Math 4700/6700 – Combinatorics – Spring 2019

Questions for Topic 3

3.1. (a) Formulate a recurrence relation (with initial conditions) for a_n, the number of regions into which n lines in general position (no two parallel, no three meeting at a point) divide the plane.

(b) Find a formula for a_n.

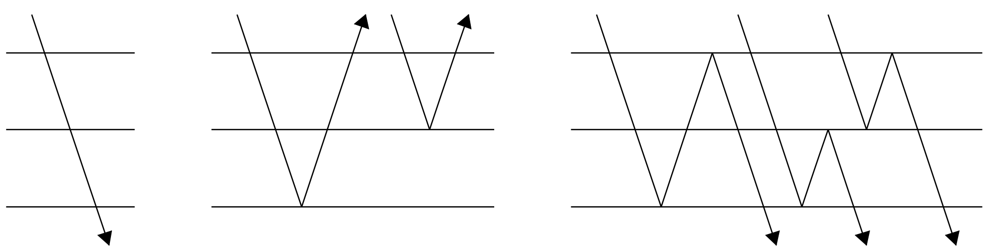
3.2. (a) Find a recurrence relation (with initial conditions) for b_n, the number of ways to arrange red, green, blue and yellow blocks in a row n units long, if red blocks are 1 unit long while green, blue and yellow blocks are 2 units long.

(b) Evaluate b_8 by repeatedly applying the recurrence relation.

3.3. Find a recurrence relation (with initial conditions) for each of the following.
(a) The number of 01-strings of length n with no pair of consecutive 1's.
(b) The number of 012-strings of length n with no pair of consecutive 1's.
(c) The number of 012-strings of length n with no pair of consecutive 1's or consecutive 2's.

3.4. Find a recurrence relation (with initial conditions) for the number of piles of red, green, blue, yellow and white poker chips of height n, with at least one red chip, and with every green chip (if there are any, there may be none) having at least one red chip somewhere below it.

3.5. Find a recurrence relation (with initial conditions) for r_n, the number of ways for an image to be reflected n times by internal faces of two adjacent panes of glass:



$r_0 = 1$ $\qquad$ $r_1 = 2$ $\qquad$ $r_2 = 3$

3.6. Find and solve a recurrence relation (with initial conditions) for the number of 0123-strings of length n in which no two consecutive characters are equal.

3.7. Solve the recurrence relation

$$b_n = 3b_{n-1} + b_{n-2} - 3b_{n-3}, \quad n \geq 3, \qquad \text{with } b_0 = 0,\ b_1 = 1,\ b_2 = 2.$$

3.8. Solve the recurrence relation

$$c_n = 6c_{n-1} - 10c_{n-2}, \quad n \geq 2, \qquad \text{with } c_0 = c_1 = 1.$$

3.9. Suppose we are breeding rabbits. Each pair of rabbits produces two pairs of young at 1 month of age and one pair of young at 2 months of age, and then dies at $2\frac{1}{3}$ months of age. At time $t = 0$ months we have one pair of rabbits, age $\frac{2}{3}$ month. How many pairs of rabbits are there at $t = n$ months?

3.10. An alternative method which can sometimes be used to solve a nonhomogeneous linear recurrence relation involves reducing it to a linear homogeneous recurrence relation.

(a) Show that if $a_n = 3a_{n-1} + 3^n$ for $n \geq 1$, then $a_n = 6a_{n-1} - 9a_{n-2}$ for $n \geq 2$.

(b) Use (a) to solve

$$a_n = 3a_{n-1} + 3^n \text{ for } n \geq 1, \text{ with } a_0 = -1$$

<u>without</u> finding a particular solution (i.e. solve the homogeneous relation $a_n = 6a_{n-1} - 9a_{n-2}$ instead of the given inhomogeneous relation).

3.11. Solve the recurrence relation
$$a_n = 2^n - 2a_{n-1} - a_{n-2}, \quad n \geq 2, \qquad \text{with } a_0 = 0,\ a_1 = 1.$$

3.12. Solve the recurrence relation
$$t_n = 5t_{n-1} - 6t_{n-2} + (2n+3)2^n, \quad n \geq 2, \qquad \text{with } t_0 = 1,\ t_1 = 4.$$

3.13. A standard loan with compound interest and fixed payments can be modelled by a simple recurrence relation. If the initial loan amount is L, the multiplier per time period is I (for example, if the interest is 3% per time period then $I = 1.03$) and the payment per time period is Y, then the principal owing after n time periods is
$$p_n = Ip_{n-1} - Y, \quad n \geq 1 \qquad \text{with} \quad p_0 = L.$$

(a) Solve the recurrence relation to find p_n as a function of n.

(b) Suppose the loan runs for exactly N time periods, so that $p_N = 0$. Find a formula for the payment Y in terms of L, I and N.

(c) Suppose you have a 30-year mortgage for \$400,000, at 4.8% per annum compounded monthly. (If we have an interest rate of t% per annum compounded monthly, that means the monthly interest rate is $(t/12)$%. The real annual interest rate can be quite a bit more than t%!) Find (i) your monthly payment, and (ii) the total amount you pay back to the bank over 30 years.

(d) Suppose you can afford to spend \$2000 per month on mortgage payments (ignore taxes and insurance), and your bank is willing to give you a 30-year loan, at 4.5% per annum compounded monthly, with monthly payments. The bank will lend you up to 80% of the value of your house and you have a rich aunt who will give you the other 20% downpayment. What is the most expensive house you could buy?

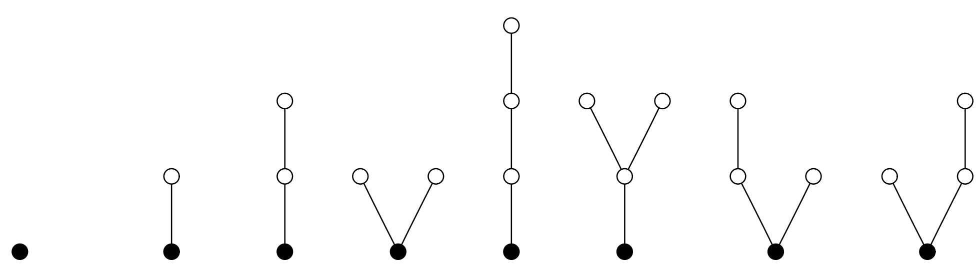
3.14. Use generating functions to solve the following problem: How many piles of discs n cm high can we construct from red and green discs of height 1 cm and black, white and yellow discs of height 2 cm?

3.15. (a) Show that
$$x + 2x^2 + 3x^3 + \ldots \;=\; \sum_{n=0}^{\infty} nx^n \;=\; x(1-x)^{-2}.$$

(b) Use generating functions to solve the recurrence relation
$$\begin{aligned} &p_n = p_{n-1} + 6p_{n-2} + 42n - 97, \quad n \geq 2, \\ &p_0 = 3, \quad p_1 = 15. \end{aligned}$$

3.16. A planted plane tree is a tree which is rooted at some vertex and drawn in the plane with all edges going upwards as you travel out from the root. The updegree of any vertex (including the root) is the number of edges going upwards from that vertex.

Let T be the set of all planted plane trees in which each vertex has updegree 0, 1 or 2. Let t_n be the number of trees in T with n vertices.



$t_0 = 0$ $\qquad$ $t_1 = 1$ $\qquad$ $t_2 = 1$ $\qquad$ —$t_3 = 2$— $\qquad$ ————$t_4 = 4$————

(Notice that the last two trees with 4 vertices are considered to be different.)

(a) Show that t_n satisfies the recurrence relation

$$t_n = t_{n-1} + (t_1 t_{n-2} + t_2 t_{n-3} + \ldots + t_{n-3} t_2 + t_{n-2} t_1) \quad (n \geq 2).$$

(b) Find an expression for the generating function $T(x) = \sum_{n=0}^{\infty} t_n x^n$. (Do not try to find t_n from $T(x)$!)

3.17. A permutation matrix is a square 01-matrix with exactly one 1 in every row and every column, e.g.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

A matrix $[a_{ij}]$ is symmetric if $a_{ij} = a_{ji}$ for all i and j - the above permutation matrix is symmetric. [The $n \times n$ symmetric permutation matrices are in one-to-one correspondence with permutations σ of $\{1, 2, \ldots, n\}$ that are involutions, i.e., σ^2 is the identity.]

Let s_n be the number of symmetric permutation matrices of dimension $n \times n$. (Define s_0 to be 1).

(a) Show that $s_n = s_{n-1} + (n-1)s_{n-2}$ for all $n \geq 2$.

(b) By finding and solving a differential equation, show that the exponential generating function $\overline{S}(x) = \sum_{n=0}^{\infty} s_n x^n/n!$ is equal to $e^{x+x^2/2}$.

(c) Using (b), show that

$$s_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{k!\,(n-2k)!\,2^k}.$$

(d) (BONUS) Can you give a combinatorial (counting argument) proof for (c)?

3.18. Consider the recurrence relation

$$\begin{aligned} n a_n &= 2a_{n-1} + 2a_{n-2}, \quad n \geq 2, \\ a_0 &= 1, \quad a_1 = 2. \end{aligned}$$

(a) By finding and solving a differential equation, show that the ordinary generating function $A(x) = \sum_{n=0}^{\infty} a_n x^n$ is equal to e^{2x+x^2}.

(b) Using (a), show that

$$a_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{2^{n-2k}}{k!\,(n-2k)!}.$$