NETWORK FLOWS

Reading: 7.1-3

Want to think about moving something (water, messages, vehicles) through a network. Important practical applications. Also several other graph theory results (Menger's theorem in vertex and edge forms, bipartite matching algorithm) are consequences.

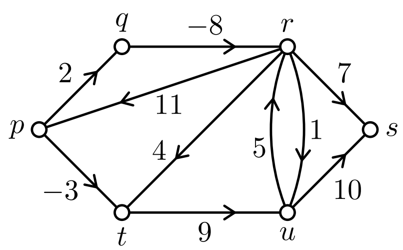
General flows

○ *flow* in a digraph: $f : A(D) \to \mathbf{R}$ Many books including ours add extra conditions but we allow any function at this point.

Recall: $A(X, Y)$ = arcs from X to Y; $\overline{X} = V(D) - X$; $\delta^+ X = A(X, \overline{X})$, $\delta^- X = A(\overline{X}, X)$. Also write $\delta^+ v$, $\delta^- v$ for individual vertices v.

Notation: Given a flow f, we define
$f(S) = \sum_{a \in S} f(a)$ for $S \subseteq A(D)$,
$f^+(X) = f(\delta^+ X)$, $f^-(X) = f(\delta^- X)$ for $X \subseteq V(D)$ (total flow on arcs out of or into X),
$\partial_D f(X) = \partial f(X) = f^+(X) - f^-(X)$ for $X \subseteq V(D)$, *net flow out of* X. Notation: using *boundary* symbol ∂, not *coboundary* δ.

(F1) $\partial f(\overline{X}) = -\partial f(X)$ because $\delta^+ \overline{X} = \delta^- X$, $\delta^- \overline{X} = \delta^+ X$.



A general flow:

$f^+(r) = 11 + 4 + 1 + 7 = 23$, $f^-(r) = -8 + 5 = -3$, $\partial f(r) = 23 - (-3) = 26$

$X = \{p, q\}$, $f^+(X) = -8 + (-3) = -11$, $f^-(X) = 11$, $\partial f(X) = -11 - 11 = -22$

$\partial f(p) + \partial f(q) = -12 + (-10) = -22 = \partial f(X)$ (flow on arc pq cancels)

Lemma F2: ∂f is additive, i.e., for $X \subseteq V(D)$ we have $\partial f(X) = \sum_{v \in X} \partial f(v)$.

Proof: By definition,

$$\sum_{v \in X} \partial f(v) = \sum_{v \in X} (f^+(v) - f^-(v)) = \sum_{v \in X} f^+(v) - \sum_{v \in X} f^-(v) = \sum_{v \in X} \sum_{a \in \delta^+ v} f(a) - \sum_{v \in X} \sum_{a \in \delta^- v} f(a).$$

Consider the final expression and $a \in A(D)$. The net contribution of a is
$0 - 0 = 0$ if $a \in A(\overline{X}, \overline{X})$,
$f(a) - f(a) = 0$ if $a \in A(X, X)$,
$f(a) - 0 = f(a)$ if $a \in A(X, \overline{X}) = \delta^+ X$,
$0 - f(a) = -f(a)$ if $a \in A(\overline{X}, X) = \delta^- X$.
Therefore,

$$\sum_{v \in X} \partial f(v) = \sum_{a \in \delta^+ X} f(a) - \sum_{a \in \delta^- X} f(a) = f(\delta^+ X) - f(\delta^- X) = f^+(X) - f^-(X) = \partial f(X). \blacksquare$$

Note: Means that $\sum_{v \in V(D)} \partial f(v) = \partial f(V(D)) = 0$ because $\delta^+(V(D)) = \delta^-(V(D)) = \emptyset$.

So now we just need to worry about how ∂f behaves on individual vertices.

(F3) $\partial : \mathbf{R}^A \to \mathbf{R}^V$ is a linear operator (linear transformation, vector space homomorphism) mapping flows (arc weights) to vertex weights: $\partial(\alpha f + \beta g) = \alpha \partial f + \beta \partial g$.

Two-terminal flows and circulations

For applications, we generally want flow to be 'balanced' at most vertices: outflow = inflow.

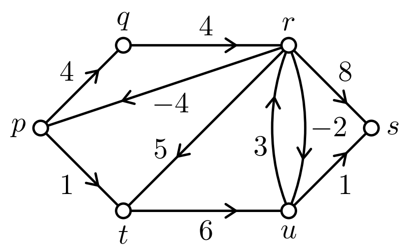
- ∘ f *conserved* at v: $f^+(v) = f^-(v)$, i.e. $\partial f(v) = 0$.
- ∘ *xy-flow* f: $\partial f(v) = 0 \ \forall\, v \in V(D) - \{x, y\}$ (conserved except at x and y, think of flowing from x to y. x is *supply vertex* and y is *demand vertex*. (Book and other sources call x the *source* and y the *sink*, but confusing: no need to assume x has indegree 0 or y has outdegree 0.)

(F4) By linearity of ∂, xy-flows also form a vector space: if $\partial f(v) = \partial g(v) = 0$ for $v \notin \{x, y\}$, then $\partial(\alpha f + \beta g)(v) = \alpha\partial f(v) + \beta\partial g(v) = 0$ for $v \notin \{x, y\}$.

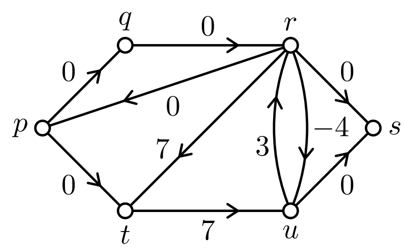
∘ *circulation* f: $\partial f = 0$, i.e., $\partial f(v) = 0 \ \forall\, v \in V(D)$. Conserved everywhere, so xy-flow for any x, y.

(F5) Set of circulations is kernel/nullspace of linear operator ∂, so is also vector subspace of all flows.

ps-flow (and sp-flow)



circulation

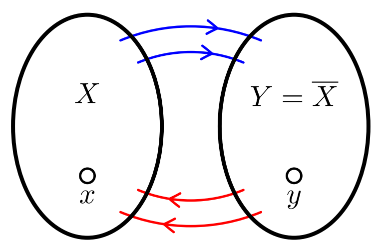


∘ *value* of an xy-flow is $\operatorname{val} f = \partial f(x)$. E.g. ps-flow above has value 9, or −9 as sp-flow. Circulation also ps-flow, value 0. Circulations always have value 0.

(F6) Value is linear on xy-flows: $\operatorname{val}(\alpha f + \beta g) = \alpha \operatorname{val} f + \beta \operatorname{val} g$. Follows from linearity of ∂.

Corollary F7: Suppose f is an xy-flow and X, Y is a partition of $V(D)$ with $x \in X$, $y \in Y = \overline{X}$. Then

$$\operatorname{val} f = \partial f(x) \overset{(1)}{=} \partial f(X) \overset{(2)}{=} -\partial f(Y) \overset{(3)}{=} -\partial f(y).$$



Proof: For (1) and (3) use additivity of ∂f and $\partial f(v) = 0$ if $v \neq x, y$. For (2) use (F1). ∎

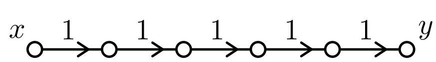
Special xy-flows: ∘ *characteristic function* χ_S of $S \subseteq A(D)$: $\chi(a) = 1$ if $a \in S$, 0 otherwise. For subdigraph H write χ_H for $\chi_{A(H)}$.

∘ *signed characteristic function* $\overrightarrow{\chi}_T$ of direction-insensitive trail T: $\overrightarrow{\chi}_T(a) = 1$ if T uses a forwards, −1 if T uses a backwards, 0 otherwise. Simpler for trails since use each edge at most once, but could extend to walks. If T is a directed trail then $\overrightarrow{\chi}_T = \chi_T$ thinking of T as subgraph.

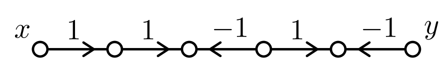
If P is a direction-insensitive (directed) xy-path, then $\overrightarrow{\chi}_P$ ($\overrightarrow{\chi}_P = \chi_P$) is an xy-flow of value 1.

If C is a direction-insensitive (directed) cycle then $\overrightarrow{\chi}_C$ ($\overrightarrow{\chi}_C = \chi_C$) is a circulation, and hence an xy-flow of value 0. Irrelevant whether or not C contains x or y.

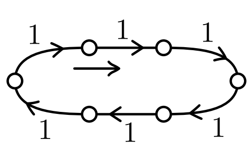
$\overrightarrow{\chi}_P = \chi_P$



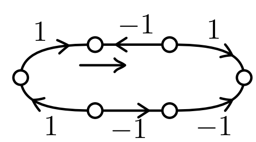
$\overrightarrow{\chi}_P$



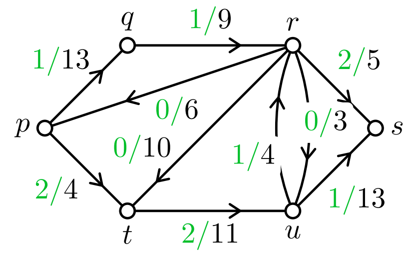
$\overrightarrow{\chi}_C = \chi_C$



$\overrightarrow{\chi}_C$



Networks and feasible flows



$f(a)/c(a)$; feasible ps-flow

∘ *network* (D, c): digraph D ($V = V(D)$, $A = A(D)$), each arc has nonnegative *capacity* $c(a)$.
Book adds distinguished vertices x, y but we do not.
Since c is just a function on arcs, can use notation developed for flows like $c(S)$, $c^+(X)$, etc.

∘ *feasible* flow in (D, c): flow (any flow, not necessarily xy-flow) that satisfies $0 \le f \le c$, i.e., $0 \le f(a) \le c(a)$ $\forall$ $a \in A(D)$.

Maximum Flow Problem: Find a *maximum xy-flow*, a feasible xy-flow in (D, c) of maximum value.

Cuts and flows

∘ *xy-edge cut* or just *xy-cut*: set of arcs K such that there exists $X \subseteq V(D)$ with $x \in X$, $y \notin X$, and $K = \delta^+X$ (*only* outward arcs). Removal destroys all directed xy-walks. *Capacity* of δ^+X in (D, c) means $c(\delta^+X) = c^+(X)$.

Theorem F8: Suppose f is a feasible xy-flow in (D, c), and δ^+X is an xy-cut. Then
(a) $\mathrm{val}\, f \le c(\delta^+X)$ and
(b) $\mathrm{val}\, f = c(\delta^+X)$ if and only if $f(a) = c(a)$ $\forall$ $a \in \delta^+X$ and $f(a) = 0$ $\forall$ $a \in \delta^-X$.

Example: Above, if $X = \{p, q, t\}$, $\mathrm{val}\, f = \partial f(X) = f^+(X) = 1 + 2 \le c^+(X) = 9 + 11 = 20$.

Proof: By Corollary F7

$$\begin{aligned}\mathrm{val}\, f &= \partial f(X) = f(\delta^+X) - f(\delta^-X)\\ &\le f(\delta^+X) \quad (1) \text{ since } f(\delta^-X) \ge 0 \text{ because } f \ge 0\\ &\le c(\delta^+X) \quad (2) \text{ since } f(\delta^-X) \le c(\delta^+X) \text{ because } f \le c.\end{aligned}$$

Equality holds if and only if it holds at (1), so $f(a) = 0$ for all $a \in \delta^-X$, and it holds at (2), so $f(a) = c(a)$ for all $a \in \delta^+X$. ∎

Corollary F9: If f is a feasible xy-flow, δ^+X is an xy-cut and $\mathrm{val}\, f = c(\delta^+X)$, then f is a maximum xy-flow, and δ^+X is a minimum (capacity) xy-cut. Does converse hold, or could maximum flow have value less than capacity of minimum cut? Goal: prove converse by constructing maximum flow and minimum cut. Do this in small steps.

Improving flow

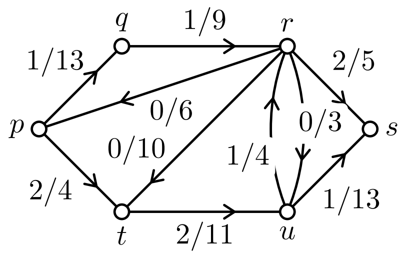
Basic idea: try to improve flow. If cannot, try to prove current flow is maximum by finding a corresponding cut. How to improve flow?
Treatment here different from book: we use explicit *residual network*, makes it easier to prove things using standard reachability arguments.

∘ *residual network* $(D^*, c^*) = \mathrm{Res}(D, c, f)$ for feasible flow f: shows how to modify f and stay feasible. $V(D^*) = V(D)$. Up to two arcs in D^* for each arc of D, with capacity function c^*:
if $f(a) < c(a)$ add a^+, copy of a, with $c^*(a^+) = c(a) - f(a)$ (indicates we can push extra flow along a);
if $f(a) > 0$ add arc a^-, opposite to a, with $c^*(a^-) = f(a)$ (indicates we can push some flow backwards along a, i.e., reduce flow in a).

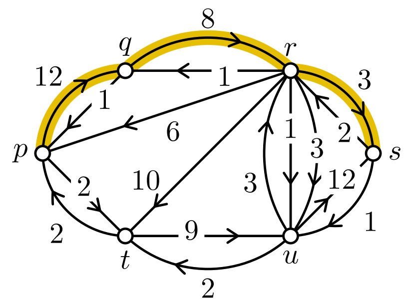
∘ $[b] = a$ for arc $b = a^+$ or a^- in D^*; $[W]$ = direction-insensitive walk in D corresponding to walk W in D^*.

Example: (0) Initial flow f_0 of value 3 and residual network:

$(D,c,f)_0$



$(D^*,c^*)_0$



∘ *f-augmenting path* for feasible xy-flow f: directed xy-path in D^*. Easy to find: use Directed Local TCM, e.g., Directed BFS or DFS, to see if y reachable from x in D^*.

(F10) If P is an f-augmenting path and $\rho = \min\{c^*(b) \mid b \in A(P)\}$ (which is > 0) then we get a better feasible xy-flow f' in (D,c) by *augmenting f along P (by ρ)*:

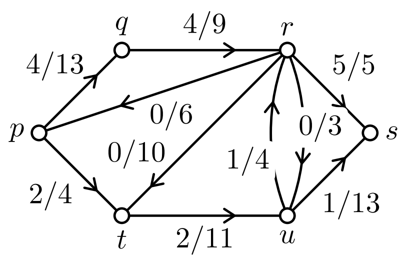
$f'(a) = f(a) + \rho \quad \forall\ a^+ \in A(P)$;

$f'(a) = f(a) - \rho \quad \forall\ a^- \in A(P)$.

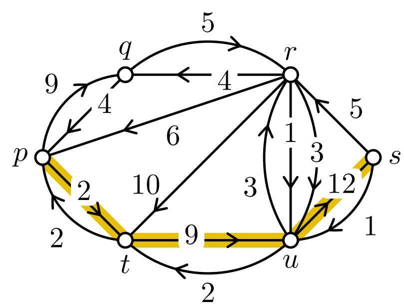
I.e., $f' = f + \rho \overrightarrow{\chi}_{[P]}$. Then f' is an xy-flow since xy-flows are a vector space which is feasible by choice of ρ and $\operatorname{val} f' = \operatorname{val} f + \rho \operatorname{val} \overrightarrow{\chi}_{[P]} = \operatorname{val} f + \rho$ by linearity of val. Could have augmented by amount less than ρ, but will be greedy and improve as much as possible.

Example (ctd): (1) Augment along $pqrs$ by $3 = \min\{12, 8, 3\}$, get f_1 of value 6 (flow and residual network only change along $pqrs$):

$(D,c,f)_1$



$(D^*,c^*)_1$



Ford-Fulkerson (F-F) Algorithm: (very complicated!)

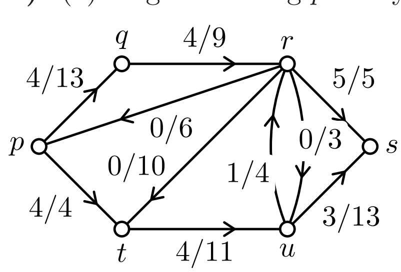
start with some feasible xy-flow f (maybe $f = 0$);
while there is an f-augmenting path P
augment f along P;

(F11) If we can find several *arc-disjoint* f-augmenting paths in D^* then can augment along all of them simultaneously.

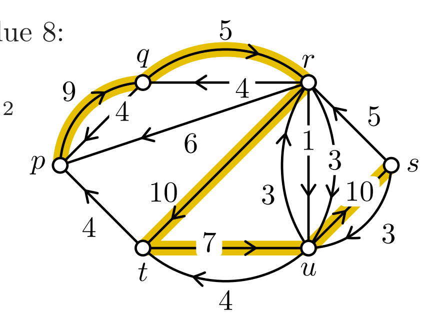
We hope that F-F (a) terminates and (b) gives a maximum flow.

Example (ctd): (2) Augment along $ptus$ by 2, get f_2 of value 8:

$(D,c,f)_2$



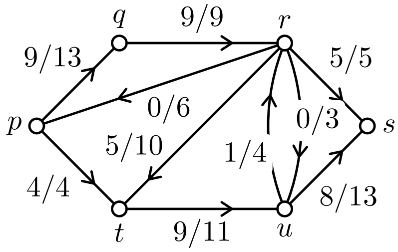
$(D^*,c^*)_2$



Could have combined (1) and (2) into single step augmenting along two edge-disjoint paths.

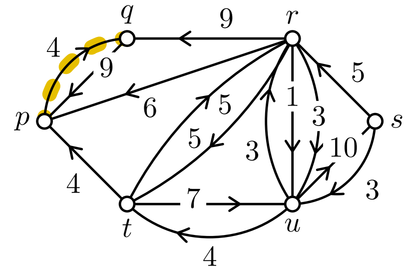
(3) Augment along $pqrtus$ by 5, get f_3 of value 13:

$(D, c, f)_3$



$(D^*, c^*)_3$

Dir. Loc. TCM



No apparent ps-path in residual network. Verified by Directed Local TCM from p. Will use output of Directed Local TCM again, shortly. So stop.

Theorem F12: Suppose f is a feasible xy-flow.
(i) If f is a maximum xy-flow then there is no f-augmenting path.
(ii) If there is no f-augmenting path then there is an xy-cut δ^+X with $\operatorname{val} f = c(\delta^+X)$. Thus, f is a maximum xy-flow, and δ^+X is a minimum xy-cut.

Proof: (i) If we had an f-augmenting path we could increase $\operatorname{val} f$.

(ii) Let $X = R^+_{D^*}(x)$. Since there is no directed xy-path in D^*, $y \in \overline{X}$, so δ^+X is an xy-cut.
By (D1), $\delta^+_{D^*}X = \delta^+_{D^*}(R^+_{D^*}(x)) = \emptyset$. So if $a \in \delta^+_D X$ then $f(a) = c(a)$, otherwise $a^+ \in \delta^+_{D^*}X$. And if $a \in \delta^-_D X$ then $f(a) = 0$, otherwise $a^- \in \delta^+_{D^*}X$. But these are exactly the conditions of Theorem F8(b), so $\operatorname{val} f = c(\delta^+X)$ and by Corollary F9 f is maximum and δ^+X is minimum. ∎

(F13) So *if* F-F terminates, by (ii) we have a maximum xy-flow. And we know how to find a minimum xy-cut δ^+X: $X = R^+_{D^*}(x)$ which we have probably already constructed with our Directed Local TCM to look for an xy-path in D^*.

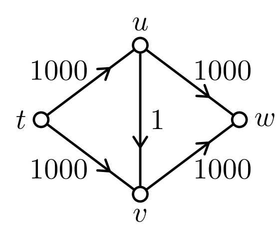
Example (ctd): Our Directed Local TCM found $R^+_{D^*}(p) = \{p, q\} = X$ so $\delta^+X = \{pt, qr\}$. Then $c(\delta^+X) = 9 + 4 = 13 = \operatorname{val} f_3$, so we have a maximum ps-flow f_3 and minimum ps-cut δ^+X. ∎

Theorem F14: If all capacities are integral then there exists an integer-valued maximum xy-flow.

Proof: Start F-F with $f = 0$. Since all capacities integral, all computations are integral, and F-F must terminate (since value increases by at least 1 each time). ∎

If all capacities rational, can scale to make them integers, so similar conclusion holds.

(F15) F-F may take many steps even for small graph. In example (at right), want tw-flow, use Directed DFS processing neighbours in some weird order. Augments along $tuvw$, $tvuw$, $tuvw$, ...: 2000 iterations.



(F16) There are examples with irrational capacities where F-F never terminates.

Edmonds-Karp (E-K) Algorithm: In F-F always choose an f-augmenting path with fewest arcs (shortest path in digraph, D^*) (can use Directed BFS in D^*). Then terminates, even with irrational capacities, and in polynomial time. Proof not too hard but omit due to time constraints.

Max-Flow Min-Cut (MFMC) Theorem: The value of a maximum xy-flow equals the capacity of a minimum xy-cut.

Proof: First, a maximum xy-flow f exists, since E-K finds one. Book fails to address existence. [Alternatively, val is a continuous function on the closed bounded (compact) space of feasible xy-flows, defined by conservation and feasibility conditions. Therefore, from topology, a flow of maximum value exists.]

Now f has no augmenting path, so by Theorem F12(ii) there is a corresponding minimum xy-cut. ∎

Comments on F-F/E-K algorithms: (1) May combine effect of parallel arcs in residual network into single arc.

(2) Need not explicitly construct the residual network: work directly with original network. Many books, inc. B&M, do not define residual network, just work with original network. More efficient for implementation. But residual network aids understanding and proofs can use familiar ideas like $R^+(\cdot)$.

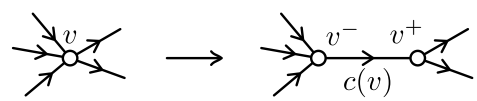
Variations

(F17) **Infinite capacities:** If (D, c) is a network where c can have values of $+\infty$, let D_∞ be the spanning subdigraph with only the infinite capacity arcs. Let $X_\infty = R^+_{D_\infty}(x)$. Then either

(i) $y \in X_\infty$, in which case we can find feasible xy-flows of arbitrarily large value, and there is no finite capacity xy-cut. No maximum flow value, but supremum of values of feasible xy-flows $= \infty$ $=$ capacity of minimum xy-cut, so MFMC Theorem holds in some sense.

(ii) $y \notin X_\infty$, in which case there is a finite capacity xy-cut $(\delta^+_D X_\infty)$, a maximum xy-flow of finite value, and MFMC Theorem holds. Can prove by replacing infinite capacities by a large finite value.

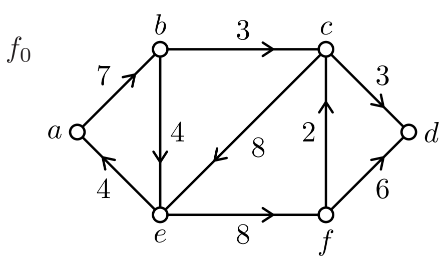
(F18) **Vertex capacities:** Suppose at flow-conserving v we want $f^+(v) = f^-(v) \le c(v)$. To implement, split v into v^- with all in-arcs, v^+ with all out-arcs (both flow-conserving), and arc v^-v^+ of capacity $c(v)$.



Decomposing flows

Will show any nonnegative flow is actually positive linear combination of flows along directed cycles and directed paths. Important for applications. Book does this in halfhearted way in §7.3 but jumps to Cor. 7.15 without proper explanation.

Example: Remove flow first along directed cycles, then along maximal directed paths.



Remove
4 along $C_1 = (abe)$,
2 along $C_2 = (cef)$,
3 along $P_1 = abcd$,
6 along $P_2 = cefd$,
now all flow is gone.

So $f_0 = 4\chi_{C_1} + 2\chi_{C_2} + 3\chi_{P_1} + 6\chi_{P_2}$.

∘ *support* of f, $\operatorname{supp} f = \{a \in A(D) \mid f(a) \neq 0\}$.
∘ *acyclic* flow f: $D[\operatorname{supp} f]$ is acyclic.

Flow Decomposition Algorithm: What we did in example.

$f = f_0$, a nonnegative flow; — here $f = f_0$
Phase 1:
while $\operatorname{supp} f$ has a directed cycle C {
  $\alpha = \min\{f(a) \mid a \in A(C)\}$; $f = f - \alpha\chi_C$; — remove $f_C = \alpha_1\chi_{C_1} + \ldots + \alpha_s\chi_{C_s}$
} — here $f = f_A = f_0 - f_C$, acyclic

Phase 2:
while supp $f \neq \emptyset$ {
  take maximal nontrivial directed path P;
  $\beta = \min\{f(a) \mid a \in A(P)\}$; $f = f - \beta\chi_P$;   remove $f_A = \beta_1\chi_{P_1} + \ldots + \beta_t\chi_{P_t}$
}   here $f = 0$

Gallai's Flow Decomposition Theorem (FDT): Every nonnegative flow f_0 may be written

$$f_0 = \overbrace{\alpha_1\chi_{C_1} + \alpha_2\chi_{C_2} + \ldots + \alpha_s\chi_{C_s}}^{f_C} + \overbrace{\beta_1\chi_{P_1} + \beta_2\chi_{P_2} + \ldots + \beta_t\chi_{P_t}}^{f_A}$$

where
(i) f_C is a nonnegative circulation, $s \geq 0$, $\alpha_1, \ldots, \alpha_s > 0$, and $C_1, \ldots, C_s$ are directed cycles;
(ii) f_A is a nonnegative acyclic flow, $t \geq 0$, $\beta_1, \ldots, \beta_t > 0$, and each P_i is a directed x_iy_i-path with $\partial f_0(x_i) > 0$, $\partial f_0(y_i) < 0$; and
(iii) if f_0 is integer-valued then we may choose $\alpha_1, \ldots, \alpha_s, \beta_1, \ldots, \beta_t$ to all be integers, so that f_C and f_A are also integer-valued.

Proof: Apply the algorithm. It terminates because at each iteration in each phase supp f loses at least one arc. Everything is then obvious except for the claims about ∂f_0 in (ii).

For each vertex v, $\partial f(v)$ does not change during Phase 1. During Phase 2, by maximality of P_i, x_i is a source and y_i a sink in $D[\text{supp}\, f]$ when P_i is chosen. Therefore, whenever $\partial f(v)$ changes in Phase 2 it moves towards 0. Moreover, when P_i is chosen $\partial f(x_i) > 0$ and $\partial f(y_i) < 0$, so we must have had $\partial f_0(x_i) > 0$, $\partial f_0(y_i) < 0$. ∎

Note: In general flow decomposition is not unique. E.g. in example could have started by removing 3 along $(abce)$.

Consequences:

(F19) For every feasible xy-flow f_0, there is a feasible acyclic xy-flow f_A of equal value. In particular, there exists a maximum xy-flow that is acyclic.

(F20) Every nonnegative acyclic xy-flow f_0 of positive value can be written as a positive linear combination of flows along directed xy-paths.

(F21) Every nonnegative circulation f_0 can be decomposed entirely as $f_0 = f_C = \alpha_1\chi_{C_1} + \alpha_2\chi_{C_2} + \ldots + \alpha_s\chi_{C_s}$: $t = 0$ in FDT(ii) because there are no vertices v with $\partial f_0(v) \neq 0$.

(F22) An acyclic digraph D has no nonnegative circulation f_0 except the zero flow: by (F21) $f_0 = \alpha_1\chi C_1 + \ldots + \alpha_s\chi C_s$, but $s = 0$ since D is acyclic.

(F23) A digraph without isolated vertices in which every vertex has indegree = outdegree is the union of arc-disjoint directed cycles.

(F24) A graph without isolated vertices in which every vertex has even degree is the union of edge-disjoint cycles.