Math 4700/6700 – Combinatorics – Spring 2019

Questions for Topic 1

1.1. How many ways are there to pick two successive cards (without replacement) from a normal 52 card deck so that:
(a) the first is an ace and the second is not a queen?
(b) the first is a spade and the second is not a queen (beware the queen of spades!)?

1.2. (a) In 2018 regular Tennessee licence plates had one of three forms: DDD-LLL, LDD-DDL, or DLD-DLD, where L represents a position occupied by a letter, and D a position occupied by a digit. How many distinct plates could be issued? (Assume all letters A-Z are valid.)

(b) In the country Complexia licence plates have 1, 2 or 3 letters and 1, 2 or 3 digits, which may be arranged in any way as long as the digits appear in a consecutive group. How many distinct licence plates can be issued?

1.3. Assume you are given 7 German, 4 Spanish and 11 Russian books, all different.
(a) How many ways are there to select one book?
(b) How many ways are there to select three books, one of each language?
(c) How many ways are there to arrange three books on a shelf so that exactly one language is left out?

1.4. Consider the set of 5-letter 'words', i.e. sequences of letters.
(a) How many are there with both the first and last letter being consonants?
(b) How many are there in which consonants appear (if at all) only in first and/or last position?
(c) Give a problem, involving 5-letter words with conditions involving the first and last positions, whose answer is:
(i) $21^2 5^3$, (ii) $21^2 26^3$,
(iii) $5^3 26^2$, (iv) $26^5 - 5^2 26^3$.

1.5. Two shelves are to have 14 books arranged between them. Either shelf will hold all the books. How many different arrangements of the books are possible? Each shelf may contain all, some or none of them.

1.6. How many anagrams (arrangements of the letters) of MISSISSIPPI are there?

1.7. (a) How many ways are there to seat 6 boys and 6 girls around a circular table with 12 seats? Two arrangements are to be considered the same if one can be rotated to give the other.

(b) How many ways are there if the boys and girls must alternate seats?

1.8. How many ways are there to divide 22 people into two soccer teams (11 each)?

1.9. A 5-card poker hand is dealt from a 52 card deck. What is the probability of:
(a) four aces?
(b) four of a kind?
(c) a full house (three of a kind and a pair)?
(d) a straight (five cards with consecutive values, not necessarily in the same suit, e.g. 4H, 5D, 6C, 7C, 8S - an ace can occur either in A2345 or in 10JQKA)? (Poker players: your answer should include ALL straights, including straight flushes and so on.)

1.10. A committee is to be chosen from 5 men and 7 women. How many ways are there to form it if:
(a) it has 5 people, 3 men and 2 women?
(b) it can be any positive size but must have equal numbers of men and women?
(c) it has 4 people of whom at least 2 are women?
(d) it has 3 members, at least 2 of whom are men, and Mr Brooks and Ms Campbell may not be on a committee with each other?

1.11. How many ways are there to choose 10 coins from
(a) \$1 in pennies, \$1 in nickels and \$1 in dimes?
(b) 50c in pennies, 50c in nickels and 50c in dimes?
(Coins of the same value are indistinguishable.)

1.12. How many ways are there to distribute 5 (identical) Mars bars and 8 lollipops to 4 children if
(a) the lollipops are all identical?
(b) each lollipop is a different flavour?

1.13. In how many ways can 12 people be distributed into three rooms (one pink, one blue and one white) so that no room is empty? Note that people are distinct objects!

1.14. How many integral solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 11$$

if
(a) each x_i is nonnegative?
(b) x_1, x_2, x_3 and x_4 are positive while $x_5 \geq -1$ and $x_6 \geq -5$?
(c) each x_i is nonnegative and at least one is zero?

1.15. How many arrangements are there of the letters

A, A, A, A, A, A, A, A, A, C, Q, S, V, W

if no two consonants can be consecutive?

1.16. In the country of Utopia the House of Representatives has 39 seats, which are contested by 4 parties. The House is elected by proportional representation: each party receives a number of seats proportional to its number of votes. In how many ways can the seats be distributed to the parties if
(a) every party gets at least 2 seats?
(b) no party gets a majority (20 or more) of the seats?
(c) the Silly party and the Sensible party together win exactly 30 seats?

1.17. How many ways are there to split 4 red, 5 blue and 7 black balls among
(a) 2 distinct boxes?
(b) 2 distinct boxes with 8 balls in each box?
(c) 4 distinct boxes, with exactly 2 empty boxes?

PROVE THE FOLLOWING IDENTITIES BY ANY SUITABLE METHOD:

1.18. $\binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \cdots + \binom{n+r}{r} = \binom{n+r+1}{r}$

1.19. $\binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1}$

1.20. $\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$

1.21. $\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}$

1.22. $\sum_{k=0}^{m} \binom{m}{k}\binom{n}{r+k} = \binom{m+n}{m+r}$

1.23. $\sum_{i=r}^{p-s} \binom{i}{r}\binom{p-i}{s} = \binom{p+1}{r+s+1}$

1.24. $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$ for $n \geq 1$. (Hint: use Binomial Theorem)

1.25. Prove that

$$\sum_{k=\ell}^{n} \binom{n}{k}\binom{k}{\ell} = \binom{n}{\ell} 2^{n-\ell}$$

in two ways:

(a) by a direct counting argument (i.e., find a set of objects that is counted by both the left and right sides of the equation; do not change the problem by applying other identities first);

(b) by taking the derivative ℓ times of both sides of the binomial expression for $(1+x)^n$, and then letting x have some particular value.

1.26. Give simple asymptotic estimates for

(a) $3n^2 + n^4 + e^n$,

(b) $n^n + e^{5n} + 3$,

(c) $P(2n, 4)$.

1.27. Prove that if $a_n \sim b_n$ and $c_n \sim d_n$ as $n \to \infty$ and b_n and d_n are both positive for all n, then $a_n + c_n \sim b_n + d_n$ as $n \to \infty$. (Warning: this problem is tricky.)

1.28. Consider all stacks consisting of $3n$ poker chips, where each chip may be red, green or blue. What <u>fraction</u> of these stacks have the same number of chips of each of the three colours? Find an asymptotic estimate for this fraction.

1.29. Suppose $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are sequences of positive real numbers.

(a) Find an example where $a_n, b_n > 1$ for all n and $a_n \sim b_n$ as $n \to \infty$, but $\log a_n \not\sim \log b_n$ as $n \to \infty$. ('$\log x$' means $\log_e x = \ln x$.) [Hint: think about things close to 1.]

(b) Prove that if $a_n \sim b_n$ as $n \to \infty$ and there is a constant $c > 1$ so that $b_n \geq c$ for all sufficiently large n, then $\log a_n \sim \log b_n$ as $n \to \infty$.

1.30. Suppose we have $4n$ distinct objects, coloured with four colours red, green, blue and yellow, so that there are n objects of each colour. Find the probability that if we randomly choose a subset of size $2n$, it contains all the red objects. Find an asymptotic estimate for this probability as $n \to \infty$, expressing your answer as simply as possible.