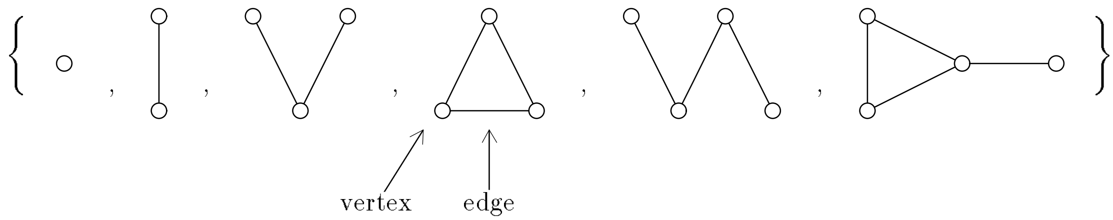
Math 4700/6700 – Combinatorics – Spring 2019

(First Few) Questions for Topic 2

2.1. Give generating functions for the set of graphs below:
(a) with the weight of a graph being its number of vertices, and
(b) with the weight of a graph being its number of edges.



2.2. Show that the generating function for all even subsets of $\mathbb{N}_n = \{1, 2, ..., n\}$ (with weight = cardinality) is

$$E(x) = \frac{(1+x)^n + (1-x)^n}{2}.$$

(An even subset is a subset of even cardinality. Hint: use the Binomial Theorem twice.)

2.3. Find the generating function for all 5-card poker hands with the following weight functions.
(a) Number of cards.
(b) Number of aces.
(c) Number of hearts.

2.4. Let $A(x)$ be the formal power series

$$A(x) \;=\; 1 - x + x^2 - x^3 + ... \;=\; \sum_{k=0}^{\infty} (-1)^k x^k \;.$$

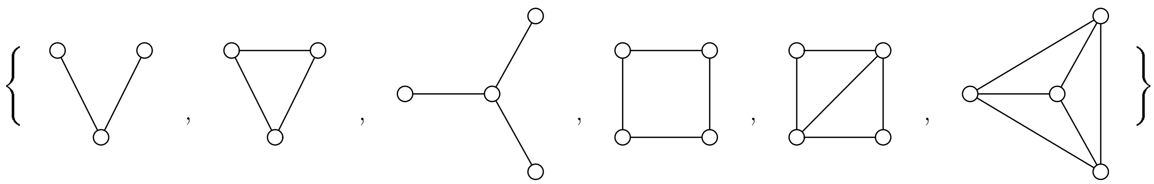
(a) Calculate $A(x)^2 = A(x)A(x)$ and express it using summation notation, in the form $\sum_{k=0}^{\infty} \ldots$.
(b) Show that $A(x)^2$ is the inverse in the ring of formal power series of $(1 + 2x + x^2)$.

2.5. (a) Prove that $\mathbb{C}[[x]]$ has no zero divisors: if $A(x)B(x) = 0$, then $A(x) = 0$ or $B(x) = 0$. (Hint: prove by contradiction, look at leading term of product.)
(b) Use (a) to show that if $A(x) \neq 0$ and $A(x)B(x) = A(x)C(x)$, then $B(x) = C(x)$.

2.6. By using the definition of multiplication and working backwards we can compute things like reciprocals, quotients and roots for formal power series.
(a) Suppose $A(x) = \sum_{k=0}^{\infty} k!\, x^k$. Use the fact that $A(x)A^{-1}(x) = 1$ to compute the first six terms (x^0 up to x^5) of $A^{-1}(x)$.
(b) Suppose the first few terms of $C(x)$ and $D(x)$ are $C(x) = 1 - 3x + 2x^2 - 4x^3 + \ldots$ and $D(x) = 1 + 5x - 8x^2 + 7x^3 + \ldots$. Compute the first four terms (x^0 up to x^3) of $C(x)/D(x)$.
(c) Suppose the first few terms of $E(x)$ are $E(x) = 1 + 6x - 8x^2 + 14x^3 + \ldots$. Compute the first four terms (x^0 up to x^3) of $\sqrt{E(x)}$. (Take the constant term to be positive.)

2.7. We defined a 'norm' for formal power series by $\|A(x)\| = 2^{-k}$, where x^k is the first power of x with a nonzero coefficient, if $A(x) \neq 0$, or $\|A(x)\| = 0$ if $A(x) = 0$. In order to use this to define limits, it must satisfy the *triangle inequality*: $\|A(x) + B(x)\| \leq \|A(x)\| + \|B(x)\|$. Prove that it does.

2.8. (a) Find the generating function $A(x, y)$ for the set of graphs below with x indicating edges and y indicating vertices.



(b) Evaluate $A(1, 1)$. What does it represent?

2.9. Find

(a) $[x^{21}](1 + x)^{24}$

(b) $[x^3](1 + 2x)^{15}$

2.10. Find

(a) $[x^{10}](1 - x)^{-3}$

(b) $[x^{12}]x^2(1 - x^2)^{-4}$

(c) $[x^k](1 - x^{10})^{-3}$

(d) $[x^i](2 - x)^{-8}$

2.11. Using the fact that

$$[x^k y^\ell]A(x)B(y) = [x^k]A(x) \,.\, [y^\ell]B(y),$$

evaluate the following.

(a) $[x^2y^3](1 - x)^{-4}(1 + y)^6$

(b) $[x^5y](1 - 2x)^{-2}(1 - \frac{y}{2})^{-2}$

2.12. Classify the following operations $*$ on $A \times B$ as injective or not. If the operation is injective, prove it. If it is not, give an example of something that does not have a unique decomposition.

(a) $A = B = \{0, 01, 010, 0101, 01010, 010101, ...\}$,
$* =$ concatenation

(b) $A = B = \{0, 010, 01010, 0101010, ...\}$,
$* =$ concatenation

(c) $A = \{0, 4, 8, 12, 16, 20, ...\}$, $B = \{0, 1, 6, 11\}$,
$* =$ addition

(d) $A = \{1, 4, 16, 64, 256, ...\}$, $B = \{3, 6, 12, 24, 48, ...\}$,
$* =$ multiplication

(e) $A = \{$all collections of people with blue eyes and brown hair$\}$,
$B = \{$all collections of people with brown eyes or brown hair$\}$,
$* =$ union

(f) $A = \{$all collections of people with blue eyes and brown hair$\}$,
$B = \{$all collections of people with brown eyes and brown hair$\}$,
$* =$ union

2.13. The 12 face cards (K, Q, J) are removed from a deck of cards. You are dealt one card from the remaining 40, and then allowed to throw two ordinary 6-sided dice, one red and one green. Express as a generating function coefficient the number of outcomes in which the sum of the two dice and the spots on your card (A=1, 2=2, 3=3, ..., 10=10) is 14, and evaluate this number.