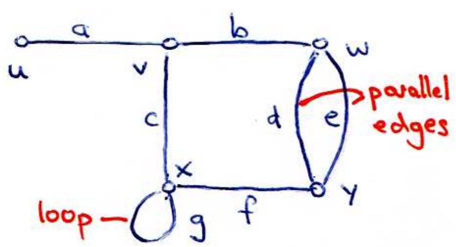
BASIC GRAPH THEORY DEFINITIONS

If book and instructor disagree, follow instructor!

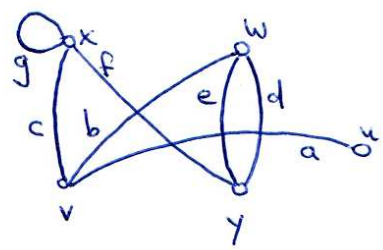
Graphs

○ *graph* G consists of
    vertex set $V(G)$,
    edge set $E(G)$,
    incidence relation ψ_G mapping each edge
        to unordered pair of vertices.

In example:
    $V(G) = \{a, b, c, d, e, f, g\}$,
    $E(G) = \{u, v, w, x, y\}$,
    ψ_G maps $a \mapsto uv$ (i.e. $\psi_G(a) = \{u, v\} = uv$), $b \mapsto vw$, $c \mapsto vx$, $d \mapsto wy$, $e \mapsto wy$, $f \mapsto xy$.



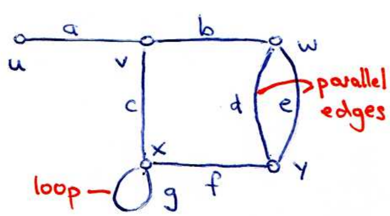
identical to

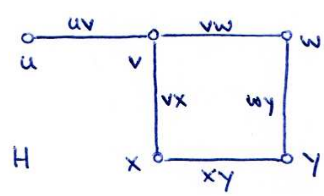


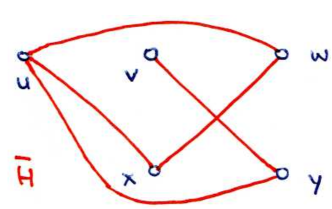
○ *incident* vertex and edge, related by ψ_G (in example, v and b).
○ vertices u and v are *adjacent* if there is an edge e whose two *ends* are u and v, i.e., $\psi_G(e) = uv$ (in example, $w \sim y$ and $x \sim x$).
○ *neighbours* are **distinct** adjacent vertices, $N_G(v)$ or just $N(v)$ is set of neighbours of v (in example, $N_G(x) = \{v, y\}$).

Conventions: If G understood, write V for $V(G)$, E for $E(G)$, n for $|V|$, m for $|E|$.
○ *null graph* has $V(G) = E(G) = \emptyset$.
Assume all graphs are **finite** and **non-null** unless otherwise stated.

○ *loopless* graph: no loops.
○ *simple* graph: no loops or parallel edges: each edge associated by ψ_G with unique pair of distinct vertices. So may as well assume an edge **is** a pair of distinct vertices.
∗ *standard (model) simple graph*: $E(G) \subseteq \binom{V(G)}{2}$ (pairs of distinct vertices), ψ_G is identity map. (My terminology, not standard, not in book.)
○ *underlying (standard) simple graph*: delete loops, reduce parallel edges to single edge.







○ *complement* $\overline{G}$ of standard simple graph G: $V(\overline{G}) = V(G)$, $E(\overline{G}) = \binom{V(G)}{2} - E(G)$: put edges between **nonadjacent** vertices.

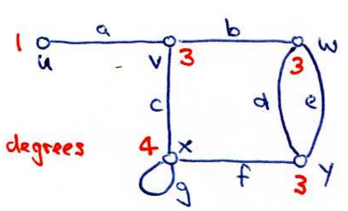
Observe: $\overline{\overline{G}} = G$.

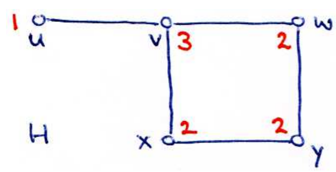
Convention: All simple graphs are assumed to be standard unless otherwise stated. B&M do this implicitly, but do not say so.

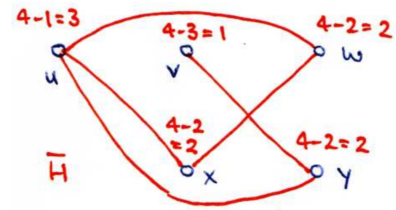
Degrees

○ *degree* $d_G(v)$ or $d(v)$: number of ends of edges incident with v (so loops count as 2).

- *k-regular*: all degrees are k.
- *cubic* = 3-regular.
- *maximum degree* $\Delta(G)$, *minimum degree* $\delta(G)$, *average degree* $d(G)$.
- *degree sequence* of G is list of all degrees (in no particular order, usually put in descending or ascending order). E.g., $4, 3, 3, 3, 1$ for example.







Degree-Sum Formula: $\sum_{v \in V(G)} d_G(v) = 2|E(G)| = 2m$.
Proof: Count ends of edges in two ways. ∎ Typical counting proof idea: count same thing two different ways.) Book proof adds entries in M_G, whichis why they do matrices first.

Example: Is there a 3-regular graph on 47 vertices?
Solution: No, $\sum_{v \in V(G)} d_G(v) = 47 \times 3$, odd so cannot be $2m$. In general sum of degrees must be even, i.e. must have even number of odd degree vertices.

Corollary: $d(G) = 2m/n$.

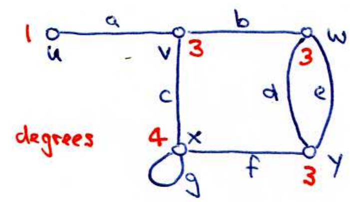
Complement degrees: for simple G, $d_{\overline{G}}(v) = n - 1 - d_G(v)$.

Matrices

- *incidence matrix* ($V \times E$)

$$M_G = \begin{array}{c} \\ u \\ v \\ w \\ x \\ y \end{array} \begin{array}{c} \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

loop indicated by 2 .



- *adjacency matrix* ($V \times V$)

$$A_G = \begin{array}{c} \\ u \\ v \\ w \\ x \\ y \end{array} \begin{array}{c} \begin{array}{ccccc} u & v & w & x & y \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 2 & 1 & 0 \end{bmatrix} \end{array}$$

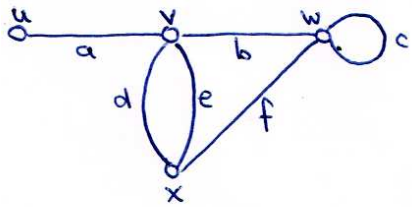
each loop adds 2 (2 ends inc, with vertex)
for parallel edges give number.

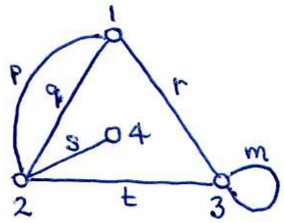
For some purposes may want each loop to count as just 1 in A_G, but counting as 2 means row/column sums are degrees, see below.

Isomorphisms and automorphisms

- *identical* graphs: $V(G) = V(H)$, $E(G) = E(H)$, $\psi_G = \psi_H$.
- *isomorphism* (θ, φ) from G to H:
  - bijection $\theta : V(G) \to V(H)$,
  - bijection $\varphi : E(G) \to E(H)$,
  - preserve incidence: $\psi_G(e) = uv \Leftrightarrow \psi_H(\varphi(e)) = \theta(u)\theta(v)$ for all $e \in E(G)$.

  Then G and H are *isomorphic*, $G \cong H$.





	θ	
u	$\mapsto$	4
v	$\mapsto$	2
w	$\mapsto$	3
x	$\mapsto$	1

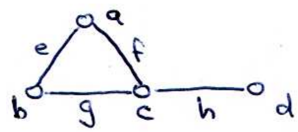
	φ	
a	$\mapsto$	s
b	$\mapsto$	t
c	$\mapsto$	m
d	$\mapsto$	q
e	$\mapsto$	p
f	$\mapsto$	r

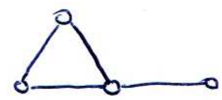
For standard simple graphs enough to give θ. Isomorphism is bijection $\theta : V(G) \to V(H)$ so that $uv \in E(G) \Leftrightarrow \theta(u)\theta(v) \in E(H)$. Then $\phi(uv)$ defined implicitly as $\theta(u)\theta(v)$.

Note: In general definition, enough to show $\Rightarrow$ because we know φ is a bijection. For simple graph definition, $\Rightarrow$ is not enough unless we know $|E(G)| = |E(H)|$ and this is finite.

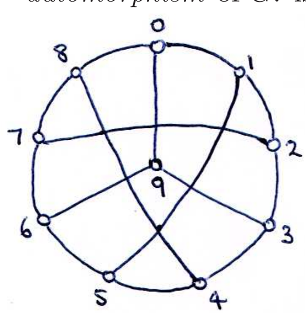
Properties: (1) $(\mathrm{id}_{V(G)}, \mathrm{id}_{E(G)})$ is an isomorphism of G to itself.
(2) If (θ, φ) is an isomorphism from G to H, then $(\theta^{-1}, \varphi^{-1})$ is an isomorphism from H to G.
(3) If (θ_1, φ_1) is an isomorphism from G to H, and (θ_2, φ_2) is an isomorphism from H to J, then $(\theta_2 \circ \theta_1, \varphi_2 \circ \varphi_1)$ is an isomorphism from G to J.
(1), (2) and (3) show that isomorphism is reflexive, symmetric, and transitive, so is an *equivalence relation*.

- *labelled* graph = graph.
- *unlabelled* graph = *isomorphism class* (equivalence class under isomorphism) of graphs (draw as graph without names on vertices or edges).

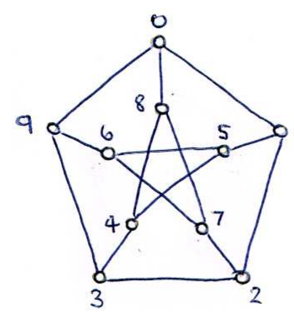




- *invariant* = quantity or property that is same for all isomorphic graphs (does not depend on names of vertices, edges).
- *automorphism* of G: isomorphism from G to G.



$$\begin{array}{lll} & \theta & \\ i & \mapsto & (i+3) \bmod 9 \quad \text{for } i \neq 9 \\ 9 & \mapsto & 9 \end{array}$$

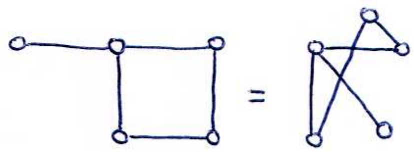


Order 3 automorphism of *Petersen graph*; after redrawing see also has order 5 automorphism.

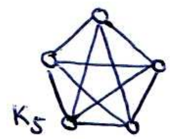
By (1), (2), (3) and associativity of composition, automorphisms form a group, *automorphism group of G*, $\mathrm{Aut}\, G$. (1) gives identity, (2) gives inverse, (3) gives closure.
- *asymmetric* graph: only automorphism is identity.
- *similar* vertices u and v: some automorphism maps u to v (equivalence relation on vertices; also have similarity for edges).
- *vertex-transitive* graph: all vertices similar.

Common invariant properties

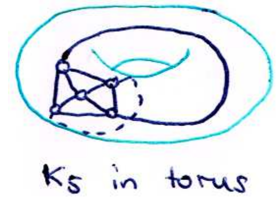
∘ *planar* graph: can be drawn in plane without edge crossings.



planar



not planar



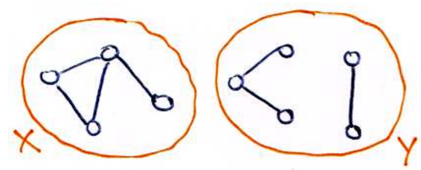
∘ *embeddable in surface* Σ: can be drawn in Σ without edge crossings. E.g., K_5 not planar, but embeddable in torus.

Connectedness

∗ *coboundary* δX of $X \subseteq V(G)$: set of all edges with one end in X, other end in $V(G) - X$. Book uses notation ∂X, perhaps trying to keep δ for minimum degree, but usually no confusion.

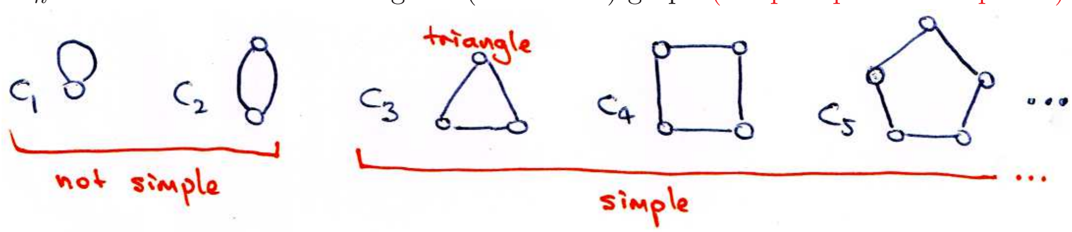
∘ *disconnected* graph has $X \subseteq V(G)$ with $X \neq \emptyset, V(G)$ and $\delta X = \emptyset$. So can partition $V(G)$ into two nonempty sets X, $Y = V(G) - X$ with no edges from X to Y.

∘ *connected* = not disconnected.

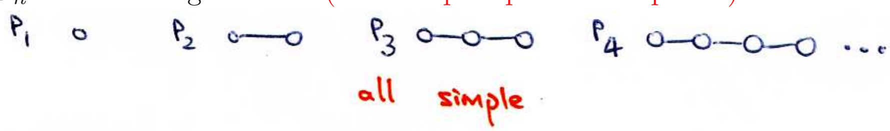


Cycles and paths (definitions slightly different from book but equivalent)

∗ *cycle* C_n = connected n-vertex 2-regular (unlabelled) graph (unique up to isomorphism).



∗ *path* P_n = C_n with one edge deleted (also unique up to isomorphism).



(Book defines these using vertices in cyclic or linear order, join pairs of consecutive vertices.)

Independent sets and cliques

∘ *independent* or *stable* set $S \subseteq V(G)$: no loops on S and vertices of S pairwise nonadjacent.

∘ *independence number* $\alpha(G)$ = maximum number of vertices in an independent set.

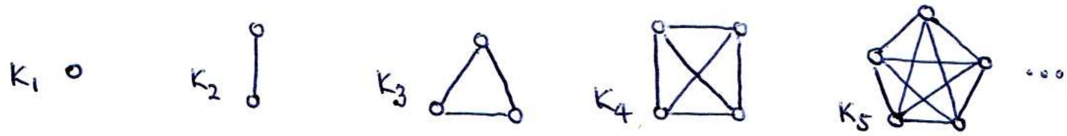
∘ *clique* $S \subseteq V(G)$: any two distinct vertices of S are adjacent (clique may refer to subgraph as well as vertex set).

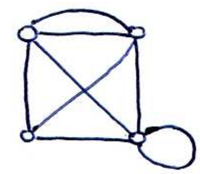
∘ *clique number* $\omega(G)$ = maximum number of vertices in a clique.

For simple graphs $\alpha(\overline{G}) = \omega(G)$, $\omega(\overline{G}) = \alpha(G)$.

Finding $\alpha(G)$, $\omega(G)$ in general is hard (NP-complete for decision version).

∘ *complete graph* K_n: simple n-vertex, any two distinct vertices adjacent (whole vertex set is clique). K_1 is *trivial graph*.

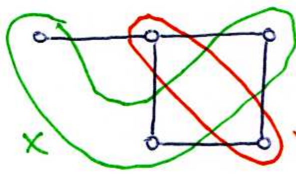




∗ *supercomplete graph*: not necessarily simple, any two distinct vertices are adjacent. Nonstandard, but useful concept later, particularly for vertex connectivity.

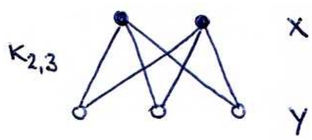
∘ *empty* or ∗ *edgeless graph* $\overline{K_n}$: n vertices, no edges.

∘ *bipartite* graph: can partition $V(G)$ into X, Y (may be empty) so every edge has one end in X, other end in Y. Write $G = G[X, Y]$. So X, Y are independent. Can think of colouring vertices with 2 colours, every edge has ends of different colours.

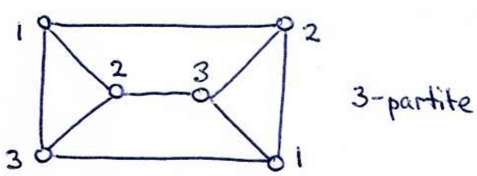


X = solid
Y = open

∘ *complete bipartite* graph $K_{m,n}$: simple $G[X, Y]$, $|X| = m$, $|Y| = n$, every $x \in X$ adjacent to every $y \in Y$. **Not complete and bipartite!**

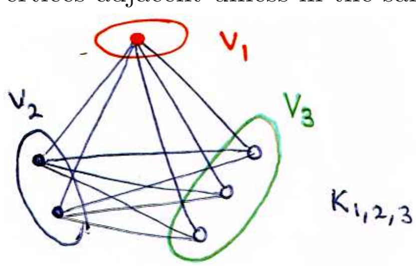


∘ *k-partite* graph: can partition $V(G)$ into $V_1, V_2, \ldots, V_k$ (may be empty) so each V_i is independent. I.e., all edges have ends in two different V_i's. Can think of in terms of k-colouring vertices.



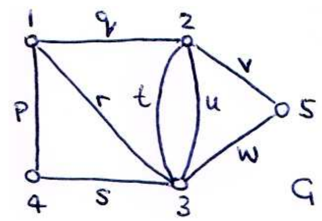
vertices of V_i labelled with i

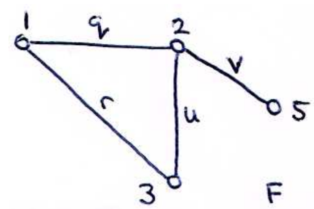
∘ *complete k-partite* graph $K_{n_1,n_2,\ldots,n_k}$: k-partite simple graph with $|V_1| = n_1$, $|V_2| = n_2$, …, $|V_k| = n_k$, simple, any two vertices adjacent unless in the same V_i.



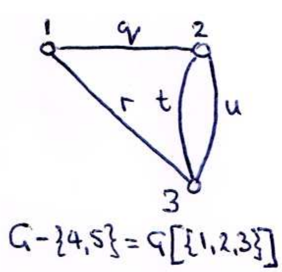
Subgraphs

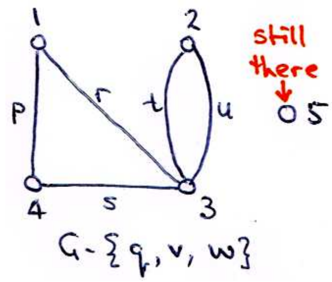
∘ *subgraph* F of G, $F \subseteq G$: graph F, $V(F) \subseteq V(G)$, $E(F) \subseteq E(G)$, and $\psi_F = \psi_G|_{E(F)}$. Also say G is *supergraph* of F.

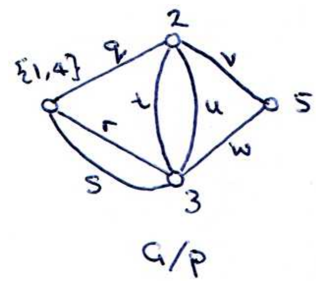




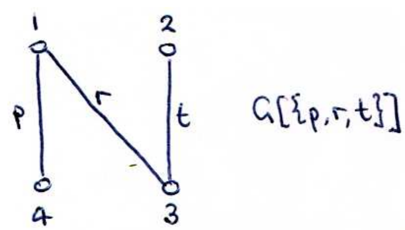
- *proper* subgraph, $F \subset G$: $F \subseteq G$, $F \neq G$. Say G is proper supergraph of F.
- $G - v$: delete v and all incident edges. Repeat: $G - S$, $S \subset V(G)$.
- $G - e$: delete edge e (do not delete any vertices). Repeat: $G - T$, $T \subseteq E(G)$.
- G/e (G *contract* e): delete e, identify ends of e if distinct. Repeat: G/T, $T \subseteq E(G)$.

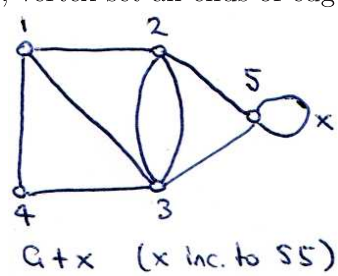






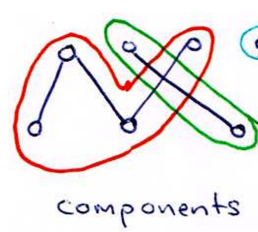
- $G/\{u, v\}$: identify vertices u and v.
- $G + e$: add edge e with known incidences. Repeat: $G + T$, T set of edges.
- *induced* subgraph $G[S]$, $S \subseteq V(G)$: has vertex set S, all edges of G with both ends in S (equivalent to deleting vertices not in S).
- *edge-induced* subgraph $G[T]$, $T \subseteq E(G)$: has edge set T, vertex set all ends of edges in T.



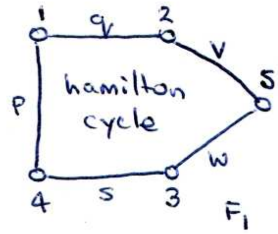


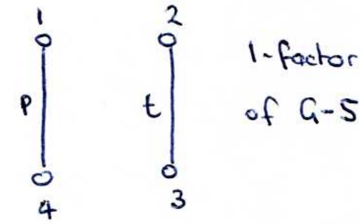
- *disjoint* subgraphs: no common vertices; *edge-disjoint* subgraphs: no common edges (may share vertices).

$*$ *component* = maximal connected subgraph (maximal under subgraph ordering). Book defines in indirect way, hard to use.



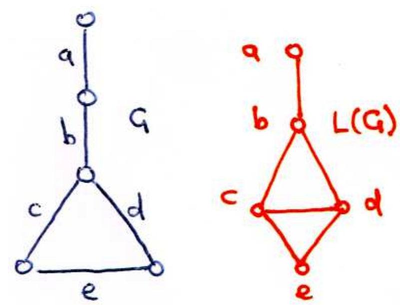
- *spanning* subgraph F of G: $V(F) = V(G)$;
  - spanning path/cycle = *hamilton* path/cycle,
  - spanning k-regular subgraph = k-*factor*.



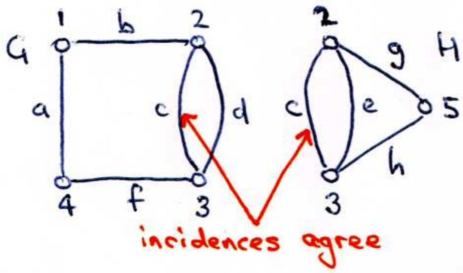


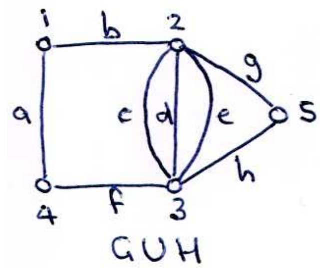
Graphs from other graphs

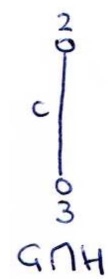
○ *line graph* $L(G)$ of simple G: $V(L(G)) = E(G)$, join e and f if incident with a common vertex ('line' is old term for 'edge').
- Can extend to loopless graphs.
- Not every graph is a line graph.
- Many vertex properties have edge counterparts: vertex property of line graph.



∗ *union and intersection* of graphs G, H can be defined (in obvious way) if *consistent*: $\psi_G(e) = \psi_H(e)\ \forall\ e \in E(G) \cap E(H)$. If G, H both subgraphs of some larger graph then are consistent.

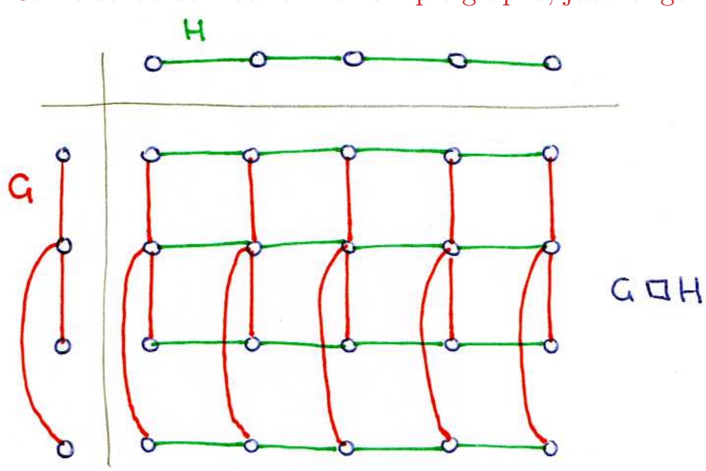




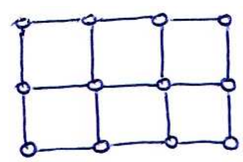


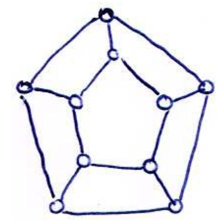
○ *cartesian product* $G\square H$ of simple G, H:
$V(G\square H) = V(G) \times V(H) = \{(u,v) \mid u \in V(G), v \in V(H)\}$;
$E(G\square H) = \{(u_1,v)(u_2,v) \mid u_1u_2 \in E(G), v \in V(H)\} \cup \{(u,v_1)(u,v_2) \mid u \in V(G), v_1v_2 \in E(H)\}$

Old notation $G \times H$. Can also be defined for non-simple graphs, just longer to write down.



○ $m \times n$ *grid* $= P_m\square P_n$; *n-prism* $= C_m\square K_2$.





∘ *join* $G \vee H$ of disjoint G, H: join every vertex of G to every vertex of H. (Also denoted $G + H$.) If simple, $\overline{G \vee H} = \overline{G} \cup \overline{H}$.