Math 4700/6700 – Combinatorics – Spring 2019

Questions for Topic 4

4.1. How many rearrangements are there of MURMUR with no two consecutive letters the same?

4.2. Use inclusion-exclusion to find $\phi(252)$, the number of integers between 1 and 252 (inclusive) that are relatively prime to 252 (having no common divisor with 252 except 1). ($\phi(n)$ is an important function in number theory called "Euler's phi function.")

4.3. Out of 143 people, 34 speak none of French, Russian or German. 90 speak no French, 85 no German and 85 no Russian. 56 speak neither French nor German, 55 speak neither French nor Russian, and 54 speak neither German nor Russian.

(a) How many speak all of French, Russian and German?

(b) How many speak French and German?

(c) How many speak French and German but not Russian?

4.4. The card game Five Hundred uses the usual 52-card pack, modified by adding one joker and deleting ten cards: the twos, threes and black fours. A Five Hundred hand has ten cards. Find the number of Five Hundred hands that have at least one card in every suit (a hand is an unordered set of cards; the joker is not in any suit). Find the probability that a Five Hundred hand has at least one card in every suit and state it with an accuracy of 4 significant figures.

4.5. How many rearrangements of the alphabet ABC...XYZ are there that contain

(a) either "THE" or "AND" or both?

(b) neither "THE" nor "MATH"?

4.6. Recall that 12 people can be distributed into 5 rooms so that none is empty in $M = 5!S(12,5)$ ways. Use inclusion-exclusion to calculate M, and from your value of M find the value of $S(12,5)$.

4.7. How many ways are there to seat n couples around a circular table so that no couple sits together? Express your answer as using summation notation. (Note: Rotated seatings are considered to be the same, so that *abcd* is the same as *dabc*, but reflected seatings are considered to be different, so that *abcd* is not the same as *adcb*).

4.8. How many rearrangements of INTELLIGENT are there with

(a) exactly three pairs of identical consecutive letters?

(b) at least two pairs of identical consecutive letters?

4.9. How many integers between 1 and 500 (inclusive) are divisible by exactly two of 3, 4, 7 and 10? How many are divisible by at most two?

4.10. Suppose that we have properties P_1, P_2, ..., P_n. For each $m \geq 0$, let c_m be the number of objects with at least m of the properties (note that $c_m = 0$ if $m > n$).

(a) Let $C(x) = \sum_{m=0}^{n} c_m x^m$. Using the facts that (i) $c_{m-1} - c_m = e_{m-1}$ for $1 \leq m \leq n$, and (ii) $E(x) = S(x-1)$, show that $C(x) = s_0 + x \sum_{k=1}^{n} s_k (x-1)^{k-1}$.

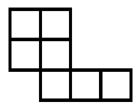
(b) From (a), show that for $1 \leq m \leq n$,

$$c_m = s_m - \binom{m}{m-1} s_{m+1} + \binom{m+1}{m-1} s_{m+2} - \ldots + (-1)^{n-m} \binom{n-1}{m-1} s_n.$$

4.11. Use the formula for e_m and a combinatorial argument to show that for $0 \leq m \leq n$,

$$\binom{n}{m} 9^{n-m} = \sum_{k=m}^{n} (-1)^{k-m} \binom{k}{m} \binom{n}{k} 10^{n-k}.$$

[Hint: think about strings of digits.]

4.12. Find the rook polynomial of  .

4.13. I have four friends, Andrew, Bill, Cynthia and Deborah. Andrew likes yellow and green, Bill likes yellow and red, Cynthia likes white, and Deborah likes white, green and red.

(a) Use the multiplication and/or expansion formulae to find the rook polynomial for the board whose squares denote acceptable person-colour combinations.

(b) In how many ways can I give each of them a hat so that no two people get a hat of the same colour, and so the colour of each hat is acceptable to the recipient?

(c) If I can only afford to give hats to two people, in how many acceptable ways can I do it?

4.14. How many permutations $x_1 x_2 ... x_6$ are there of $\mathbf{N}_6 = \{1, 2, ..., 6\}$ with $x_i \not\equiv 2i \pmod 6$ for $1 \leq i \leq 6$?