Math 4710/6710 – Graph Theory – Fall 2019

Assignment 5, due in class, Friday 22nd November

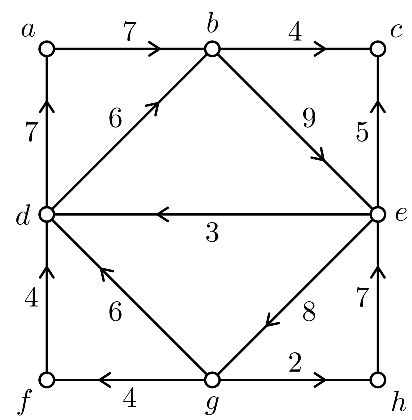
Please note:

- Solutions to problems should be fully explained, using clear English sentences where necessary.
- Solutions to 5.1 may be handwritten. **Solutions to the proof problems 5.2, 5.3 and 5.4 must be typed or written in pen, NOT PENCIL.** All solutions are to be submitted in hard copy, not electronic form, on the due date.
- Solutions should be written or typed (as appropriate) neatly on one side only of clean paper with straight (not ragged) edges.
- Multiple pages should be stapled (not clipped or folded) together.

5.1. Use the Flow Decomposition Algorithm described in class to decompose the flow f shown in the digraph below into a positive linear combination of flows along directed cycles and directed paths. Document your working as follows. At each step show the current remaining flow on a copy of the graph, highlight the edges of the cycle or path along which you are removing flow, say how much flow you are removing along that cycle or path, and then show the new flow on a new copy of the graph. Summarize your results by writing $f = \alpha_1\chi_{C_1} + \ldots + \alpha_s\chi_{C_s} + \beta_1\chi_{P_1} + \ldots + \beta_t\chi_{P_t}$ where you give explicit values for $\alpha_1, \ldots, \alpha_s, \beta_1, \ldots, \beta_t$ and explicit descriptions of $C_1, \ldots, C_s, P_1, \ldots, P_t$.

Based on your decomposition, show on two copies of the digraph a nonnegative circulation f_C and a nonnegative acyclic flow f_A that sum to f.

Note: the class web site contains a PDF file with a page containing multiple copies of this digraph, which you may use for your answer.



5.2. Use the Flow Decomposition Theorem to show that in a network $N = (D, c)$ with distinct vertices x and y there is always a maximum xy-flow that has zero flow on every arc entering x or leaving y.

5.3, corrected version of B&M (2nd pr.) 9.2.2. Show that a 3-connected nonbipartite loopless graph contains at least four odd cycles. [Hints: Make sure you consider the case where G has only three vertices. You can do this question by considering lots of cases, but there is an easier way: pair up cycles so that one of each pair must be odd.]

5.4, adapted from 7.2.35 from West's book (2nd ed.). (Ore, 1963) A graph is *hamilton-connected* if for every pair of distinct vertices x, y there is a hamilton xy-path.

(a) Prove that a simple graph G with $n \geq 2$ vertices is hamilton-connected if $d(u) + d(v) \geq n + 1$ for all distinct nonadjacent vertices u, v. [Hint: Prove that appropriate graphs related to G are hamiltonian by considering their Bondy-Chvátal closures.]

(b) [**Math 6710 only**] Prove that (a) is sharp by constructing, for each even $n \geq 4$, a simple n-vertex graph with minimum degree $n/2$ that is not hamilton-connected. [Hint: try something with a vertex cut of size 2.]