Burnside's Lemma: Suppose S is closed under the action of G. Then

$$\#\text{ orbits in } S = \frac{1}{|G|}\sum_{g\in G}|I(g)|.$$

True for any group action (satisfying properties in problem [5M], not just our setup with colourings). Idea is that invariant sets are usually easy to count, as we will see.

Proof: $$\frac{1}{|G|}\sum_{g\in G}|I(g)| = \frac{1}{|G|}|\{(g,s)\in G\times S \mid g\triangle s = s\}| = \frac{1}{|G|}\sum_{s\in S}|S(s)|$$
$$\overset{\text{Lemma 1}}{=} \sum_{s\in S}\frac{1}{|G\triangle s|} = \#\text{ orbits}$$
because each orbit $P = G\triangle s$ contributes $1/|P|$, $|P|$ times, so is counted once overall. ∎

Example: How many 012-strings are there of length 8 that are inequivalent under reversal? (E.g., 01201201 $\sim$ 10210210.).

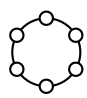
Solution: Think of $S = \{$012-strings of length 8$\}$ as colourings of 8 positions with colours $0, 1, 2$. Have group $G = \{e, r\}$, r = reversal.

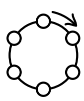
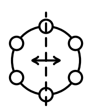
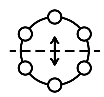
$|I(e)| = |S| = 3^8$,

$|I(r)| = \#$ of strings of form $\alpha\beta\gamma\delta\delta\gamma\beta\alpha = 3^4$

so $\#$ orbits $= (3^8+3^4)/2 = (81^2+81)/2 = (82\cdot 81)/2 = 41\cdot 81 = 3321$.

Example: How many inequivalent 6-bead necklaces can be made from red, green and yellow beads if all three colours are used.

Solution: $S = \{$all colourings of  using all of $r, g, y\}$.

$G = \{e,$  $r, r^2, r^3, r^4, r^5,$  $f_1, f_2, f_3,$  $g_1, g_2, g_3\}$ (r = 1/6 turn rotation, f_i = reflection about axis through two opposite beads, g_i = reflection about axis not going through beads).

$|I(e)| = |S| = T(6,3)$ (distribute 6 beads to 3 colours using all colours) $= 3^6 - \binom{3}{2}2^6 + \binom{3}{1}1^6 - \binom{3}{0}0^6 = 729 - 3\cdot 64 + 3 - 0 = 540$.

$|I(r)| = |I(r^5)| = 0$ $(= T(1,3))$

$|I(r^2)| = |I(r^4)| = 0$ $(= T(2,3))$

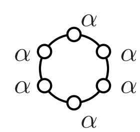
$|I(r^3)| = 3! = 6$ $(= T(3,3))$ all three colours, in any order

$|I(f_i)| = T(4,3) = 3^4 - \binom{3}{2}2^4 + \binom{3}{1}1^4 - \binom{3}{0}0^4 = 81 - 3\cdot 16 + 3 - 0 = 36$

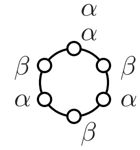
(Alternate argument: choose which of $\alpha, \beta, \gamma, \delta$ are equal in $\binom{4}{2}$ ways, then arrange colours on 3 locations: $\binom{4}{2}3! = 6\cdot 6 = 36$.)

$|I(g_i)| = 3! = 6$ $(= T(3,3))$ all three colours, in any order

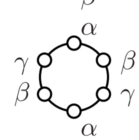
So $\#$ orbits $= (540+4\cdot 0+4\cdot 6+3\cdot 36)/12 = 672/12 = 56$.

$I(r)$: 

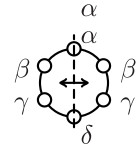
only one colour

$I(r^2)$: 

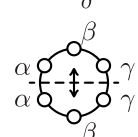
at most two colours

$I(r^3)$: 

3! choices

$I(f_i)$: 

$T(4,3)$ choices

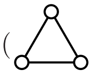
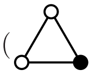
$I(g_i)$: 

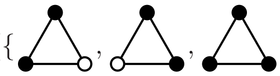
3! choices

Now: want more information than just number of inequivalent colourings; want number with given colour distributions.

- A *pattern* is an orbit of colourings.
- If colours are $c_1, c_2, \dots, c_m$ and $s \in S$ has n_i locations of colour c_i, then the *weight* of s is $w(s) = c_1^{n_1} c_2^{n_2} \dots c_m^{n_m}$. If s, t in same pattern then $w(s) = w(t)$.

  For any $A \subseteq S$, *inventory* of A is
  $w(A) = \sum_{s \in A} w(s)$.

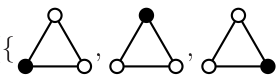
$w($  $) = w^3$, $w($  $) = bw^2$, etc.

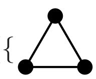
$w(\{$  $\} = b^3 + 2b^2w$

$w(A)$ is multivariable generating function of A with one variable for each colour. Term "weight" for $w(s)$ is a bit confusing because from g.f. perspective, n_i's are the weights. And why use different term "inventory" for (total) weight of a set? – because soon will see sets whose "weight" is not its inventory! All of this bad terminology is standard, unfortunately.

- Want weights of *inequivalent* colourings. If P is a pattern then the *weight* of P is $\widetilde{w}(P) = w(s)$ for any $s \in P$.

  The *pattern inventory* is
  $\widetilde{W} = \sum_{\text{patterns } P} \widetilde{w}(P)$.
  What we want to find.

$\widetilde{w}(\{$  $\}) = bw^2$

$\widetilde{w}(\{$  $\}) = b^3$

For $S = \{T_1, \dots, T_8\}$, $\widetilde{W} = b^3 + b^2w + bw^2 + w^3$

Weighted Burnside's Lemma: If $S \subseteq C^L$ is closed under the action of G then $\widetilde{W} = \dfrac{1}{|G|} \sum_{g \in G} w(I(g))$.

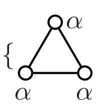
Proof: Just insert weights in proof of Lemma 2 and original Burnside's Lemma.

$$\frac{1}{|G|}\sum_{g\in G} w(I(g)) = \frac{1}{|G|}\sum_{g\in G}\sum_{s\in I(g)} w(s) = \frac{1}{|G|}\sum_{(g,s)\ \text{with}\ g\triangle s = s} w(s) = \frac{1}{|G|}\sum_{s\in S}\sum_{g\in S(s)} w(s)$$
$$= \frac{1}{|G|}\sum_{s\in S}|S(s)|w(s) \overset{\text{Lemma 1}}{=} \sum_{s\in S}\frac{w(s)}{|G\triangle s|} = \sum_{\text{patterns } P}\sum_{s\in P}\frac{\widetilde{w}(P)}{|P|} = \sum_{\text{patterns } P}\widetilde{w}(P)$$
$$= \widetilde{W}. \blacksquare$$

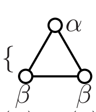
Actually works for any G-invariant weight function $w : S \to V$ where V is a rational vector space; G-invariant means $w(g\triangle s) = w(s)$ for all $g \in G$, $s \in S$.
Original B.L. follows by substituting all $c_i = 1$, so $w(s) = 1$ for every s.
Can apply directly, but if $S = C^L$ then will see a clever way to find the $w(I(g))$'s.

Example: S = coloured triangles, $G = \{e, r, r^2, m, n, p\}$.

$I(e) = S$ $\qquad w(I(e)) = b^3 + 3b^2w + 3bw^2 + w^3$

$I(r) = I(r^2) = \{$ 

$\}$ $\qquad w(I(r)) = w(I(r^2)) = b^3 + w^3$

$I(m) = \{$ 

$\}$ $\qquad w(I(m)) = w(I(n)) = w(I(p)) = b^3 + b^2w + bw^2 + w^3$

and $I(n)$, $I(p)$ similar

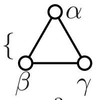
So $\widetilde{W}(b, w) = \frac{1}{6}\left(b^3 + 3b^2w + 3bw^2 + w^3 + 2(b^3 + w^3) + 3(b^3 + b^2w + bw^2 + w^3)\right) = \frac{1}{6}(6b^3 + 6b^2w + 6bw^2 + 6w^3) = b^3 + b^2w + bw^2 + w^3$, as before. Hence number of inequivalent colourings is $\widetilde{W}(1, 1) = 4$.

Important assumption: Henceforth, $S = C^L$ (all possible colourings of our locations).

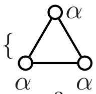
For given $g \in G$ can represent $w(I(g))$ using *cycles* of g acting on L: replace each cycle of length k by term $c_1^k + c_2^k + \ldots + c_m^k$, take product over all cycles. Works because

(1) for $s \in I(g)$ all locations on same cycle of g must be same colour, and

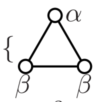
(2) since $S = C^L$, colourings of different cycles are independent so can just multiply together g.f. for each cycle.

$I(e) = \{$  $\}$,

$w(I(e)) = b^3 + b^2 w + b w^2 + w^3 = (b+w)^3$.

$I(r) = \{$  $\}$,

$w(I(r)) = b^3 + w^3$.

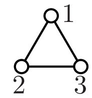
$I(m) = \{$  $\}$,

$w(I(m)) = b^3 + b^2 w + b w^2 + w^3 = (b+w)(b^2+w^2)$ ($b+w$ for α, b^2+w^2 for β).

Formally: represent g as permutation of L by listing its cycles: $g = (a_1 a_2 \ldots a_h)(b_1 b_2 \ldots b_j)\ldots$ means g sends $a_1 \mapsto a_2 \mapsto a_3 \ldots \mapsto a_h \mapsto a_1$ and $b_1 \mapsto b_2 \ldots \mapsto b_j \mapsto b_1$, etc.

- The *cycle structure* of g is $Z_g(x_1, x_2, x_3, \ldots) = x_1^{q_1} x_2^{q_2} x_3^{q_3} \ldots$ where there are q_i cycles of length i.

Then $w(I(g))$ is obtained from Z_g by replacing each x_k by $c_1^k + c_2^k + \ldots + c_m^k$, i.e. $w(I(g)) = Z_g(\sum_{i=1}^m c_i, \sum_{i=1}^m c_i^2, \sum_{i=1}^m c_i^3, \ldots)$.

:

$e = (1)(2)(3)$, $\quad Z_e = x_1^3$,
$w(I(e)) = Z_e(b+w, b^2+w^2, b^3+w^3, \ldots) = (b+w)^3$.

$r = (123)$, $\quad Z_r = x_3$,
$w(I(r)) = Z_r(b+w, b^2+w^2, b^3+w^3, \ldots) = b^3 + w^3$.

$m = (1)(23)$, $\quad Z_m = x_1 x_2$,
$w(I(r)) = Z_m(b+w, b^2+w^2, b^3+w^3, \ldots) = (b+w)(b^2+w^2)$.

Now use this in the Weighted Burnside's Lemma. Need one more definition first.

- The *cycle index* of G is the "average cycle structure" $P_G(x_1, x_2, \ldots) = \dfrac{1}{|G|}\displaystyle\sum_{g \in G} Z_g(x_1, x_2, \ldots)$ (a gen. function in variables $x_1, x_2, \ldots$; only finitely many for any given G).

$$P_G = \tfrac{1}{6}(x_1^3 + x_3 + x_3 + x_1x_2 + x_1x_2 + x_1x_2)$$
$$= \tfrac{1}{6}(x_1^3 + 2x_3 + 3x_1x_2).$$

Polya's Theorem: If $S = C^L$ then $\widetilde{W} = P_G(\sum_{i=1}^m c_i, \sum_{i=1}^m c_i^2, \sum_{i=1}^m c_i^3, \ldots)$.

Proof: $\widetilde{W} \overset{\text{W.B.L.}}{=} \dfrac{1}{|G|}\displaystyle\sum_{g \in G} w(I(g))$

$= \frac{1}{|G|}\sum_{g \in G} Z_g(\sum_{i=1}^m c_i, \sum_{i=1}^m c_i^2, \sum_{i=1}^m c_i^3, \ldots)$

$\overset{\text{defn of } P_G}{=} P_G(\displaystyle\sum_{i=1}^m c_i, \sum_{i=1}^m c_i^2, \sum_{i=1}^m c_i^3, \ldots)$.

For $S = \{$all 8 triangles$\}$,
$$\widetilde{W} = P_G(b+w, b^2+w^2, b^3+w^3, \ldots)$$
$$= \tfrac{1}{6}\big((b+w)^3 + 2(b^3+w^3) + 3(b+w)(b^2+w^2)\big)$$
which we computed before.

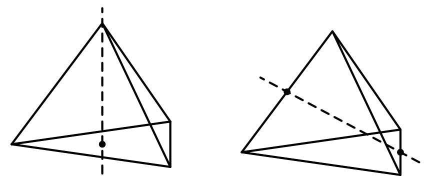
Corollary: If $S = C^L$ and $|C| = m$ then the number of inequivalent colourings is
$P_G(m, m, m, \ldots) = \frac{1}{|G|}\sum_{g \in G} m^{z(g)}$
where $z(g)$ is the number of cycles in g.

Proof: We get this if we substitute $c_i = 1$ for every colour c_i, which makes each $w(s)$ and $\widetilde{w}(P)$ equal to 1, so we are just counting. ∎

For our triangles with two colours, $\widetilde{W}(1,1) = P_G(2, 2, \ldots) = \frac{1}{6}(2^3 + 2\cdot 2 + 3\cdot 2\cdot 2) = 24/6 = 4$.

If we use m colours then we get $\frac{1}{6}(m^3 + 2\cdot m + 3\cdot m\cdot m) = (m^3 + 3m^2 + 2m)/6$.

Example: (a) Find pattern inventory for colourings of faces of a regular tetrahedron with red, yellow and blue, inequivalent up to rotation in space. (b) How many inequivalent tetrahedra with two blue faces? (c) How many tetrahedra with three "hot" (red or yellow) faces? (d) How many inequivalent tetrahedra altogether?



Solution: (a) If we label the faces $1, 2, 3, 4$ then our group has three types of element:

e: $Z_e = x_1^4$.

8 rotations r_i about axis through a vertex, e.g. $r_1 = (123)(4)$: $Z_{r_i} = x_1 x_3$.

3 rotations s_j about an axis through midpoints of two opposite edges, e.g. $s_1 = (14)(23)$: $Z_{s_j} = x_2^2$.

So $P_G = (x_1^4 + 8x_1x_3 + x_2^2)/12$ and hence $\widetilde{W} = ((r+y+b)^4 + 8(r+y+b)(r^3+y^3+b^3) + 12(r^2+y^2+b^2)^2)/12$ ... could expand out, but won't!

(b) For g.f. just by number of blue faces, set $r = y = 1$, and we want $[b^2]$:

$[b^2]\widetilde{W}(1,1,b) = [b^2]((2+b)^4 + 8(2+b)(2+b^3) + 3(2+b^2)^2)/12 = \frac{1}{12}(\binom{4}{2}2^2 + 0 + 3\cdot 4) = 36/12 = 3$

(which agrees with intuitive answer).

(c) Could work out number with one blue face, but will do more directly. For g.f. by number of hot faces, set $r = y = h$, and we want $[h^3]$:

$[h^3]\widetilde{W}(h,h,1) = [h^3]((2h+1)^4 + 8(2h+1)(2h^3+1) + 3(2h^2+1)^2)/12 = \frac{1}{12}(\binom{4}{3}2^3 + 8\cdot 2 + 0)$
$= 48/12 = 4$

(which again agrees with intuitive answer).

(d) Total number of inequivalent tetrahedra is just

$\widetilde{W}(1,1,1) = (3^4 + 8\cdot 3^2 + 3\cdot 3^2)/12 = (81+99)/12 = 15.$