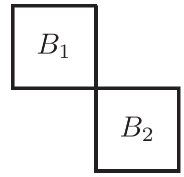
To calculate rook polynomials, have two main results.

Multiplication Formula: If B can be divided into B_1 and B_2 which are *disjunct* (no rook on B_1 threatens any square of B_2, and vice versa) then

$$R(x, B) = R(x, B_1)R(x, B_2).$$

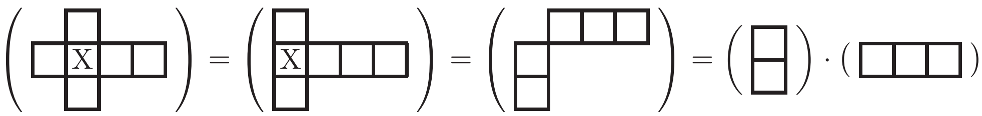


Proof: Just an application of the Product Lemma for ordinary generating functions: to get a placement of rooks on B we can combine placements on B_1 and B_2 in a uniquely decomposable way, and weights (numbers of rooks) add. ∎

Note: Extends to several disjunct boards.

Notation: (B) denotes $R(x, B)$.

Example: Note that we can rearrange rows and columns however we like.



$$= (1 + 2x)(1 + 3x) = 1 + 5x + 6x^2.$$

Now second (very important) result.

Expansion formula: Let s be a square of B. Let

B_s = board obtained by deleting s from B, and

B_s^* = board obtained by deleting s and its row and column from B

Then

$$R(x, B) = R(x, B_s) + xR(x, B_s^*).$$

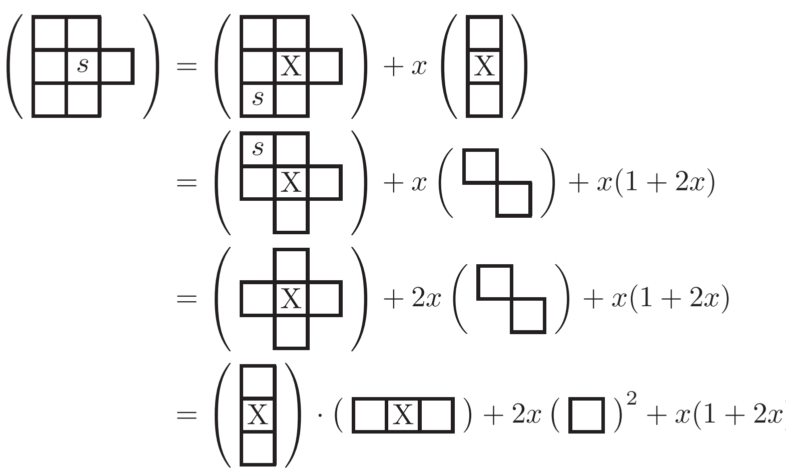
Proof: Consider putting k rooks on B. Two cases:

(1) no rook on s: then k rooks on B_s, $r_k(B_s)$ ways;

(2) one rook on s: then $k-1$ rooks on B_s^*, $r_{k-1}(B_s^*)$ ways.

So $r_k(B) = r_k(B_s) + r_{k-1}(B_s^*)$ for $k \geq 1$, and $r_0(B) = r_0(B_s) = 1$, which yields the generating function equation. ∎

Example: Actually a quicker solution than the one shown is to expand by taking s to be the square on the right.



$$= (1 + 2x)^2 + 2x(1 + x)^2 + x(1 + 2x)$$

$$= 1 + 4x + 4x^2 + 2x + 4x^2 + 2x^3 + x + 2x^2$$
$$= 1 + 7x + 10x^2 + 2x^3$$

Assignments with forbidden positions

Example: We have five children A, B, C, D, E and six presents 1, 2, 3, 4, 5, 6. We can afford to give only three presents. No child should get more than one present, and no type of present should be given to more than one child. How many ways are there to do this if A dislikes 2 and 3, B dislikes 2, and D dislikes 1, 4 and 5?

	1	2	3	4	5	6
A		X	X			
B		X				
C						
D	X			X	X	
E						

Solution 1: Find $r_3(B) = [x^3]R(x, B)$, B = 24 non-X squares. Complicated! We omit this.

Solution 2: Use inclusion-exclusion ideas.

$U = \{$all arrangements of 3 rooks on the 5×6 board$\}$.

P_i = 'there is a rook in column i on an X', $1 \leq i \leq 6$

We want to find $e_0 = N(\overline{P}_1\overline{P}_2\overline{P}_3\overline{P}_4\overline{P}_5\overline{P}_6) = s_0 - s_1 + s_2 - s_3 + s_4 - s_5 + s_6$.

$$s_0 = |U| = \binom{5}{3}\binom{6}{3}3! = 10.20.6 = 1200$$

$$\left.\begin{array}{l} N(P_1) = N(P_3) = N(P_4) = N(P_5) = 1.\binom{4}{2}\binom{5}{2}2! \\ N(P_2) = 2.\binom{4}{2}\binom{5}{2}2! \\ N(P_6) = 0 \end{array}\right\} \begin{array}{l} s_1 = (1+2+1+1+1+0)\binom{4}{2}\binom{5}{2}2! \\ \quad = r_1(\overline{B})\binom{4}{2}\binom{5}{2}2! \\ \quad = 6.6.10.2 = 720 \end{array}$$

where $\overline{B}$ = board of X's. More generally, consider s_k:

$N(P_{i_1}P_{i_2}\dots P_{i_j})$ = number of ways to put j *black* rooks (bad!) on X's in columns $i_1, i_2, \dots, i_j$ and $3 - j$ *grey* rooks (can be good or bad) on any squares in $6 - j$ remaining columns.

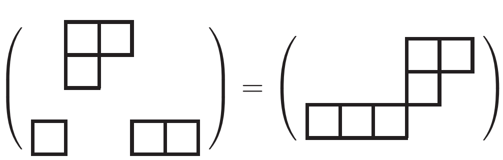
s_j = number of pairs (u, Y), $u \in U$, Y set of j properties that u has

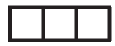
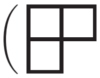
= (number of ways to put j black rooks (for Y) on board of X's)×

(number of ways to put $3 - j$ grey rooks (for $\overline{Y}$) in other $5 - j$ rows and $6 - j$ columns)

$$= r_j(\overline{B})\binom{5-j}{3-j}\binom{6-j}{3-j}(3-j)!$$

Need to distinguish black and grey rooks: s_k may count same distribution of rooks several times depending on which properties they satisfy. So want $R(x, \overline{B})$:



$= ($  $)\ ($  $)$

$= (1+3x)(1+3x+x^2) = 1+6x+10x^2+3x^3$

Thus,

$$\begin{aligned}
s_0 &= r_0(\overline{B})\binom{5}{3}\binom{6}{3}3! &= 1.10.20.6 &= 1200\\
s_1 &= r_1(\overline{B})\binom{4}{2}\binom{5}{2}2! &= 6.6.10.2 &= 720\\
s_2 &= r_2(\overline{B})\binom{3}{1}\binom{4}{1}1! &= 10.3.4.1 &= 120\\
s_3 &= r_3(\overline{B})\binom{2}{0}\binom{3}{0}0! &= 3.1.1.1 &= 3\\
s_4 &= s_5 = s_6 = 0
\end{aligned}$$

and therefore $e_0 = s_0 - s_1 + s_2 - s_3 + s_4 - s_5 + s_6 = 1200 - 720 + 120 - 3 = 1320 - 723 = 597$.

Notes: (1) Use whenever $\overline{B}$ easier than B for finding rook polynomial.

(2) General formula: if we are putting k rooks on board, and $\overline{B}$ is the complement of B in an $m \times n$ board, then

$$s_j = r_j(\overline{B})\binom{m-j}{k-j}\binom{n-j}{k-j}(k-j)!$$

(3) Once we have the s_j's we can find the e_j's and c_j's, etc.

(4) Special case: permutations with forbidden positions: $k = m = n$, get $s_j = r_j(\overline{B})(n-j)!$ so that

$r_n(B) = e_0 = \sum_{j=0}^{n}(-1)^j s_j = \sum_{j=0}^{n}(-1)^j r_j(\overline{B})(n-j)!$ (many books just list this special case).

Example: In above example, how many ways are there to give exactly m people presents they don't like, $0 \le m \le 3$?

Solution: We want e_m, $0 \le m \le 3$, so calculate $E(x) = S(x-1)$:

$$\begin{aligned}
S(x) &= 1200 + 720x + 120x^2 + 3x^3\\
E(x) &= S(x-1) = 1200 + 720(x-1) + 120(x-1)^2 + 3(x-1)^3\\
&= (1200 - 720 + 120 - 3) + (720 - 2.120 + 3.3)x + (120 - 3.3)x^2 + 3x^3\\
&= 597 + 489x + 111x^2 + 3x^3
\end{aligned}$$

and so we have

m	0	1	2	3
e_m	597	489	111	3

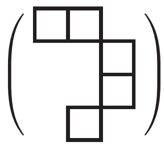
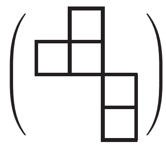
Terminology: $E(x)$ = *hit polynomial* – counts *hits*, i.e. things in forbidden positions.

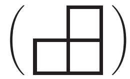
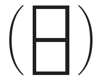
Example: Our original rook polynomial question:

	1	2	3	4
A	X	X		
B			X	
C			X	
D		X		

Solution: B has 11 squares, $\overline{B}$ has only 5, so use $\overline{B}$. Special case (4) from above, so $s_j = r_j(\overline{B})(n-j)!$ where $n = 4$, $0 \le j \le 4$.

Now

$R(x, \overline{B}) =$  $=$ 

$=$  

$= (1 + 3x + x^2)(1 + 2x) = 1 + 5x + 7x^2 + 2x^3$

Therefore

$$\begin{aligned} s_0 &= 4!r_0(\overline{B}) = 24.1 = 24 \\ s_1 &= 3!r_1(\overline{B}) = 6.5 = 30 \\ s_2 &= 2!r_2(\overline{B}) = 2.7 = 14 \\ s_3 &= 1!r_3(\overline{B}) = 1.2 = 2 \\ s_4 &= 0!r_4(\overline{B}) = 1.0 = 0 \end{aligned}$$

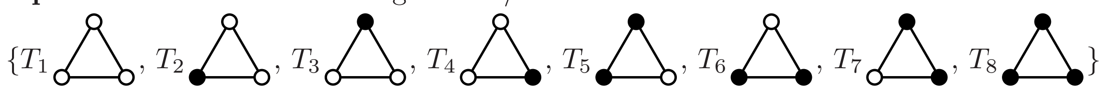
and hence the number of ways to fill all jobs is

$$e_0 = s_0 - s_1 + s_2 - s_3 + s_4 = 24 - 30 + 14 - 2 + 0 = 6.$$

5. POLYA THEORY

Aim: Count "inequivalent" objects.

Example. Colour vertices of triangle black/white.



Possible equivalences:

(a) $T_i \sim T_j$ if T_j is a rotation of T_j. Equivalence classes $\{T_1\}$, $\{T_2, T_3, T_4\}$, $\{T_5, T_6, T_7\}$, $\{T_8\}$; four inequivalent objects.

(b) $T_i \sim T_j$ if T_j obtained by reflection of T_i about a vertical axis. Equivalence classes $\{T_1\}$, $\{T_2, T_4\}$, $\{T_3\}$, $\{T_5, T_7\}$, $\{T_6\}$, $\{T_8\}$; six inequivalent objects.

(c) $T_i \sim T_j$ if they have the same "contrast pattern" (can swap colours white $\leftrightarrow$ black). Equivalence classes $\{T_1, T_8\}$, $\{T_2, T_7\}$, $\{T_3, T_6\}$, $\{T_4, T_5\}$; four inequivalent objects.

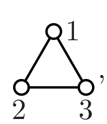
Counting "inequivalent objects" really means counting *equivalence classes*.

We will examine situations like (a) and (b), but not (c). Our symmetries will not permute colours, just positions. But more general version of theory includes permutations of colours.

General setup:

- Object with set L of *locations*.
- Set G of symmetries of L, each a bijection $g : L \to L$, g carries ℓ to $g(\ell)$.
  Must form *group* under composition $(g_2 \circ g_1)(\ell) = g_2(g_1(\ell))$. – closed, associative, identity (e), inverses.

Example, triangles with (a):



, $L = \{1, 2, 3\}$, $G = \{e, r, r^2\}$.

	$g(\ell)$ \ ℓ	1	2	3
	e	1	2	3
g	r	2	3	1
	r^2	3	1	2

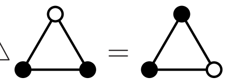
	$g_2 \circ g_1$ \ g_1	e	r	r^2
	e	e	r	r^2
g_2	r	r	r^2	e
	r^2	r^2	e	r

- Set C of *colours*, $C^L = \{s : L \to C\}$, all *colourings* of L.

- *Action* of G on C^L by $g\triangle s = s \circ g^{-1}$, satisfies $g_2\triangle(g_1\triangle s) = (g_2 \circ g_1)\triangle s$. And two other basic properties of action, see problem on this.

$C = \{b = \bullet, w = \circ\}$

$C^L = \{T_1, T_2, \ldots, T_8\}$.

$r\triangle$  , $\quad r\triangle s = s'$

To figure out colour of a location in s' need to look backwards to see where that location came from: so need to consider r^{-1}.
$s'(i) = s(r^{-1}(i))$, i.e. $s' = s \circ r^{-1}$.

We will be concerned with $S \subseteq C^L$ closed under action of G: $g\triangle s \in S\ \forall\ g \in G$, $s \in S$. Sometimes S is all of C^L, sometimes not.

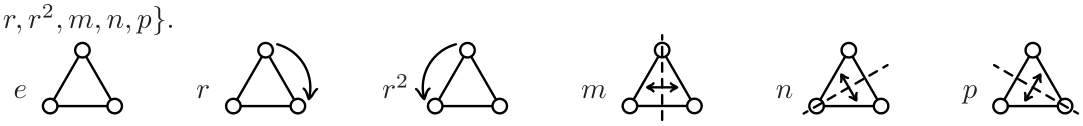
- Action of G breaks C^L up into *equivalence classes* called *orbits*: orbit of s is $G\triangle s = \{g\triangle s \mid g \in G\}$. Apply all possible symmetries to s.

$S = C^L$: orbits $\{T_1\}$, $\{T_2, T_3, T_4\}$, $\{T_5, T_6, T_7\}$, $\{T_8\}$.
4 orbits $\leftrightarrow$ 4 inequivalent coloured triangles.

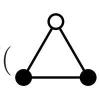
Or could take $S = \{T_2, T_3, T_4, T_8\}$, union of 2 orbits $\{T_2, T_3, T_4\}$, $\{T_8\}$.

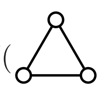
Notes: (1) S closed under action of $G \Leftrightarrow S$ is a union of orbits.
(2) s, s' in same orbit $\Leftrightarrow G\triangle s = G\triangle s'$.
(3) Inequivalent objects $\leftrightarrow$ orbits.
(4) In each orbit, all colourings have same distribution of colours. So can we count orbits according to colour distributions?

New example: before introduce more concepts extend group on triangle to include reflections; now $G = \{e, r, r^2, m, n, p\}$.



- Have *stabiliser* of colouring s: $S(s) = \{g \in G \mid g\triangle s = s\}$. Symmetries that leave s unchanged. Often denoted G_s.

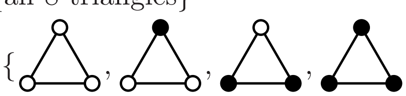
$S($$) = \{e, m\}$

$S($$) = \{e, r, r^2, m, n, p\}$

- Have *invariant set* of symmetry g: $I(g) = \{s \in S \mid g\triangle s = s\}$. Colourings not altered by g.

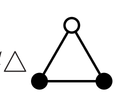
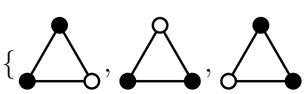
We use these ideas to help count orbits.

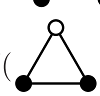
$I(e) = \{$all 8 triangles$\}$

$I(m) = \{$$\}$

Observe: For each $g \in G$ and $s, s' \in C^L$, $s = s' \Leftrightarrow g\triangle s = g\triangle s'$. $\Rightarrow$ is clear; for $\Leftarrow$ act on both with g^{-1}.

Lemma 1: For any $s \in S$, $|G\triangle s|\ |S(s)| = |G|$.

$G\triangle$$= \{$$\}$,

$S($$) = \{e, m\}; \quad 3 \cdot 2 = 6.$

Proof: For any $t \in S$, let $S(s,t) = \{g \in G \mid g\triangle s = t\}$ (the symmetries that send s to t). Every $g \in G$ sends s to a unique $t \in G\triangle s$, so $G = \bigcup_{t \in G\triangle s} S(s,t)$. Now idea is to show that $|S(s,t)| = |S(s)|$ for every t. Given $t \in G\triangle s$, there exists $h \in G$ with $h\triangle s = t$, and

$$k \in S(s,t) \quad \Leftrightarrow \quad k\triangle s = t = h\triangle s \quad \Leftrightarrow \quad h^{-1}\triangle(k\triangle s) = h^{-1}\triangle(h\triangle s)$$

$\Leftrightarrow \quad (h^{-1} \circ k) \triangle s = s \quad \Leftrightarrow \quad h^{-1} \circ k \in S(s)$

$\Leftrightarrow \quad k \in h \circ S(s)$ (a left coset of $S(s)$, sort of a translation of $S(s)$ by h).

So $|S(s,t)| = |h \circ S(s)| = |S(s)|$. Therefore, $|G| = \bigcup_{t \in G \triangle s} |S(s,t)| = |G \triangle s|\, |S(s)|$, as required. ∎