Math 4700/6700 – Combinatorics – Spring 2019

Extra Questions for Topic 2

Laurent series, compositions and the Lagrange inversion theorem

A *Laurent series* is like a formal power series but allows finitely many negative (integer) powers of x, so it looks like $P(x) = p_{-n}x^{-n} + p_{-n+1}x^{-n+1} + \ldots + p_{-1}x^{-1} + p_0 + p_1x + p_2x^2 + \ldots$ for some nonnegative integer n. Another way to say this is that a Laurent series looks like $x^{-n}A(x)$ where $A(x)$ is a formal power series. Formal power series themselves are Laurent series (take $n = 0$). The set of Laurent series with complex coefficients is denoted $\mathbb{C}((x))$ and elements can be added, multiplied, and multiplied by scalars in the expected way. We sometimes just write a Laurent series as $\sum_k p_kx^k$ where the sum is assumed to be over a set of integers with only finitely many negative elements.

The addition and multiplication operations for formal power series can be extended to Laurent series in the natural way. Term-by-term differentiation can also be defined for Laurent series by $P'(x) = D_xP(x) = \frac{d}{dx}P(x) = \sum_k kp_kx^{k-1}$. This is a linear operator and can be shown to obey rules like the Product Rule and Chain Rule.

The *valuation* $\operatorname{val} P(x)$ of a Laurent series $P(x)$ is the smallest power of x with a nonzero coefficient. So $\operatorname{val}(1 + x + x^2 + x_3 \ldots) = 0$ (lowest power is x^0), $\operatorname{val}(4x^{-2} + 5 + 6x^2 + 7x^4 + \ldots) = -2$ (lowest power is x^{-2}), and so on.

Computations using formal power series, or Laurent series, are well defined provided only finitely many nonzero terms contribute to the coefficient of each power of x. Given this, we can extend things like linearity (which preserves finite linear combinations) to well-defined infinite sums. For example, if $T : \mathbb{C}[[x]] \to \mathbb{C}[[x]]$ (or $T : \mathbb{C}((x)) \to \mathbb{C}((x))$) is a linear operator then $T(\sum_{k=0}^{\infty} a_kx^k) = \sum_{k=0}^{\infty} a_kT(x^k)$.

Below are some problems. In solving them, you may use results from an earlier problem when proving a later problem. For example, when solving W3 you may use W1 and W2 (even if you did not solve W1 and W2 yourself).

The rule for these is as follows. You may substitute up to four W problems (any four) for four regular homework problems, as follows:

- You may substitute a W problem for any of 2.20, 2.27, or 2.35.
- You may substitute a W problem for one of 2.29 or 2.30.

Warning: This problem sheet is in its first iteration and there may be bugs (i.e., mistakes!) in some of the problems.

W1. (a) Using the fact that a formal power series $A(x) \in \mathbb{C}[[x]]$ with $\operatorname{val} A(x) = 0$ (i.e., having a nonzero constant term) always has a reciprocal (multiplicative inverse) $A(x)^{-1} \in \mathbb{C}[[x]]$ (also with $\operatorname{val} A(x)^{-1} = 0$), show that every nonzero Laurent series $P(x) \in \mathbb{C}((x))$ has a reciprocal $P(x)^{-1} \in \mathbb{C}((x))$ (also a Laurent series).

(Then we can also form arbitrary negative, as well as positive, integer powers of any nonzero Laurent series. This includes all nonzero formal power series, providing the answer can be a Laurent series.)

(b) Suppose that $P(x) \in \mathbb{C}((x))$. Show that $\operatorname{val} P(x)^n = n \operatorname{val} P(x)$ for any integer n. You should treat positive and negative n as separate cases.

W2. Suppose $P(x) \in \mathbb{C}((x))$ is a Laurent series and $Q(x) = \sum_{k=-n}^{\infty} q_kx^k \in \mathbb{C}((x))$ is also a Laurent series. We define the composition $(Q \circ P)(x) = Q(P(x))$ to mean $\sum_{k=-n}^{\infty} q_kP(x)^k$ provided this makes sense, i.e., provided we are only summing a finite number of nonzero terms to determine the coefficient of any given power of x (positive, zero or negative) and provided only finitely many negative powers of x have a nonzero coefficient.

(a) Explain why $Q(A(x))$ is always defined if $A(x) \in \mathbb{C}[[x]]$ is a formal power series with $\operatorname{val} A(x) \geq 1$ and $Q(x) \in \mathbb{C}((x))$ is an arbitrary Laurent series. (It may be helpful to talk about valuations.)

(b) Prove that if $A(x) \in \mathbb{C}[[x]]$ has $\operatorname{val} A(x) = 1$, then there is a unique formal power series $B(x)$ so that $B(A(x)) = x$. Show also that $\operatorname{val} B(x) = 1$. (Hints: Show that the coefficients of $B(x)$ can be calculated inductively. It may be helpful to write $A(x) = xD(x)$ for some $D(x)$.)

(c) Suppose $A(x) \in \mathbb{C}[[x]]$ has $\operatorname{val} A(x) = 1$. We can apply (b) to $A(x)$ to get $B(x) \in \mathbb{C}[[x]]$ with $B(A(x)) = x$, and then (b) again to $B(x)$ to get $C(x) \in \mathbb{C}[[x]]$ with $C(B(x)) = x$. Assuming composition is associative (which is tedious but not difficult to verify) show that $A(x) = C(x)$.

It follows that each $A(x) \in \mathbb{C}[[x]]$ with $\operatorname{val} A(x) = 1$ has a unique compositional inverse $B(x) = A^{(-1)}(x) \in \mathbb{C}[[x]]$, with $\operatorname{val} B(x) = 1$, such that $A^{(-1)}(A(x)) = A(A^{(-1)}(x)) = x$. Moreover, to show that $B(x) = A^{(-1)}(x)$ it is enough to show one of $B(A(x)) = x$ or $A(B(x)) = x$.

W3. (Proof of Product Rule and Chain Rule) If $P(x) = \sum_k p_k x^k$ is a Laurent series then we define its derivative as $P'(x) = D_x P(x) = \frac{d}{dx} P(x) = \sum_k k p_k x^{k-1}$. This is a linear operator. In this problem $P(x), Q(x)$ are arbitrary Laurent series.

(a) Prove the Product Rule $\frac{d}{dx}(P(x)Q(x)) = P'(x)Q(x) + P(x)Q'(x)$ as follows: (i) first prove it when $P(x) = x^m$ and $Q(x) = x^n$; then use infinite linearity (twice) to show the full Product Rule.

(b) Use the Product Rule and induction to show that $\frac{d}{dx} P(x)^n = nP'(x)P(x)^{n-1}$ for any nonnegative integer n (don't forget to deal with $n = 0$).

(c) Use the Product Rule, part (b) above, and the fact that $P(x)^n P(x)^{-n} = 1$, to show that $\frac{d}{dx} P(x)^n = nP'(x)P(x)^{n-1}$ for any negative integer n.

(d) Use the results above and infinite linearity to prove the Chain Rule for differentiation of Laurent series: if $Q(P(x))$ is defined then $\frac{d}{dx} Q(P(x)) = P'(x)Q'(P(x))$.

W4. (Lagrange Inversion Theorem, First Version).

(a) Prove that $[x^{-1}]P'(x) = 0$ for any Laurent series $P(x)$.

(b) Suppose that $A(x) \in \mathbb{C}[[x]]$ has $\operatorname{val} A(x) = 1$, so that it has a compositional inverse $B(x) = A^{(-1)}(x)$, which also has $\operatorname{val} B(x) = 1$. Prove that for any integers n, k we have

$$n[x^n]B(x)^k = k[x^{n-k}] \left(\frac{x}{A(x)} \right)^n = k[x^{-k}]A(x)^{-n} \quad (*)$$

(where $A(x)^{-n}$ may be a Laurent series), by using the following steps:

- Explain why we may assume $n \geq k$.
- Explain why we can write $C(x) = B(x)^k = \sum_{i \geq k} c_i x^i$. Then use $C(A(x))$ to write x^k as a sum involving powers of $A(x)$.
- Differentiate both sides of your formula for x^k, assuming normal rules for differentiation (linearity, Product Rule, Chain Rule).
- Multiply both sides by $A(x)^{-n}$ to get an equation for two Laurent series.
- Apply the coefficient operator $[x^{-1}]$ to both sides of your equation, and simplify using (a).
- Derive $(*)$ above.

W5. Suppose that $B(x), C(x) \in \mathbb{C}[[x]]$.

(a) Suppose that $B(x) = x\,C(B(x))$ is a valid equation. Show that $\operatorname{val} B(x) = 1$ and $\operatorname{val} C(x) = 0$.

(b) Suppose that $\operatorname{val} C(x) = 0$. Prove that $B(x) = x\,C(B(x))$ if and only if $B(x) = A^{(-1)}(x)$ where $A(x) = x/C(x)$.

Therefore, given $C(x)$ with $\operatorname{val} C(x) = 0$, we can always find $A(x)$ and then $B(x)$, so there is a unique solution to $B(x) = xC(B(x))$.

W6. (Lagrange Inversion Theorem, Second Version).

(a) Suppose that $B(x), C(x) \in \mathbb{C}[[x]]$ and $B(x) = x\,C(B(x))$. Use the Lagrange Inversion Theorem (first version) to prove that for $n, k \in \mathbb{Z}$ we have

$$n[x^n]B(x)^k = k[x^{n-k}]C(x)^n.$$

(b) Deduce that for $B(x)$ and $C(x)$ as in (a), and any $P(x) \in \mathbb{C}((x))$ and $n \geq \operatorname{val} P(x)$, $n \neq 0$, we have

$$[x^n]P(B(x)) = \frac{1}{n}[x^{n-1}]\big(P'(x)C(x)^n\big).$$

(c) Show that if $B(x), C(x) \in \mathbb{C}[[x]]$ satisfy $B(x) = \alpha + xC(B(x))$ where $\alpha \in \mathbb{C}$, then for any $P(x) \in \mathbb{C}((x))$ and $n \geq \operatorname{val} P(x)$, $n \neq 0$, we have

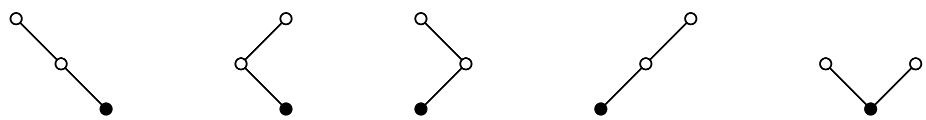
$$[x^n]P(B(x)) = \frac{1}{n}[x^{n-1}]\big(P'(x+\alpha)C(x+\alpha)^n\big).$$

So in future problems we can use the following theorem, which helps when we have a functional equation for a generating function $B(x)$, usually found through some kind of recursive argument.

Lagrange Inversion Theorem: Suppose $B(x), C(x) \in \mathbb{C}[[x]]$, $P(x) \in \mathbb{C}((x))$, $\alpha \in \mathbb{C}$, and we have $B(x) = \alpha + x\,C(B(x))$. Then

$$[x^n]P(B(x)) = \frac{1}{n}[x^{n-1}]\big(P'(x+\alpha)C(x+\alpha)^n\big) \quad \text{provided } n \geq \operatorname{val} P(x) \text{ and } n \neq 0.$$

W7. A *binary tree* has a root vertex, and at each vertex there are zero, one or two upward branches. Each upward branch is either a left branch or a right branch; left branches are different from right branches. For example, all five binary trees with three vertices are shown below (the solid vertex is the root).



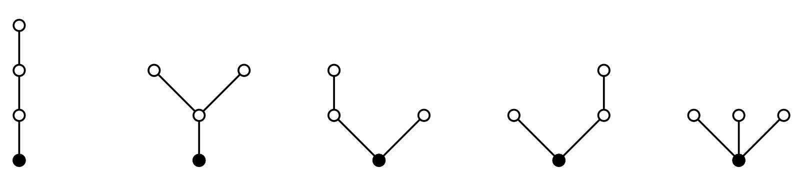
(a) If $B(x)$ is the generating function for binary trees weighted by number of vertices, explain why $B(x) = x(1 + B(x))^2$ (think about left and right branches).

(b) From (a), we can say that $B(x) = xC(B(x))$ where $C(x) = (1+x)^2$. Use this and the Lagrange Inversion Theorem to find the number of n-vertex binary trees for $n \geq 1$.

(c) Now use the Lagrange Inversion Theorem to find the number of n-vertex ternary trees, where a *ternary tree* has between zero and three upward branches at each vertex, and the three possible upward branches, left, centre and right, are all considered to be distinct.

W8. A *planted plane tree* is a tree which is rooted at some vertex and drawn in the plane with all edges going upwards as you travel out from the root. The *updegree* of any vertex (including the root) is the number of edges going upwards from that vertex.

The five planted plane trees on four vertices are shown below (the solid vertex is the root). Note that the third and fourth trees shown are considered distinct. So the order of branches matters, but (unlike binary or ternary trees) branches do not have a designated position (left or right, for example).



(a) If $T(x)$ is the generating function for all planted plane trees, explain why $T(x) = x/(1 - T(x))$, and use this with the Lagrange Inversion Theorem to calculate the total number of planted plane trees on n vertices. Does your answer give 5 trees on 4 vertices?

(b) Explain why the generating function for planted plane trees in which the root has updegree m is $x\,T(x)^m$, and use the Lagrange Inversion Theorem again to find the number of planted plane trees on n vertices in which the root has updegree m. Check your answer for $n = 4$ and $m = 1, 2, 3$ against the figure above.