) Thus $p(x, y) \le c(x, y)$. And if we can find $\mathcal{P}$ and U with $|\mathcal{P}| = |U|$, $\mathcal{P}$ must be maximum and U must be minimum, and then $p(x, y) = c(x, y)$.

Vertex Version of Menger's Theorem really, UNIT Version!**:** If x, y are distinct vertices of a graph G, $p(x, y) = c(x, y)$. So if x, y are nonadjacent then $p(x, y) = c(x, y) = c^{\mathrm{v}}(x, y)$.

Proof: We find $\mathcal{P}$ and U as in (K8) by the MFMC Theorem and flow decomposition in a digraph D using idea for vertex capacities: split each vertex into input, output sides:
for each vertex v of G, D has two vertices v^-, v^+ and an arc v^-v^+,
for each edge uv of G, D has two arcs u^+v^- and u^-v^+,
but omit x^- and y^+.
All arcs get capacity 1. (For just VERTEX cuts, arcs from edges get capacity ∞.) An acyclic integer-valued maximum x^+y^--flow in (D, c) can be decomposed into directed xy-paths in D corresponding to internally disjoint xy-paths in G. A minimum x^+y^--cut in D corresponds to a minimum xy-unit cutset in G. ∎

Notes: Again, also directed version. Again, can use max flow algorithm to compute.

- G is *k-connected* (or k-vertex-connected, really should be k-UNIT-connected) if $G-U$ is connected for every set of vertices and edges U with $|U| < k$. Equivalent to $c/p(x, y) \ge k$ $\forall$ distinct $x, y \in V(G)$.

- *connectivity* (or vertex connectivity, should be UNIT connectivity) $\kappa(G)$ is the largest k for which G is k-connected. I.e., $\kappa(G)$ is the minimum size of any cutset of vertices and edges. Thus, $\kappa(G) = \min\{c/p(x, y) \mid x, y \in V(G), x \ne y\}$. Observe that $\kappa(G) \le \kappa'(G) \le \delta(G)$.

Note: Connectivity of a one-vertex graph is conventionally taken as 1,

Global Vertex Version of Menger's Theorem: G with $n \ge 2$ is k-connected if and only if there are k internally disjoint paths between any pair of distinct vertices.

Standard approach also makes sense: treat supercomplete graphs specially, and just look at non-adjacent vertices in other graphs. Now will see how this works.

Connectivity for supercomplete graphs

- $m(x, y)$ = number of edges between x and y;
  $m_{\min}(G) = \min\{m(x, y) \mid x, y \in V(G),\ x \ne y\}$, the *minimum edge multiplicity*.

(K9) If G is supercomplete then $p(x, y) = n - 2 + m(x, y)$ whenever $x \ne y$, so
$$\kappa(G) = \min_{x \ne y} p(x, y) = \min_{x \ne y} \big(n - 2 + m(x, y)\big) = n - 2 + \min_{x \ne y} m(x, y) = n - 2 + m_{\min}(G).$$

Connectivity for non-supercomplete graphs

In this case really turns out to just depend on VERTEX cuts.

- $x \sim y$ means x, y adjacent.

Lemma K10: If G is not supercomplete and $c(u,v) \geq k$ for all distinct nonadjacent u,v, then $c(x,y) \geq k$ for all distinct adjacent x,y.

Proof: Assume $c(u,v) \geq k$ for all distinct nonadjacent u,v. Since $c(u,v) \leq n-2$ for all such pairs u,v, and at least one such pair u,v exists, $k \leq n-2$.

Suppose there are adjacent x,y with $c(x,y) \leq k-1$. Let $H = G - E(x,y)$. In H, $x \not\sim y$, so there is a minimum xy-unit cutset S consisting only of vertices. So

$$k-1 \geq c(x,y) = p(x,y) = p_H(x,y) + m(x,y) = c_H(x,y) + m(x,y) = |S| + m(x,y).$$

Hence, $|S| \leq k-1-m(x,y) \leq k-2 \leq n-4$, and there is some $z \in V(G) - (S \cup \{x,y\})$. Now in $H - S$, z is in a different component from at least one of x or y, say from x. Therefore $z \not\sim x$ in H and hence in G. Now $S \cup E(x,y)$ separates z from x in G, but $|S \cup E(x,y)| = |S| + m(x,y) \leq k-1$, contradicting $c(u,v) \geq k$ whenever $u \not\sim v$.

Corollary K11: If G is not supercomplete then

$$\kappa(G) = \min\{c(x,y) \mid x \not\sim y\} = \min\{p(x,y) \mid x \not\sim y\} = \min\{c^{\mathrm{v}}(x,y) \mid x \not\sim y\}.$$

Only considering nonadjacent vertices reduces work, sometimes significantly. And only need to think about vertex cuts, not unit cuts.

To show connectivity of non-supercomplete graph: show $\kappa \leq k$ by finding vertex cut of size k; show $\kappa \geq k$ by finding k internally disjoint xy-paths for all nonadjacent distinct x,y.

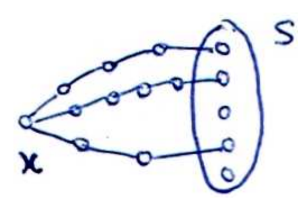
Disjoint paths between sets of vertices

Often want paths between *sets* of vertices: have useful trick to handle this.

Lemma K12: Suppose G is k-connected, and $S \subseteq V(G)$ has $|S| \geq k$. If we form G' by adding a new vertex v adjacent to all vertices of S, then G' is also k-connected.

Proof: If G' is supercomplete then it is k-connected because $|V(G')| \geq k+1$. Otherwise, every vertex cut in G' contains a vertex cut in G (if it separates two vertices different from v), or contains S (if it separates v from other vertices), and so has $\geq k$ vertices. ∎

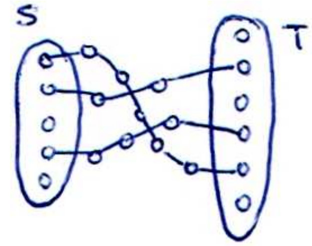
Fan Lemma: Suppose G is k-connected and $S \subseteq V(G)$ with $|S| \geq k$, and $x \in V(G)$. Then there are k paths from x to S that are vertex-disjoint except at x and have no internal vertices in S (a k-*fan* from x to S).



Note: Can have $x \in S$, in which case we get trivial path x and $k-1$ paths xy_i, $y_1, \ldots, y_{k-1}$ distinct vertices of S. Book restricts to $x \notin S$.

To prove, add extra vertex y adjacent to S, use Lemma K12 and vertex version of Menger for xy-paths. Stop paths as soon as hit vertex of S.

Corollary K13: Suppose G is k-connected and $S, T \subseteq V(G)$ with $|S|, |T| \geq k$. Then there are k vertex-disjoint paths from S to T (with no internal vertices in $S \cup T$).



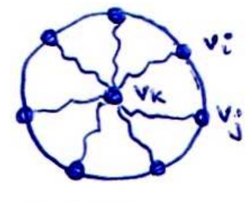
Note: S, T can overlap.

To prove, add extra vertices adjacent to S, T, use Lemma K12 and vertex version of Menger.

Don't know which vertex of S connected to which vertex of T. If can control this, have k-linkage. Needs more than k-connected.

Now see an application of this.

Corollary K14 (Dirac): Suppose G is k-connected, $k \geq 2$, and $S \subseteq V(G)$ with $|S| = k$. Then there is a cycle C in G that includes all vertices of S.

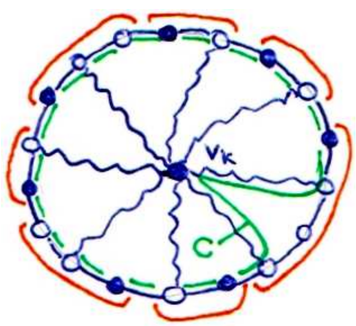


$|V(C')| = k-1$

Proof: By induction on k. Write $S = \{v_1, v_2, \ldots, v_k\}$.

If $k = 2$ then by the vertex version of Menger's Theorem there are two internally disjoint v_1v_2-paths, which together form the required cycle.

Suppose $k \geq 3$, and the result holds for $k-1$. Since a k-connected graph is also $(k-1)$-connected, G has a cycle C' containing $S' = \{v_1, v_2, \ldots, v_{k-1}\}$. We may assume $v_k \notin V(C')$ or we are done.



$|V(C')| \geq k$

The vertices of S' divide C' into $k-1$ segments. If $|V(C')| = k-1$ then since G is $(k-1)$-connected the Fan Lemma gives a $(k-1)$-fan from v_k to $V(C')$ and we replace some segment $v_i \ldots v_j$ by $v_i \ldots v_k \ldots v_j$ to get C. So $|V(C')| \geq k$ and there is a k-fan from v_k to $V(C')$. By the pigeonhole principle two of the k paths in the fan must end on the same one of the $k-1$ segments, and we use these to obtain a cycle C containing $S' \cup \{v_k\} = S$. ∎

Note: This is best possible: $K_{k,k+1}$ is k-connected but has no cycle containing the $k+1$ vertices in the second part of the bipartition.