Math 4700/6700 – Combinatorics – Spring 2019

Questions for Topic 5

5.1. (a) Describe and name the eight symmetries of a square.

The symmetries in (a) form a group which acts on squares whose corners are labelled (i.e. "coloured") with 1, 2, 3 and 4. For example:

1	2
3	4

or

1	4
2	1

.

(b) Find the orbit of

1	2
3	2

.

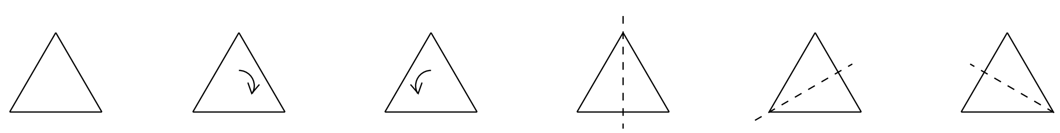
(c) Using your names from (a) for the group elements, find the stabiliser of

2	4
4	2

.

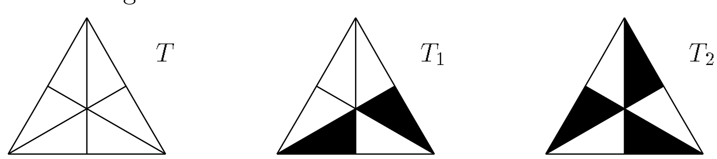
(d) Find the invariant set of the 180° degree rotation.

5.2.



e $\quad r \quad r^2 \quad m \quad n \quad p$

Consider the action of the group of symmetries of an equilateral triangle (shown above) on figures where the six regions of the triangle T shown at left below are each coloured black or white.



(a) Find the orbit of the coloured triangle T_1 shown in the centre above.

(b) Find the stabiliser of the coloured triangle T_2 shown at right above.

(c) Find the invariant set of m.

5.3. Let L and C be two sets, and $S = C^L = \{\text{all functions } s : L \to C\}$. Let $(G, \circ)$ be a permutation group on L (where $\circ$ is the usual (right to left) composition of functions). Show that if we define

$$g \triangle s \;=\; s \circ g^{-1} \qquad \forall\, s \in S,\, g \in G$$

then $\triangle$ is an action of $(G, \circ)$ on S. You must show that $\triangle$ satisfies the following three group action axioms:

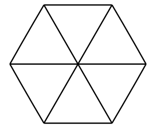
(A1) $g \triangle s$ exists in S $\forall\, s \in S,\, g \in G$;

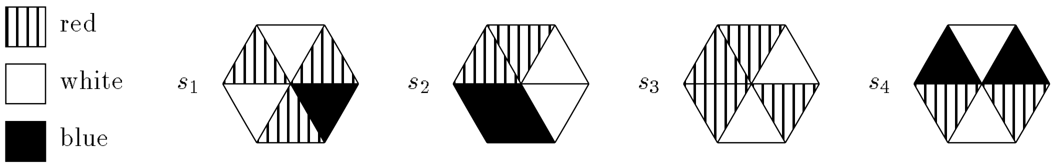
(A2) $e \triangle s = s$ (e = identity of G) $\forall\, s \in S$;

(A3) $(g \circ h) \triangle s = g \triangle (h \triangle s) \quad \forall\, s \in S$ and $g, h \in G$.

5.4. Use Burnside's Lemma to find the number of inequivalent labelled squares in 5.1 above.

5.5. Use Burnside's Lemma to find the number of inequivalent coloured triangles in 5.2 above.

5.6. Let S be the set of figures obtained by colouring the six regions of  with red, white, or blue, for example:



Use Burnside's Lemma to find the number of inequivalent figures in the following subsets of S, where S is acted on by the twelve symmetries (identity, rotations and reflections) of a regular hexagon.

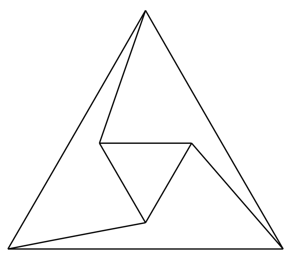
(a) $A = \{s \in S \mid \text{no colour is used more than three times}\}$ (so that $s_1, s_2, s_4 \in A$ but $s_3 \notin A$).

(b) $B = \{s \in S \mid \text{no two adjacent regions are the same colour}\}$ (so that $s_1, s_4 \in B$ but $s_2, s_3 \notin B$).

Note: Part (a) could be solved by using Pólya's Theorem and adding appropriate coefficients from the pattern inventory. However, part (b) cannot be solved using Pólya's Theorem because of the restriction on adjacent regions.

Hint: You will need to find $|A|$ and $|B|$. To do this, you may wish to use inclusion-exclusion.

5.7. Consider the S of figures obtained by colouring each of the four regions of the equilateral triangle shown below using the two colours purple and yellow.



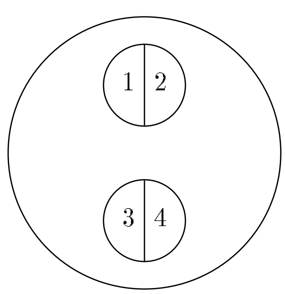
Suppose that S is acted on by G, consisting of e and the two other symmetries (both rotations) of the above object. Define the weight of a figure s with i purple and j yellow regions to be $p^i y^j$. Use the Generalised Burnside's Lemma to find the pattern inventory of S.

5.8. Suppose that G is a group, and that $g \in G$. Then the subgroup generated by g consists of all powers of g: $< g > = \{\ldots, g^{-3}, g^{-2}, g^{-1}, g^0 = e, g, g^2, g^3, \ldots\}$.

(a) Let A be the subgroup of the permutations of $\mathbf{N}_5 = \{1, 2, 3, 4, 5\}$ generated by (1325). List all elements of A.

(b) Find the cycle index of the group A.

5.9. Consider the setup shown below, where two small discs of equal size are mounted at opposite points on a large disc. Each small disc is divided into two semicircular regions. Each small disc can be rotated 180° about its own centre. The large disc can also be rotated 180° about its centre, taking the small discs with it.



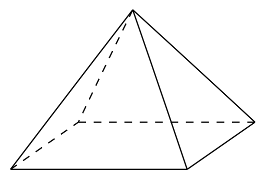
(a) List all elements of the group of permutations of the four semicircular regions corresponding to all allowable sequences of movements.

(b) Find the cycle index of this group.

5.10. Use Polya's Theorem to find the pattern inventory (where $b^i w^j$ represents a figure with i black regions and j white regions) for question 5.2 above.

5.11. Use Polya's Theorem to find the number of inequivalent colourings in the set S (not A or B) of question 5.6 above which have exactly two red regions.

5.12. Use Polya's Theorem to find the number of inequivalent colourings of the faces of a square pyramid with m colours. (Two coloured pyramids are considered equivalent if one can be rotated in space to get the other - reflections are not allowed.)



5.13. Suppose that we have two discs, one larger than the other. Each is divided into three equal sectors. They are mounted concentrically so that each can rotate 120° in either direction about its centre. Each of the six sectors is to be coloured white, red, blue, green or black. In how many inequivalent ways can this be done so that exactly three regions are of a neutral colour (black or white)? (Use Polya's Theorem. Two colourings are considered equivalent if the discs can be rotated so as to change one to the other.)