Math 4710/6710 – Graph Theory – Fall 2019

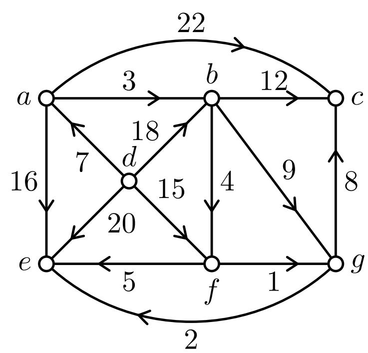
Assignment 4, due in class, Monday 28th October

Please note:

- Solutions to problems should be fully explained, using clear English sentences where necessary.
- Solutions to 4.1 and 4.2 may be handwritten. **Solutions to the proof problem 4.3 must be typed or written in pen, NOT PENCIL.** All solutions are to be submitted in hard copy, not electronic form, on the due date.
- Solutions should be written or typed (as appropriate) neatly on one side only of clean paper with straight (not ragged) edges.
- Multiple pages should be stapled (not clipped or folded) together.

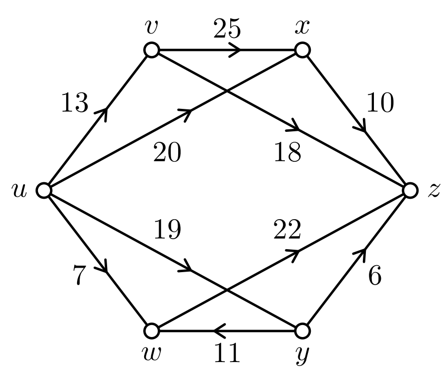
4.1. Apply Dijkstra's Algorithm to the weighted digraph shown below to find shortest paths from d to every other vertex. At each step draw a *separate copy* of the graph, showing the current outbranching, with permanent arcs solid and tentative arcs dashed, and with the value of $\ell(v)$ (distance or tentative distance) shown next to each vertex v.

Note: the class web site contains a PDF file with a page containing multiple copies of this digraph, which you may use for your answer.



4.2. In the network below, use the Ford-Fulkerson algorithm, starting from the zero flow, to find a maximum flow from u to z. You may augment along more than one path at each step if the paths are arc-disjoint. At each step you should **explicitly show the residual network**, show in the residual network the path or paths you are augmenting along, say by how much you are augmenting along each path, and then show the new flow. Prove that your answer is optimal by finding the set reachable from u in the residual network, and using that to find a corresponding minimum cut.

Note: the class web site contains a PDF file with a page containing multiple copies of this network, which you may use for your answer.



4.3. All students should do parts (a)–(d). Students in Math 6710 should also do (e).

(a) Let D be the directed path $xaby$ of length 3. If $S = \{x, a\}$ then since $x \in S$ and $y \notin S$, $\delta^+ S = \{ab\}$ is by definition an xy-cut. Find another set of vertices T with $x \notin T$ and $y \in T$ (the opposite of what we had for S!) so that $\delta^+ T = \delta^+ S$.

Part (a) means we cannot just say 'Let $\delta^+ S$ be an xy-cut' and assume that this means that $x \in S$ and $y \notin S$. We will have to be specific about this in the rest of the question.

For the rest of this problem, suppose D is a digraph, and suppose c is a nonnegative real-valued function defined on the arcs of D.

(b) If $S, T \subseteq V(D)$, show that $c(\delta^+(S \cap T)) + c(\delta^+(S \cup T)) \leq c(\delta^+ S) + c(\delta^+ T)$. [Hint: Write $\overline{S}$ for $V(D) - S$ and so on. Break the vertex set into four parts, $V_0 = S \cap T$, $V_1 = S \cap \overline{T}$, $V_2 = \overline{S} \cap T$ and $V_3 = \overline{S} \cap \overline{T}$, and break each of the sets of arcs in the inequality down into arcs going between pairs of these four sets. A picture may be helpful.]

(c) Prove that if each of S and T contains x but not y, then each of $S \cap T$ and $S \cup T$ also contains x but not y.

(d) Using (b) and (c), prove that if $N = (D, c)$ is a network, each of S and T contains x but not y, and each of $\delta^+ S$ and $\delta^+ T$ is a minimum capacity xy-cut, then each of $\delta^+(S \cap T)$ and $\delta^+(S \cup T)$ is also a minimum capacity xy-cut.

(e) [**Math 6710 only**] Prove that if $N = (D, c)$ is a network where the capacities in c are all positive, each of S and T contains x but not y, and each of $\delta^+ S$ and $\delta^+ T$ is a minimum capacity xy-cut, then D has no arcs in either direction between $S \cap \overline{T}$ and $\overline{S} \cap T$.