Math 4700/6700 – Combinatorics – Spring 2019

(Last Few) Questions for Topic 2

2.23. In the country of Complexia the unit of currency is the glob, which is divided into 7 franths. One franth is divided into 4 quinks. There are three types of 1 quink coin available: one round, one hexagonal and one square. There are also round coins available in denominations of 2 quinks, 1 franth, 6 quinks, 2 franths and 4 franths. Express the number of ways of making change for a glob as a generating function coefficient, but do not evaluate it.

2.24. In the country of Excellenzia white eggs are sold in packages of one dozen, both white and brown eggs are sold in packages of half a dozen, and brown eggs are also sold in four-packs. In how many different ways can I buy four gross of eggs (a gross is 144)? Express as a generating function coefficient and then find an exact answer.

2.25. Use generating functions to show that every nonnegative integer has a unique representation of the form

$$\sum_{k=1}^{\infty} x_k.k! \quad ,$$

where x_k is an integer between 0 and k (inclusive), for each k. (You need to look at an infinite product of generating functions.)

2.26. The following proofs may be done using Ferrers diagrams, or by other means.

(a) Show that the number of partitions of n into three parts is equal to the number of partitions of $2n$ into three parts of size less than n.

(b) Show that the number of partitions of n is equal to the number of partitions of $2n$ into n parts.

(c) Show that the number of *self-conjugate* partitions of n (partitions that are conjugate to themselves) is equal to the number of partitions of n into distinct odd parts.

2.27. Use generating functions to find the number of partitions of 100 into exactly three parts.

2.28. Suppose that

$$\begin{aligned} A &= \{\epsilon, \text{aaaa}, \text{aaaaaaaa}, ...\} \\ &= \{\text{sequences of a multiple of 4 a's}\}. \end{aligned}$$

Show that $\overline{A}(x) = (\cosh x + \cos x)/2$ (where $\cosh x$ and $\cos x$ refer to the Maclaurin series for $\cosh x$ and $\cos x$, considered as formal power series).

2.29. Consider the set of strings made from a's, b's and c's so that there is at least one a, the number of b's is not exactly one, and the number of c's is not exactly two. How many such strings of length 15 are there? (Give an exact single number.)

2.30. Find the number of 'words' of length n made from a's, b's, c's and d's, so that the number of a's plus the number of b's is even, and the number of c's plus the number of d's is also even.

2.31. Find the number of ways to give n people red, green or blue hats so that an even number of people (possibly none) have green hats, and an odd number have red hats.

2.32. Find the number of ways of distributing 30 people into 4 distinct rooms so that rooms 1 and 2 are nonempty, and rooms 3 and 4 each contain an even number of people (possibly none).

2.33. Suppose we arrange n distinct books on 3 shelves, so that each shelf has at least one book, with the proviso that if a shelf has two or more books, the first two books on that shelf must be in alphabetical order by title. Use string interleavings and exponential generating functions to determine the number of ways to do this. Make sure you have the correct answers for small values of n.

2.34. Prove (using a combinatorial argument) that

$$n^r = \sum_{k=0}^{r} S(r,k)P(n,k).$$

(Define $P(n,k) = 0$ when $k > n$.)

2.35. Prove using a combinatorial argument that if $r, k \geq 1$ then $S(r,k) = kS(r-1,k) + S(r-1,k-1)$.