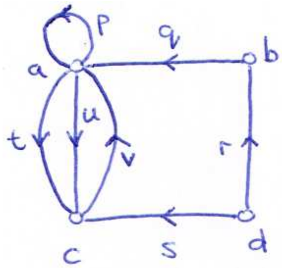
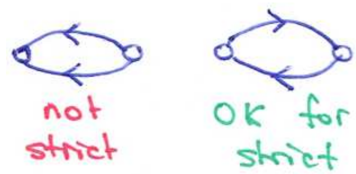
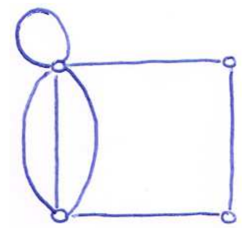
DIRECTED GRAPHS

Reading: 1.5, 2.1, 2.5, 3.4, 6.3



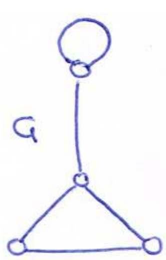


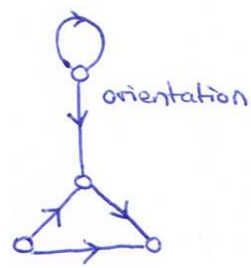


$\psi_D(q) = (b, a) = ba$
$d^+(c) = 1$, $d^-(c) = 3$
$N^+(a) = \{c\}$, $N^-(a) = \{b, c\}$

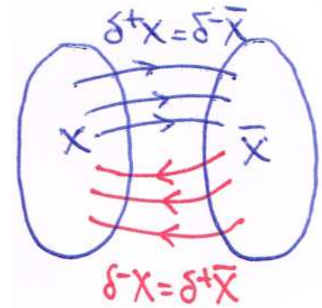
underlying graph

- ◦ *directed graph* or *digraph*: D has vertex set $V(D)$, set of *arcs/* or *directed edges* $A(D)$, incidence function ψ_D mapping each arc to *ordered* pair of vertices.
- ◦ *strict* digraph: no loops or parallel arcs (but opposite arcs are allowed); denote arc as uv.
- ◦ Arc from u to v: *head* v, *tail* u, u *dominates* v.
- ◦ *outdegree* $d^+(v)$, *indegree* $d^-(v)$.
- ◦ Set of *outneighbours* $N^+(v) = \{u \in V(D) \mid u \neq v,\ v \text{ dominates } u\}$; *inneighbours* $N^-(v)$.
- ◦ *underlying graph*: ignore directions.



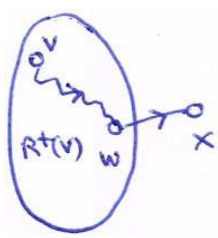


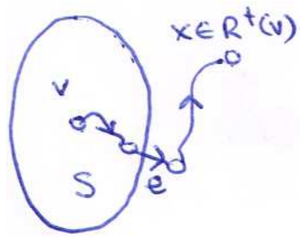
- ◦ *associated digraph* of graph G: replace each edge by pair of opposite arcs.
- ◦ *orientation* of graph G: replace each edge by *one* of possible arcs; *oriented graph* = orientation of simple graph.
- ◦ *tournament*: orientation of complete graph K_n.
- ◦ *source*: vertex of indegree 0; *sink*: vertex of outdegree 0.
- ◦ *converse* of D: reverse all arcs.

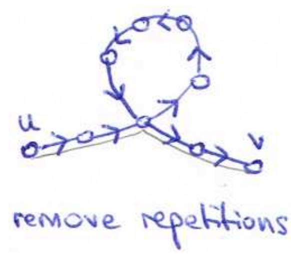


Moving around in digraphs: Have directed versions of walks, trails, paths, cycles: must follow edges in correct direction. Directed uv-walk goes *from u to v*.

- ◦ *connected*: underlying graph connected.
- ◦ $A_D(X, Y) = A(X, Y)$ = edges with tail in X, head in Y for $X, Y \subseteq V(D)$, not necessarily disjoint.
- ◦ $\delta^+_D(X) = \delta^+(X) = \delta^+X = A(X, \overline{X})$ and $\delta^-X = A(\overline{X}, X)$, where $\overline{X} = V(D) - X$. Notation in book is confused (p. 59, 62 of 2nd pr.) Standard symbol here is δ (coboundary), book uses ∂ (boundary). Probably trying to keep δ for minimum degree, but generally there's no confusion.
- ◦ *strong* or *strongly connected*: $\delta^+X \neq \emptyset$ for all proper nonempty subsets X of $V(D)$. (Or equivalently, $\delta^-X \neq \emptyset$ for all such X.)
- ◦ *reachability* in digraphs means directed reachability: uR^+v if there is a directed uv-walk; say *u can reach v* or *v is reachable from u*. Not necessarily an equivalence relation now: not symmetric. Can define converse (transpose) relation R^-.
- ◦ $R^+_D(v)$ means vertices reachable from v; $R^-_D(v)$ means vertices that can reach v.







(D1) $\delta^+(R^+_D(v)) = \emptyset$. If $e \in \delta^+(R^+_D(v))$ has tail w, head x, then $x \in R^+_D(v)$, a contradiction.

(D2) If $v \in S$ and $\delta^+S = \emptyset$ then $R^+_D(v) \subseteq S$. If there was $w \in R^+_D(v) \cap \overline{S}$ then the edge following the last vertex of S on a vw-path would contradict $\delta^+S = \emptyset$.

(D3) (directed M9) $\exists$ a directed uv-walk if and only if $\exists$ a directed uv-path. Remove repetitions.

Theorem D4 (directed M3/M10)**:** For a digraph D, the following are equivalent.
(i) G is strongly connected;
(ii) $\forall\, u, v \in V(D)$ $\exists$ a directed uv-walk (i.e., uRv);
(iii) $\forall\, u, v \in V(D)$ $\exists$ a directed uv-path.

Proof: (i) $\Rightarrow$ (ii) by (D1), (ii) $\Rightarrow$ (i) by (D2), and (ii) $\Leftrightarrow$ (iii) by (D3). ∎

- ◦ *strong component*: maximal strongly connected subgraph (not definition in B&M).

Can use Theorem D4 to show vertex set of strong component is an equivalence class for bidirectional reachability, intersection of R^+ and R^-.

Lemma D5: If D has a nontrivial closed directed walk, then D has a directed cycle.

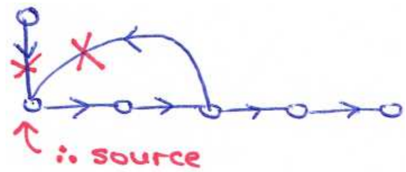
Proof: Proceed until first repeated vertex, must form cycle. ∎

Note: Corresponding result for undirected graphs NOT true because can use same edge in both directions, so get unavoidable repeated edges.

◦ *branching* or *arborescence*: rooted tree where all edges directed outward from root. Can be constructed via *Directed Local TCM*; special cases *Directed BFS* and *Directed DFS* (only consider edges going outward from root).

(D6) (need for flows, later) Starting from v, Directed Local TCM constructs $R^+_D(v)$.

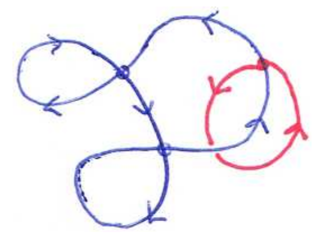
◦ *acyclic* digraph or *DAG*: no directed cycles.



Lemma D7 (directed T2)**:** An acyclic digraph has at least one source and at least one sink.

Proof: Look at ends of a maximal (cannot be extended in either direction) directed path. ∎

Can define *directed euler trail/tour*: again uses all edges **and vertices**.



Theorem D8 (directed M13)**:** A digraph D has a directed euler tour if and only if it is connected and every vertex v has $d^+(v) = d^-(v)$.

Proof similar to undirected version: if maximal trail doesn't use all edges, can find something to splice into it. Note we don't need strongly connected; follows automatically.

Shortest paths

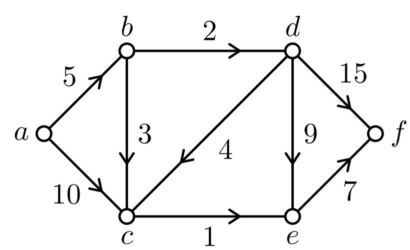
One-terminal shortest path problem: Given digraph D, nonnegative weight (distance) $w(a)$ for each arc, vertex x, find shortest (minimum total weight) directed xv-path for all vertices v (length $d(x, v)$). May assume D strict: loops don't help, for parallel arcs keep one of least weight.

Dijkstra's Algorithm: Loosely, we have a set S of vertices for which we know a shortest path from x (we start off with S empty but can immediately add x). We also have tentative shortest paths to vertices that are one arc away from vertices in S. At each step we choose the vertex u with smallest tentative distance and add it to S, making its tentative shortest path a permanent shortest path. Then we update the tentative shortest paths to other vertices by seeing if we get an improvement going via u.

Formally, we keep a set S ($\overline{S} = V(G) - S$), parent function p (predecessor on shortest path from x), estimate $\ell(v)$ of $d(x, v)$.

for all vertices v {
  $p(v) = \emptyset$; $\ell(v) = \infty$;
}
$S = \emptyset$; $\ell(x) = 0$;

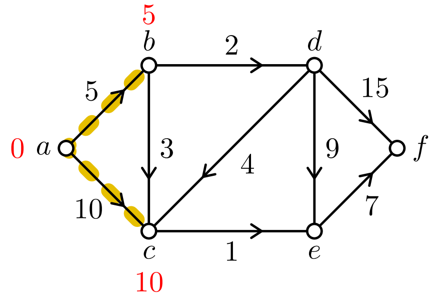
while there is $v \notin S$ with $\ell(v) < \infty$ {
  choose $u \notin S$ with $\ell(u)$ minimum;
  add u to S;
  for each $v \in N^+(u)$ with $v \notin S$ {
    if $\ell(v) > \ell(u) + w(uv)$ {
      $p(v) = u$; $\ell(v) = \ell(u) + w(uv)$;
    }
  }
}



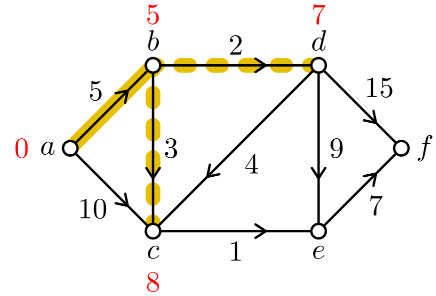
add to S	consequences
a :	b 5, c 10
b :	c 8, d 7
d :	e 16, f 22
c :	e 9
e :	f 16
f :	

Example: See graph above. At each stage we have outbranching with permanent part on vertices of S (solid) and tentative arcs from S to other vertices (dashed). Assume $\ell(v) = \infty$ if no value shown.

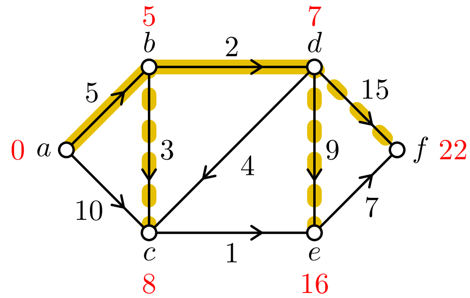
add a:



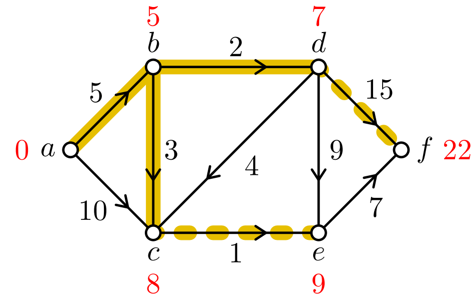
add b:



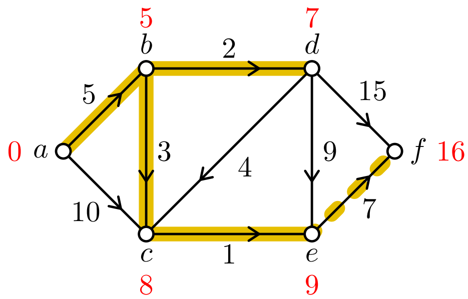
add d:



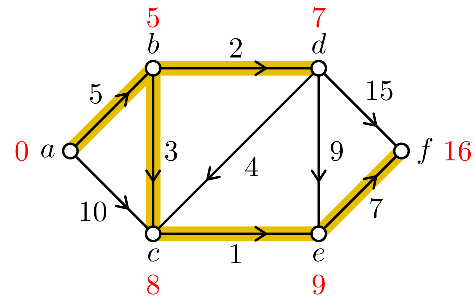
add c:



add e:



add f:

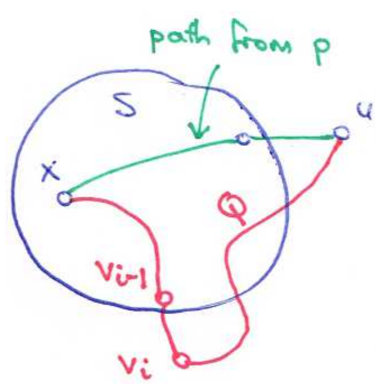


Proof this works: For brevity 'path' means directed path. Any any stage let $F = \{v \mid \ell(v) < \infty\}$. We claim that at the end of the algorithm $S = F = R^+(x)$, and that p indicates an xv-path of length $\ell(u) = d(x, u)$ for every $u \in S$. Observe:

(1) $\ell(u)$ is nonincreasing.
(2) Once we add u to S, $p(u)$ and $\ell(u)$ are fixed, and $\ell(u) < \infty$.
(3) If $u \neq x$ then $\ell(u) < \infty \Rightarrow p(u) \in S$ and following p backwards gives an xu-path of length $\ell(u)$.

By (2) and (3), $S \subseteq F \subseteq R^+(x)$. When we add u to S we ensure that $N^+(u) \subseteq F$, so $\delta^+ S \subseteq A(S, F - S)$. The algorithm ends when $F = S$, which means $\delta^+ S \subseteq A(S, S - S) = \emptyset$, so $R^+(x) \subseteq S$ by (D2). Thus, at the end $S = F = R^+(x)$.

By (3), p agrees with ℓ, and by (2), $\ell(u)$ never changes once $u \in S$. So it suffices to show that $\ell(u) = d(x, u)$ at the point u is added to S. We may assume this is true for vertices already in S. (We can let u be the first vertex for which it fails, or we can argue by induction.) (3) guarantees that $\ell(u) \geq d(x, u)$. Assume that $\ell(u) > d(x, u)$, so when we add u, there is an xu-path $Q = v_0 v_1 \ldots v_k$ of length $< \ell(u)$ ($v_0 = x$, $v_k = u$). Let v_i be the first vertex of Q in $\overline{S}$. Then



$$\begin{aligned}\ell(u) &> w(Q) \geq w(v_0 Q v_{i-1}) + w(v_{i-1}v_i) \geq d(x, v_{i-1}) + w(v_{i-1}v_i) \\ &= \ell(v_{i-1}) + w(v_{i-1}v_i) \qquad \text{since } v_{i-1} \in S, \text{ so } d(x, v_{i-1}) = \ell(v_{i-1}) \\ &\geq \ell(v_i) \qquad \text{since this was true when we put } v_{i-1} \text{ in } S, \text{ and stays true by (1) and (2).}\end{aligned}$$

Thus, $u \neq v_i$ and we should have chosen v_i rather than u, a contradiction. ∎