Math 4710/6710 – Graph Theory – Fall 2019

Basic concepts I, I, III

* indicates a definition different from the one in the book.

Graphs

Graph G consists of vertex set $V(G)$, edge set $E(G)$, incidence relation ψ_G mapping each edge to unordered pair of vertices.
Convention: $n = |V(G)|$, $m = |E(G)|$.
If $\psi_G(e) = uv$ say e *incident* with u, v.
Parallel edges are incident with same vertices. *Loop* is incident with same vertex twice.
Adjacent vertices u, v: some edge with $\psi_G(e) = uv$. Write $u \sim v$. (Can have $u \sim u$ if loop at u.)
Neighbor of v is u with $u \neq v$, $u \sim v$. (A vertex is never its own neighbor.) $N_G(v)$ or $N(v)$ is set of neighbors of v.
Null graph: $V(G) = E(G) = \emptyset$. Usually assume all graphs nonnull and finite.
Loopless graph has no loops. *Simple* graph has neither loops nor parallel edges.
* *Standard (model) simple graph* has $E(G) \subseteq \binom{V(G)}{2}$, ψ_G defined implicitly as identity map. Usually assume all simple graphs are standard model.
Underlying (standard) simple graph: delete loops, reduce parallel edges to single edge.
Complement $\overline{G}$ of standard simple graph G: $V(\overline{G}) = V(G)$, $E(\overline{G}) = \binom{V(G)}{2} - E(G)$. Note that $\overline{\overline{G}} = G$.

Matrices

Incidence matrix M_G: index rows by vertices, columns by edges, ve-entry is how many times e is incident with v (2 for a loop).
Adjacency matrix A_G: index both rows and columns by vertices. For $u \neq v$, uv-entry is number of edges incident with both u and v. For $u = v$, uu-entry is TWICE number of loops incident with u.

Degrees

Degree $d_G(v)$ or $d(v)$: number of ends of edges incident with v. (Loops count twice. Degree is sum of v's row in M_G, and sum of v's row or v's column in A_G.)
k-regular: all degrees are k. *Cubic* = 3-regular.
Maximum degree is $\Delta(G)$, *minimum degree* is $\delta(G)$, *average degree* is $d(G)$.
Degree sequence of G: list of all degrees, in no particular order. (Technically, multiset of degrees. Often convenient to list in ascending or descending order.)
Degree-Sum Formula: $\sum_{v \in V(G)} d_G(v) = 2|E(G)| = 2m$. *Proof:* Count ends of edges in two ways. ∎ (Hence sum of degrees must be even.)
Corollary: $d(G) = 2m/n$.
Complement degrees: $d_{\overline{G}}(v) = n - 1 - d_G(v)$.

Isomorphisms and automorphisms

Identical graphs G and H have $V(G) = V(H)$, $E(G) = E(H)$, $\psi_G = \psi_H$.
Isomorphism from G to H is pair (θ, φ) where $\theta : V(G) \to V(H)$ is a bijection, $\varphi : E(G) \to E(H)$ is a bijection, and incidence is preserved: $\psi_G(e) = uv \Leftrightarrow \psi_H(\varphi(e)) = \theta(u)\theta(v)$ for all $e \in E(G)$. If isomorphism exists say G and H are *isomorphic*, write $G \cong H$.
For standard simple graphs G and H, just give θ. If $u \sim v$ in $G \Leftrightarrow \theta(u) \sim \theta(v)$ in H for all $u, v \in V(G)$ then we have isomorphism; φ is given by natural extension of θ to pairs of vertices.
For finite graphs, if know $|E(G)| = |E(H)|$ and θ, φ are bijections, enough to check one direction, $\psi_G(e) = uv \Rightarrow \psi_H(\varphi(e)) = \theta(u)\theta(v)$.
Properties of isomorphisms:

(1) $(\mathrm{id}_{V(G)}, \mathrm{id}_{E(G)})$ is an isomorphism of G to itself.
(2) If (θ, φ) is an isomorphism from G to H, then $(\theta^{-1}, \varphi^{-1})$ is an isomorphism from H to G.
(3) If (θ_1, φ_1) is an isomorphism from G to H, and (θ_2, φ_2) is an isomorphism from H to K, then $(\theta_2 \circ \theta_1, \varphi_2 \circ \varphi_1)$ is an isomorphism froM G to K.

For any well-defined class of graphs, (1), (2) and (3) show that isomorphism is reflexive, symmetric, and transitive. Hence it is an equivalence relation.

Labelled graph = graph, as previously defined. *Unlabelled* graph represents equivalence class of graphs, draw picture of graph with no labels on vertices or edges.

Invariant of graphs is property that is same for all isomorphic graphs.

Automorphism of G is isomorphism of G to itself. Properties (1), (2), (3) above, together with associativity of composition, show that set of all automorphisms of G, $\operatorname{Aut} G$, is a group under composition.

Asymmetric graph: only automorphism is identity.

Similar vertices u and v: there is an automorphism mapping u to v; equivalence relation. *Vertex-transitive* graph: all vertices are similar.

Common invariant properties

* *Coboundary* $\delta_G X$ or δX of $X \subseteq V(G)$: set of edges of G with one end in X, other end in $V(G) - X$.

Disconnected if can partition $V(G)$ into two nonempty sets X, Y so no edge from X to Y. Equivalently, there is $X \neq \emptyset, V(G)$ with $\delta X = \emptyset$. *Connected* if not disconnected. (Equivalent condition in terms of paths, later.)

Planar graph: can be drawn in plane without edge crossings. *Embeddable in surface* Σ if can be drawn in Σ without edge crossings.

Cycles and paths

* *Cycle* C_n = connected n-vertex 2-regular (unlabelled) graph.

* *Path* $P_n = C_n$ with one edge deleted.

Independent sets and cliques

Independent or *stable* set $S \subseteq V(G)$: no loops on S and vertices of S are pairwise nonadjacent. *Independence number* $\alpha(G)$ = maximum number of vertices in an independent set.

Clique $S \subseteq V(G)$: any two distinct vertices of S are adjacent. *Clique number* $\omega(G)$ = maximum number of vertices in a clique. (*Clique* also used to refer to subgraph.)

For simple graphs $\alpha(\overline{G}) = \omega(G)$, $\omega(\overline{G}) = \alpha(G)$.

Complete graph K_n: simple n-vertex, any two distinct vertices are adjacent. ($E(K_n) = \binom{V(K_n)}{2}$), whole vertex set is clique.)

Trivial graph is K_1.

* *Supercomplete graph* is not necessarily simple, but any two distinct vertices are adjacent.

Empty or * *edgeless graph* $\overline{K_n}$: n vertices, no edges.

Bipartite graph: can partition $V(G)$ into X, Y (may be empty) so that every edge has one end in X, other end in Y. Write $G = G[X, Y]$. (So X, Y are independent.)

Complete bipartite graph $K_{m,n}$: simple $G[X, Y]$, $|X| = m$, $|Y| = n$, every $x \in X$ adjacent to every $y \in Y$.

k-partite graph: can partition $V(G)$ into $V_1, V_2, \ldots, V_k$ (sets may be empty) so that each V_i is independent.

Complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$: k-partite simple graph with $|V_1| = n_1$, $|V_2| = n_2$, etc., any two vertices adjacent unless in the same V_i.

Subgraphs

Subgraph F of G, $F \subseteq G$: F is a graph, $V(F) \subseteq V(G)$, $E(F) \subseteq E(G)$, and $\psi_F = \psi_G|_{E(F)}$.

Proper subgraph, $F \subset G$: $F \subseteq G$, $F \neq G$.

$G - v$: delete vertex v and all incident edges. Repeat: $G - S$, $S \subset V(G)$.

$G - e$: delete edge e (do not delete any vertices). Repeat: $G - T$, $T \subseteq E(G)$.

G/e (*G contract e*): delete e, identify ends of e if distinct. Repeat: G/T, $T \subseteq E(G)$.

$G/\{u, v\}$: identify vertices u and v.

$G + e$: add edge e with known incidences. Repeat: $G + T$, T set of edges.

Disjoint subgraphs: no common vertices. *Edge-disjoint* subgraphs: no common edges.

* *Component* is a maximal connected subgraph (maximal under subgraph ordering).

Spanning subgraph F of G: $V(F) = V(G)$. Spanning path or cycle usually called *hamilton* path or cycle. Spanning k-regular subgraph usually called *k-factor*.

Induced subgraph $G[S]$ for $S \subseteq V(G)$: vertex set is S, all edges of G with both ends in S.
Edge-induced subgraph $G[T]$ for $T \subseteq E(G)$: edge set is T, vertex set is all ends of all edges in T.

Graphs from other graphs

Line graph $L(G)$ of simple graph G: $V(L(G)) = E(G)$, make e and f adjacent in $L(G)$ if they are incident with a common vertex in G.
* *Union and intersection* of graphs G, H can be defined if they are *consistent*: incidence function agrees for any edges in $E(G) \cap E(H)$. Take union/intersection of vertex and edge sets, and inherit incidence from G or H (or both).
Cartesian product $G \square H$ of simple G, H:
$V(G \square H) = V(G) \times V(H) = \{(u, v) \mid u \in V(G), v \in V(H)\}$;
$E(G \square H) = \{(u_1, v)(u_2, v) \mid u_1u_2 \in E(G), v \in V(H)\} \cup \{(u, v_1)(u, v_2) \mid u \in V(G), v_1v_2 \in E(H)\}$
(Sometimes also denoted $G \times H$.)
$m \times n$ *grid* $= P_m \square P_n$.
Join $G \vee H$ of disjoint G, H: join every vertex of G to every vertex of H by one edge. (Sometimes also denoted $G + H$.) If simple, $\overline{G \vee H} = \overline{G} \cup \overline{H}$.

Moving around

Walk in G: alternating sequence of vertices and edges $W = v_0e_1v_1e_2v_2 \ldots v_{\ell-1}e_\ell v_\ell$ where $\psi_G(e_i) = v_{i-1}v_i$ for each i. (In simple graph can just write $W = v_0v_1v_2 \ldots v_{\ell-1}v_\ell$.) *Length* is ℓ. *Initial vertex* v_0, *terminal* or * *final vertex* v_ℓ, *ends* v_0 and v_ℓ, *internal vertices* $v_1, v_2, \ldots, v_{\ell-1}$.
Reverse of walk W: $W^{-1} = v_\ell e_\ell v_{\ell-1} \ldots v_1e_1v_0$.
uv-walk has initial vertex u, final vertex v.
Closed walk has initial vertex = final vertex.
Trail is walk with no repeated edges.
* *Path* is walk with no repeated vertices. (So defines subgraph that is path graph.)
* *Cycle* is closed walk with no repeated vertices except that initial vertex = final vertex, no repeated edges, and at least one edge. (So defines subgraph that is cycle graph.) In simple graph write $(v_0v_1v_2 \ldots v_{\ell-1})$.
* *Reachability relation* R_G: uR_Gv if there is a uv-walk in G. $R_G(u) = \{v \in V(G) \mid uR_Gv\}$. This is an equivalence relation.

The *distance* from u to v in G, $d_G(u, v)$, is the length of a shortest uv-path.
* *Euler trail* = trail using all edges **and vertices** of G,
Euler tour = closed euler trail,

Trees

acyclic graph or *forest*: no cycles;
tree: (nonnull) connected forest;
leaf: degree 1 vertex.
cutedge e: $G - e$ has more components than G;
cutvertex v: $G - v$ has more components than G.

Lemma: A nontrivial tree has at least two leaves.

Equivalent characterizations of a tree:
(i) connected and acyclic (definition!);
(ii) connected, $m = n - 1$;
(iii) acyclic, $m = n - 1$;
(iv) connected, every edge is a cutedge;
(v) loopless, unique uv-path for all vertices u, v.

Rooted trees: *r-tree* or *tree rooted at* r is tree with special designated vertex r, the *root*. In an r-tree T there is a unique rv-path rTv.
ancestor of v: any vertex of rTv (inc. v)
parent $p(v)$: immediate predecessor on rTv (root has no parent)
proper ancestor (not v itself), *descendant*, *related*

level of v: $\ell(v) = d_T(r, v)$

* **Global Tree Construction Method (Global TCM):** Start with edgeless spanning subgraph F. At each step choose an edge not forming a cycle (equivalently, joining two distinct components) with F and add it to F. When we cannot continue F is a spanning tree.

* **Local Tree Construction Method (Local TCM):** Choose particular vertex r. Apply Global TCM, at each step adding an edge leaving the component containing r.

* **Breadth First Search (BFS):** Use Local TCM, adding uv with $u \in V(T)$, $v \notin V(T)$, where u was added to T as early as possible.

Edge exchange properties: Let T, U be distinct spanning trees of a graph G, and $e \in E(T) - E(U)$.
(EE1) There is $e' \in E(U) - E(T)$ such that $T - e + e'$ is a spanning tree.
(EE2) There is $e'' \in E(U) - E(T)$ such that $U + e - e''$ is a spanning tree.

Kruskal's Algorithm: Apply Global TCM, being greedy, i.e., picking an available edge of minimum weight at each step.

Jarník-Prim Algorithm: Apply Local TCM, being greedy, i.e., picking an available edge of minimum weight at each step.

Directed graphs

directed graph or *digraph*: D has vertex set $V(D)$, set of *arcs/* or *directed edges* $A(D)$, incidence function ψ_D mapping each arc to *ordered* pair of vertices.
strict digraph: no loops or parallel arcs (but opposite arcs are allowed); denote arc as uv.
Arc from u to v: *head* v, *tail* u, u *dominates* v.
outdegree $d^+(v)$, *indegree* $d^-(v)$.
Set of *outneighbours* $N^+(v) = \{u \in V(D) \mid u \neq v,\ v \text{ dominates } u\}$; *inneighbours* $N^-(v)$.
underlying graph: ignore directions.
associated digraph of graph G: replace each edge by pair of opposite arcs.
orientation of graph G: replace each edge by *one* of possible arcs; *oriented graph* = orientation of simple graph.
tournament: orientation of complete graph K_n.
source: vertex of indegree 0; *sink*: vertex of outdegree 0.
converse of D: reverse all arcs.

Moving around in digraphs: Have directed versions of walks, trails, paths, cycles, euler trails and euler tours: must follow edges in correct direction. Directed uv-walk goes *from u to v*.

connected: underlying graph connected.
If $X, Y \subseteq V(D)$, $A(X, Y)$ = edges with tail in X, head in Y. Let $\overline{X}$ denote $V(D) - X$. $\delta^+(X) = A(X, \overline{X})$ and $\delta^-(X) = A(\overline{X}, X)$.
strong or *strongly connected*: $\delta^+(X) \neq \emptyset$ for all proper nonempty subsets X of $V(D)$. (Or equivalently, $\delta^-(X) \neq \emptyset$ for all such X.)
reachability in digraphs means directed reachability: uR^+v if there is a directed uv-walk (or equivalently a directed uv-path); say v *is reachable from* u.
$R^+_D(v)$ means vertices reachable from v; $R^-_D(v)$ means vertices that can reach v.

branching or *outbranching* or *arborescence*: rooted tree where all edges directed outward from root. Can be constructed via *Directed Local TCM*; special cases *Directed BFS* and *Directed DFS* (only consider edges going outward from root).
acyclic digraph (computer scientists call it a *DAG*): no directed cycles.

Lemma: An acyclic digraph has at least one source and at least one sink.

distance in networks: given digraph D, nonnegative weight (distance) $w(a)$ for each arc, $d(u, v)$ is minimum length (total weight) of uv-path.

Flows

Network (D, c): digraph D ($V = V(D)$, $A = A(D)$), each arc has nonnegative *capacity* $c(a)$.
δ^+X, δ^+v, δ^-X, δ^-v: arcs out of/into set of vertices X or single vertex v.
$\overline{X} = V - X$.
A *flow* is $f \in \mathbf{R}^A$, i.e., f is a function $f : A \to \mathbf{R}$.
If $S \subseteq A$ then $f(S)$ means $\sum_{a\in S} f(a)$.
If $X \subseteq V$ and $v \in V$ then $f^+(X)$, $f^+(v)$, $f^-(X)$, $f^-(v)$ mean $f(\delta^+X)$, $f(\delta^+v)$, $f(\delta^-X)$, $f(\delta^-v)$ respectively.
$\partial f(X) = f^+(X) - f^-(X)$ is *net outflow from* X and $\partial f(v)$ is defined similarly.

Proposition (Vertex additivity of net flow): For any $f : A(D) \to \mathbf{R}$ and any $X \subseteq V(D)$, $\partial f(X) = \sum_{v\in X} \partial f(v)$.
Say f *conserved* at v if $f^+(v) = f^-(v)$, i.e., $\partial f(v) = 0$.
f is a *circulation* if f is conserved at all $v \in V$.
Given *supply vertex* x and *demand vertex* y an *xy-flow* (or often just *flow*) is a flow $f : A \to \mathbf{R}$ conserved at every $v \in V - \{x, y\}$.
Feasible flow in (D, c): flow (not necessarily xy-flow) that satisfies $0 \le f(a) \le c(a)\ \forall\ a \in A(D)$.
The *value* of an xy-flow is $\operatorname{val} f = \partial f(x)$ (net flow out of x). (Linear function on xy-flows.)

Special flows: - if P directed xy-path, $\chi_P(a) = 1$ if $a \in A(P)$, 0 otherwise. $\operatorname{val}\chi_P = 1$.
- if P direction-insensitive xy-path, $\overrightarrow{\chi}_P(a) = 1$ if P uses a forwards, -1 if P uses a backwards, 0 otherwise. $\operatorname{val}\overrightarrow{\chi}_P = 1$.
- if C directed cycle (may or may not contain x or y), $\chi_C(a) = 1$ if $a \in A(C)$, 0 otherwise. $\operatorname{val}\chi_C = 0$.

An *xy-cut* is a set of arcs K for which there exists some set of vertices X with $x \in X$, $y \notin X$ and $K = \delta^+X$.
The *capacity* of the cut $K = \delta^+X$ is just $c(K) = c^+(X)$.
A *minimum xy-cut* means an xy-cut of minimum capacity.

Lemma: $\partial f(X) = \operatorname{val} f$ for any xy-cut δ^+X.

Observation: For any feasible xy-flow f and xy-cut $K = \delta^+X$, we have $\operatorname{val} f \le c(K)$. Moreover, equality holds if and only if $f(a) = c(a)\ \forall\ a \in \delta^+X$ and $f(a) = 0\ \forall\ a \in \delta^-X$.

Residual network $(D^*, c^*) = \operatorname{Res}(D, c, f)$: shows how we can modify flow f. Same vertex set as D. Up to two arcs for every arc a of D:
if $f(a) < c(a)$ add a^+, copy of a, to D^* with capacity $c^*(a) = c(a) - f(a)$ (shows we can push extra flow along a);
if $f(a) > 0$ add a^-, opposite to a, to D^* with capacity $c^*(a) = f(a)$ (shows that we can 'push some flow backwards' along a, i.e., reduce flow in a).
f-augmenting path: directed xy-path in D^*.

Observe: If P is an f-augmenting path then we can augment along f to get a new feasible xy-flow of higher value. **Ford-Fulkerson Algorithm** repeatedly searches for f-augmenting path and augments; if no f-augmenting path, vertices X reachable from x in D^* give minimum cut δ^+X. **Edmonds-Karp Algorithm** is special version guaranteed to terminate in polynomial time.

Note: If all capacities integral, F-F Algorithm shows that an integer-valued maximum flow exists.

Max Flow Min Cut Theorem: The value of a maximum xy-flow equals the capacity of a minimum xy-cut.

Note: Can allow infinite capacities, MFMC Theorem still holds.

Note: Vertex capacities $c(v)$ implemented by splitting v into v^- with all in-arcs, v^+ with all out-arcs, and arc v^-v^+ of capacity $c(v)$.

Support of f, $\operatorname{supp} f = \{a | f(a) \ne 0\}$. *Acyclic* flow has acyclic support.

Flow Decomposition Algorithm: Given nonnegative flow f_0, first remove flow around directed cycles (remove circulation f_C) to get acyclic f_A, then remove flow along maximal directed paths to get 0.

Gallai's Flow Decomposition Theorem (FDT): Every nonnegative flow f_0 may be written

$$f_0 = \overbrace{\alpha_1\chi_{C_1} + \alpha_2\chi_{C_2} + \ldots + \alpha_s\chi_{C_s}}^{f_C} + \overbrace{\beta_1\chi_{P_1} + \beta_2\chi_{P_2} + \ldots + \beta_t\chi_{P_t}}^{f_A}$$

where
(i) f_C is a nonnegative circulation, $s \geq 0$, $\alpha_1, \ldots, \alpha_s > 0$, and $C_1, \ldots, C_s$ are directed cycles;
(ii) f_A is a nonnegative acyclic flow, $t \geq 0$, $\beta_1, \ldots, \beta_t > 0$, and each P_i is a directed x_iy_i-path with $\partial f_0(x_i) > 0$, $\partial f_0(y_i) < 0$; and
(iii) if f_0 is integer-valued then we may choose $\alpha_1, \ldots, \alpha_s, \beta_1, \ldots, \beta_t$ to all be integers, so that f_C and f_A are also integer-valued. ∎

New

Connectivity

- *edge cutset:* $S \subseteq E(G)$ so $G - S$ is disconnected.
- *edge cut:* $S \subseteq E(G)$ so that there exists $X \subseteq V(G)$, $X \neq \emptyset, V(G)$, with $S = \delta X$.

$p'(x, y)$ = maximum number of *edge-disjoint* xy-paths.

xy-edge cutset: $S \subseteq E(G)$ so that $G - S$ has no xy-path; $s'(x, y)$ = minimum cardinality of an xy-edge cutset.
xy-edge cut: $S = \delta X = E(X, \overline{X})$ where $x \in X$, $y \in \overline{X}$; $c'(x, y)$ = minimum cardinality of an xy-edge cut.

Observe: Any xy-edge cut is an xy-edge cutset. Any xy-edge cutset contains an xy-edge cut. Hence $s'(x, y) = c'(x, y)$.

Also, $p'(x, y) \leq s'(x, y) = c'(x, y)$ for every distinct x, y. So if they are equal we have maximum number of edge-disjoint xy-paths, minimum xy-edge cutset and minimum xy-edge cut.

Menger's Theorem (Edge Version): If x, y are distinct vertices of a graph G, $p'(x, y) = c'(x, y) = s'(x, y)$.

G is *k-edge-connected* if $G - S$ is connected for all $S \subseteq E(G)$ with $|S| < k$. (Equivalent to $s'/c'/p'(x, y) \geq k$ $\forall$ distinct $x, y \in V(G)$.)
Edge-connectivity $\kappa'(G)$ is maximum k for which G is k-edge-connected.

$p(x, y)$ = maximum number of *internally disjoint* xy-paths (nothing in common except x and y).
xy-vertex cut(set): $S \subseteq V(G) - \{x, y\}$ so that $G - S$ has no xy-path; $c^{\text{v}}(x, y)$ = minimum cardinality of an xy-vertex cutset.

Unit in a graph is either a vertex or an edge;
xy-unit cutset: $U \subseteq (V(G) - \{x, y\}) \cup E(G)$ so $G - U$ has no xy-path;
$c(x, y)$ = minimum size of xy-unit cutset.

Notes: xy-vertex cut only exists if x, y not adjacent, and then $c^{\text{v}}(x, y) = c(x, y)$.

We have $p(x, y) \leq c(x, y)$ so if they are equal we have a maximum number of internally disjoint xy-paths and a minimum xy-unit cutset.

Menger's Theorem (Vertex or Unit Version): If x, y are distinct vertices of a graph G, $p(x, y) = c(x, y)$.

G is *k-connected* if $G - U$ is connected for every set of vertices and edges U with $|U| < k$. (Equivalent to $c/p(x, y) \geq k$ $\forall$ distinct $x, y \in V(G)$.)

Connectivity $\kappa(G)$ is the largest k for which G is k-connected.

$m(x, y)$ = number of edges between x and y.
$x \sim y$ means x, y adjacent.
G is *supercomplete* if every pair of distinct vertices are adjacent. (My term, not standard.)

Observe: If G is supercomplete then $p(x, y) = n - 2 + m(x, y)$ for distinct vertices x, y. Hence $\kappa(G) = n - 2 + \min_{x \neq y} m(x, y)$.

Lemma: If G is not supercomplete and $c(u,v) \geq k$ for all distinct nonadjacent u, v, then $c(x,y) \geq k$ for all distinct adjacent x, y. Hence $\kappa(G) = \min_{x \not\sim y} c(x,y) = \min_{x \not\sim y} p(x,y) = \min_{x \not\sim y} c^{\mathrm{v}}(x,y)$: only need to look at nonadjacent vertices (not adjacent ones) and vertex (not unit) cuts.

Lemma: If G is k-connected and add new vertex v adjacent to at least k vertices of G, result is also k-connected.

Fan Lemma: Suppose G is k-connected and $S \subseteq V(G)$ with $|S| \geq k$, and $x \in V(G)$. Then there are k paths from x to S that are vertex-disjoint except at x and have no internal vertices in S (a *k-fan* from x to S).

Corollary: Suppose G is k-connected and $S, T \subseteq V(G)$ with $|S|, |T| \geq k$. Then there are k vertex-disjoint paths from S to T (with no internal vertices in $S \cup T$).

Application (Dirac): Suppose G is k-connected, $k \geq 2$, and $S \subseteq V(G)$ with $|S| = k$. Then there is a cycle C in G that includes all vertices of S.

Hamilton cycles

hamilton path or cycle: spanning.
hamiltonian graph: has hamilton cycle.
traceable graph: has hamilton path.
$c(G)$: number of components of G.
G is *t-tough* if $c(G-S) \leq |S|/t \;\forall\; S \subseteq V(G)$ with $c(G-S) \geq 2$. Hamiltonian $\Rightarrow$ 1-tough (or just 'tough').

Theorem (Dirac): If G is a simple graph with $\delta \geq n/2$, $n \geq 3$, then G is hamiltonian.

Theorem (Ore): Suppose G is an n-vertex simple graph, $n \geq 3$, and $d(u) + d(v) \geq n$ for all distinct nonadjacent u, v. Then G is hamiltonian.

Lemma: Let G be an n-vertex simple graph with distinct nonadjacent vertices u, v. If $d(u) + d(v) \geq n$ (uv is *addable edge*) then G is hamiltonian $\Leftrightarrow G + uv$ is hamiltonian.

Bondy-Chvátal closure: Given G, repeatedly add addable edges until reach graph G^c with no more addable edges. Can show G^c is unique: *Bondy-Chvátal closure of G*. G hamiltonian $\Leftrightarrow G^c$ hamiltonian. Theorems of Dirac and Ore just cases where G^c is complete.

Chvátal-Erdős Theorem: If $n \geq 3$ and $\kappa(G) \geq \alpha(G)$ then G is hamiltonian.

Matchings

Matching M: set of independent edges (pairwise nonadjacent, no common vertices).
M-saturated vertex: incident with edge of M, otherwise *M-unsaturated*.
Perfect matching or 1-*factor*: saturates all vertices.
$\alpha'(G)$ = size of maximum matching.
M-alternating path: Edges alternately in, not in, M.
M-augmenting path: M-alternating, ends are M-unsaturated.

Berge's Theorem: A matching M is maximum if and only if there is no M-augmenting path.

Vertex cover K is set of vertices, every edge has at least one end in K.
$\beta(G)$ = cardinality of minimum vertex cover.

Observe: $\alpha'(G) \leq \beta(G)$, so if we have matching M and vertex cover K with $|M| = |K|$ then M is maximum and K is minimum.

König-Egerváry Theorem: For bipartite G, $\alpha'(G) = \beta(G)$. (Not true for general graphs.)

König-Ore Formula: For bipartite $G(X,Y)$, $\alpha'(G) = |X| - \max_{S \subseteq X}(|S| - |N(S)|)$.

Hall's Theorem: Bipartite $G(X,Y)$ has a matching saturating $X \Leftrightarrow |N(S)| \geq |S| \;\forall\; S \subseteq X$.

Corollary: Every k-regular bipartite graph, $k \geq 1$, has a perfect matching. Hence every k-regular bipartite graph, $k \geq 0$, has a partition of its edges into perfect matchings (a 1-*factorization*).

Defect of M is $\mathrm{def}(M)$ = number of M-unsaturated vertices $= n - 2|M|$.

Observe: For any matching M and $S \subseteq V(G)$, $\text{def}(M) \geq c_{\text{odd}}(G - S) - |S| = \text{shf}(S)$, *shortfall* of S (not standard term).

Berge's Formula, 1958: For any G,

$$\min_{\text{matchings } M \text{ of } G} \text{def}(M) = \max_{S \subseteq V(G)} \text{shf}(S)$$

$$\text{or equivalently} \quad \alpha'(G) = \frac{1}{2}\left(|V(G)| - \max_{S \subseteq V(G)} (c_{\text{odd}}(S) - |S|)\right).$$

Tutte's 1-Factor Theorem: G has a perfect matching $\Leftrightarrow c_{\text{odd}}(S) \leq |S| \; \forall \; S \subseteq V(G)$.

Colourings

k-colouring: $c : V(G) \to S$, $|S| = k$ (often $S = \{1, 2, \ldots, k\}$.
Proper colouring: no two adjacent vertices get the same colour.
k-colourable: G has a proper k-colouring.
Chromatic number $\chi(G)$: smallest k for which G is k-colourable.

Brooks' Theorem: If G is simple and connected and not a complete graph or odd cycle, then $\chi(G) \leq \Delta(G)$.

Chromatic polynomial $P(G, k)$ is number of proper k-colourings of G with colours $1, 2, \ldots, k$. Turns out to be polynomial in k.
$P(K_n, k) = k(k-1)(k-2)\ldots(k-n+1)$.
$P(\overline{K_n}, k) = k^n$.
$P(T, k) = k(k-1)^{n-1}$ for any n-vertex tree T.

Expansion formula: If $xy \notin E(G)$, $P(G, k) = P(G + xy, k) + P(G_{x=y}, k)$.

Deletion-contraction formula: If $xy \in E(G)$, $P(G, k) = P(G - xy, k) - P(G/xy, k)$.

Euler's formula: Let G be a *plane graph* (specific crossing-free drawing of planar graph) with r *faces* (regions determined by graph, including outside). If G is connected then $n - m + r = 2$.

- *degree* $d(f)$ of face f = total length of all boundary walks.
- $F(G)$ = set of faces of G.

Face Degree-Sum Formula: $\sum_{f \in F(G)} d(f) = 2m$. (Every face has two sides. Or apply Degree-Sum Formula to dual.)

Theorem. Let G be a simple planar graph with $n \geq 3$. Then $m \leq 3n - 6$.

Corollary. K_5 is not planar.

Corollary. (a) The average degree of a simple planar graph is less than 6. (b) Thus, a planar graph G must have a vertex of degree at most 5.

Observation: Every planar graph is 6-colourable (by greedy colouring).

Five Colour Theorem. Every planar graph G is 5-colourable.

Four Colour Theorem (Appel and Haken, 1976)**:** Every planar graph G is 4-colourable.