**Math 4710/6710 – Graph Theory – Fall 2019**

**Combinatorial induction proofs**

Combinatorial induction proofs often follow a typical framework, as follows.

**Theorem:** Every object with property P has property Q.

**Proof:** By induction on the size $s$ of an object $J$ with property P.

*Basis:* Suppose $s \in \{\text{some small numbers}\}$. Then the result holds because ... .

*Induction step:* Suppose $s \notin \{\text{some small numbers}\}$ and the result holds for all objects of size $< s$. Construct from $J$ some object $J'$ with property P of size $s' < s$. Then $J'$ has property Q. Use this to show that $J$ also has property Q. ∎

Induction proofs as above often also use a 'proof by contradiction' approach, using the idea of a 'minimum counterexample'.

**Alternative proof:** Let $J$ be an object of smallest size $s$ that is a counterexample, i.e., $J$ has property P but does not have property Q.

Then we cannot have $s \in \{\text{some small numbers}\}$ beacuse ... .

So we may suppose that $s \notin \{\text{some small numbers}\}$. Construct from $J$ some object $J'$ of size $s' < s$ with property P. Now conclude that $J'$ has property Q and use that to show that $J$ has property Q, a contradiction. (Alternatively and equivalently, use the fact that $J$ does not have property Q to show that $J'$ does not have property Q, a contradiction.) ∎

The minimum counterexample approach is often used in research when we do not actually know whether a result is true or not. By considering a hypothetical minimum counterexample and studying its properties we have a hope of either proving the result (as above) or actually figuring out how to construct a minimum counterexample and proving that the result is incorrect.

For a standard example of a minimum counterexample proof, see the Wikipedia page on the Fundamental Theorem of Arithmetic (the fact that every integer at least 2 has a unique prime factorization). This gives a minimum counterexample proof of the uniqueness part of this theorem.

A very important combinatorial example of a minimum counterexample proof is the proof of the Four Colour Theorem using an unavoidable set of reducible configurations. This is discussed in Section 15.2 of Bondy and Murty.

**HOW NOT TO DO AN INDUCTION ARGUMENT**

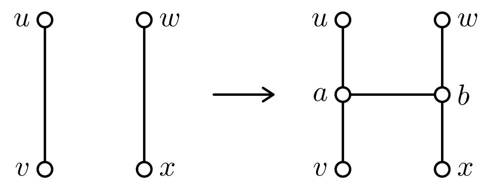
**WRONG WRONG WRONG WRONG WRONG WRONG WRONG WRONG**

**Claim:** Every edge in a connected cubic simple graph $G$ is in a cycle.

**Proof:** By induction on $n$, the number of vertices in $G$. Remember that $n$ must be even.

Basis: If $n = 4$, then $G$ is isomorphic to $K_4$, and every edge of $K_4$ is in a cycle.

Induction step: Suppose that $n \geq 6$, and the result holds for graphs on $(n-2)$ vertices. Let $G'$ be a connected cubic simple graph on $n-2$ vertices; then the result holds for $G'$. Construct $G$ from $G'$ by taking two edges $uv$, $wx$ of $G'$, subdividing these edges to get $uav$ and $wbx$, and adding the edge $ab$. Then $G$ is a connected cubic simple graph on $n$ vertices. We prove that every edge $e$ of $G$ is in a cycle.




Every cycle $C'$ in $G'$ extends to a cycle in $G$: if $uv$ is an edge of $C'$ we replace it by $uav$, and if $wx$ is an edge of $C'$, we replace it by $wbx$.

If we have an edge $e \in E(G) - \{ua, av, wb, bx, ab\}$ then $e$ is also an edge of $G'$, so by induction it is in a cycle $C'$ in $G'$, which extends to a cycle $C$ of $G$ containing $e$, as above.

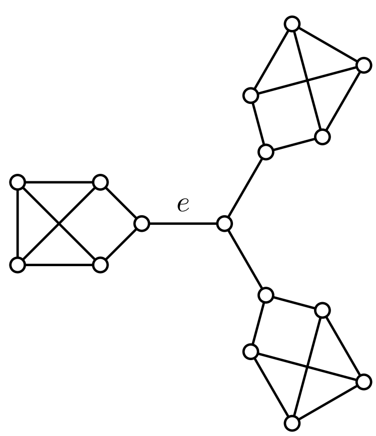
If $e = ua$ or $av$, we find by induction a cycle $C'$ in $G'$ containing $uv$, and then we extend $C'$ to a cycle $C$ in $G$, which contains $e$. Similarly, if $e = wb$ or $bx$, we extend a cycle $C'$ containing $wx$ in $G'$ to a cycle $C$ in $G$ containing $e$.

Finally, suppose $e = ab$. Since $G'$ is connected, there is a path from $\{u, v\}$ to $\{w, x\}$ in $G'$; let $P'$ be a shortest such path. Then $P'$ does not contain $uv$ or $wx$ (because if it did we could find a shorter path). Suppose the ends of $P'$ are $c \in \{u, v\}$ and $d \in \{w, x\}$. Then $C = cabd \cup P'$ is a cycle containing $e$ in $G$.

Since every edge of $G$ is in a cycle, the result follows. ∎

**WRONG WRONG WRONG WRONG WRONG WRONG WRONG WRONG**

But this is obviously false since $e$ in the graph below is not in a cycle.




What is wrong with the above proof? (You may want to circle or underline bits of the proof, if that is helpful.)