4. INCLUSION-EXCLUSION

Aim: Want to count elements of a set with certain combinations of properties.

Example: In a group of 100 students, 50 take French, 80 take Math, and 43 take both. How many take neither?

Solution:

$$\begin{aligned} N(\overline{F}\,\overline{M}) &= N - N(F) - N(M) + N(FM) \\ &= 100 - 50 - 80 + 43 = 13 \end{aligned}$$

Notation:

U = universe, set of all objects of interest
P, Q properties – $\overline{P}$ = 'not P', PQ = 'P and Q' (note: $\overline{PQ} \neq \overline{P}\,\overline{Q}$)
$N(P)$ = number of objects with property P
$N = |U|$ = total number of objects

Principle of inclusion and exclusion:

$$\begin{aligned} N(\overline{P}_1\overline{P}_2\overline{P}_3\ldots\overline{P}_n) =& N \\ & - N(P_1) - N(P_2) - \ldots - N(P_n) \\ & + N(P_1P_2) + N(P_1P_3) + \ldots + N(P_{n-1}P_n) \\ & - N(P_1P_2P_3) - N(P_1P_2P_4) - \ldots - N(P_{n-2}P_{n-1}P_n) \\ & \vdots \\ & + (-1)^n N(P_1P_2P_3\ldots P_n) \\ =& \sum_{X\subseteq\{P_1,P_2,\ldots,P_n\}} (-1)^{|X|} N(\text{all of } X) \qquad (N(\text{all of } \emptyset) = N) \end{aligned}$$

Suppose we let

$$\begin{aligned} s_0 &= N \\ s_1 &= N(P_1) + N(P_2) + \ldots + N(P_n) \\ s_2 &= N(P_1P_2) + N(P_1P_3) + \ldots + N(P_{n-1}P_n) \\ &\vdots \\ s_n &= N(P_1P_2\ldots P_n). \end{aligned}$$

so s_k = no. of pairs (u, X) where $X \subseteq \{P_1, \ldots, P_n\}$, $|X| = k$, and $u \in U$ has all of the properties in X. Then then this becomes

$$N(\overline{P}_1\overline{P}_2\ldots\overline{P}_n) = s_0 - s_1 + s_2 - s_3 + \ldots + (-1)^n s_n.$$

Proof: Look at number of times any $u \in U$ is counted. If u has none of properties, then is counted once in s_0 and not in anything else.

Suppose u has exactly $m \geq 1$ of properties P_1, P_2, …, P_n. Terms counting u:

1	in s_0
m	in s_1
$\binom{m}{2}$	in s_2
$\vdots$	
$\binom{m}{m}$	in s_m
0	in s_k for $k > m$

So, to total sum u contributes:

$$1 - m + \binom{m}{2} - \binom{m}{3} + \ldots + (-1)^m \binom{m}{m} = (1-1)^m = 0$$

using the Binomial Theorem.

Thus, total sum counts objects with none of properties once, doesn't count any other objects at all, as required. ∎

Note: Can extend to compute number with any combination of properties. E.g., for $N(\overline{P}_1\overline{P}_2\overline{P}_3P_4 P_5 \ldots P_n)$ take universe $U' = \{u \in U \mid u$ satisfies $P_4, P_5, \ldots, P_n\}$ and compute $N'(\overline{P}_1\overline{P}_2\overline{P}_3)$ in U'.

Example: Find number of solutions of $x_1 + x_2 + x_3 = 15$ in nonnegative integers with $x_1 \leq 5$, $x_2 \leq 7$ and $x_3 \leq 10$.

Solution: *Could do this using g.f.s; let's use inc-exc instead.* Let U be set of all nonnegative solutions of the equation. Let

$P_1 =$ '$x_1 \geq 6$' $\qquad P_2 =$ '$x_2 \geq 8$' $\qquad P_3 =$ '$x_3 \geq 11$'

We want $N(\overline{P}_1\overline{P}_2\overline{P}_3)$.

Recall: ways to distribute r identical objects into n distinct boxes is $\left(\!\binom{n}{r}\!\right) = \binom{n+r-1}{n-1}$.

So:

$$N = \left(\!\binom{3}{15}\!\right) = \binom{17}{2} = 136 \qquad N(P_1P_2) = \left(\!\binom{3}{15-14}\!\right) = \left(\!\binom{3}{1}\!\right) = \binom{3}{2} = 3$$

$$N(P_1) = \left(\!\binom{3}{15-6}\!\right) = \left(\!\binom{3}{9}\!\right) = \binom{11}{2} = 55 \qquad N(P_1P_3) = N(P_2P_3) = N(P_1P_2P_3) = 0$$

$$N(P_2) = \left(\!\binom{3}{15-8}\!\right) = \left(\!\binom{3}{7}\!\right) = \binom{9}{2} = 36$$

$$N(P_3) = \left(\!\binom{3}{15-11}\!\right) = \left(\!\binom{3}{4}\!\right) = \binom{6}{2} = 15$$

Thus,

$$\begin{aligned} N(\overline{P}_1\overline{P}_2\overline{P}_3) &= N - N(P_1) - N(P_2) - N(P_3) \\ &\quad + N(P_1P_2) + N(P_1P_3) + N(P_2P_3) - N(P_1P_2P_3) \\ &= 136 - 55 - 36 - 15 + 3 + 0 + 0 - 0 = 33 \end{aligned}$$

Example: How many numbers between 1 and 1000 inclusive are divisible by 4, 6 or 10?

Solution: Find number *not* divisible by any of 4, 6 or 10 and subtract from 1000: three properties

$D_4 =$ divisible by 4 $\qquad D_6 =$ divisible by 6 $\qquad D_{10} =$ divisible by 10

Number not divisible by any of these is $N(\overline{D}_4\overline{D}_6\overline{D}_{10})$. We have:

$$\begin{aligned} N(D_4) &= 1000/4 = 250 & N(D_4D_6) &= \lfloor 1000/12 \rfloor = 83 \\ N(D_6) &= \lfloor 1000/6 \rfloor = 166 & N(D_4D_{10}) &= 1000/20 = 50 \\ N(D_{10}) &= 1000/10 = 100 & N(D_6D_{10}) &= \lfloor 1000/30 \rfloor = 33 \\ & & N(D_4D_6D_{10}) &= \lfloor 1000/60 \rfloor = 16 \end{aligned}$$

So

$$\begin{aligned} N(\overline{D}_4\overline{D}_6\overline{D}_{10}) &= N - N(D_4) - N(D_6) - N(D_{10}) \\ &\quad + N(D_4D_6) + N(D_4D_{10}) + N(D_6D_{10}) - N(D_4D_6D_{10}) \\ &= 1000 - 250 - 166 - 100 + 83 + 50 + 33 - 16 = 634 \end{aligned}$$

Now $N(D_4$ or D_6 or $D_{10}) = N - N(\overline{D}_4\overline{D}_6\overline{D}_{10}) = 1000 - 634 = 366$.

Example (Derangements): A *derangement* of $\mathbb{N}_n = \{1, 2, \ldots, n\}$ is a permutation $x_1 x_2 \ldots x_n$ so that $x_i \neq i$ for all $i = 1, 2, \ldots, n$. (No i in ith position.) For example, $3\underline{2}41$ not a derangement, 4321 is a derangement. Find d_n, number of derangements of $\mathbb{N}_n$.

Solution: Use inc-exc with U=all permutations of $\mathbb{N}_n$, with property P_i meaning there is an i in ith position, $i = 1, 2, \ldots, n$. We want $N(\overline{P}_1\overline{P}_2 \ldots \overline{P}_n)$. We have

$$\begin{aligned}
&N = n! \\
&N(P_1) = N(P_2) = \ldots = N(P_n) = (n-1)! \\
&N(P_1P_2) = N(P_1P_3) = \ldots = N(P_{n-1}P_n) = (n-2)! \\
&N(P_1P_2P_3) = \ldots = N(P_{n-2}P_{n-1}P_n) = (n-3)! \\
&\vdots \\
&N(P_1P_2 \ldots P_n) = 0! = 1
\end{aligned}$$

Thus we see that $s_k = \binom{n}{k}(n-k)! = \dfrac{n!}{k!(n-k)!}(n-k)! = \dfrac{n!}{k!}$, so

$$\begin{aligned}
d_n &= N(\overline{P}_1\overline{P}_2 \ldots \overline{P}_n) \\
&= s_0 - s_1 + s_2 - s_3 + \ldots + (-1)^n s_n \\
&= \frac{n!}{0!} - \frac{n!}{1!} + \frac{n!}{2!} - \frac{n!}{3!} + \ldots + (-1)^n \frac{n!}{n!} \\
&= n!\left(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \ldots + \frac{(-1)^n}{n!}\right). \\
&= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}
\end{aligned}$$

Notice: As $n \to \infty$, we have

$$d_n \to n!\left(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \ldots\right) = n! e^{-1} = \frac{n!}{e}.$$

Often want number of objects having a given number of the properties we are interested in.

Notation: Properties $P_1, \ldots, P_n$ as before, with $s_0 = N$, $s_1 = N(P_1) + \ldots + N(P_n)$, $s_2 = N(P_1P_2) + \ldots + N(P_{n-1}P_n)$, etc.
$S(x) = \sum_{m=0}^{n} s_m x^m$ – g.f. for $(s_0, s_1, s_2, \ldots)$
e_m = number of objects having *exactly* m of the properties $P_1, \ldots, P_n$
$E(x) = \sum_{m=0}^{n} e_m x^m$ – g.f. for $(e_0, e_1, e_2, \ldots)$ – *hit polynomial (for reasons may explain later, if we get far enough with rook polynomials)*

Usually know s_i's, want to find e_i's. How can we do this? Answer from g.f.s:

Theorem: $S(x) = E(x+1)$

Proof:

$$\begin{aligned} S(x) &= \sum_{k=0}^{n} s_k x^k \\ &= \sum_{k=0}^{n} \left(\sum_{m=k}^{n} \binom{m}{k} e_m \right) x^k && \text{because each item with exactly } m \text{ properties contributes } \binom{m}{k} \text{ to } s_k \\ &= \sum_{0 \le k \le m \le n} \binom{m}{k} e_m x^k = \sum_{m=0}^{n} e_m \sum_{k=0}^{m} \binom{m}{k} x^k \\ &= \sum_{m=0}^{n} e_m (x+1)^m && \text{by Binomial Theorem} \\ &= E(x+1) \end{aligned}$$

Corollary: (a) $E(x) = S(x-1)$.
(b) $e_m = \sum_{k=m}^{n} (-1)^{k-m} \binom{k}{m} s_k = s_m - \binom{m+1}{m} s_{m+1} + \binom{m+2}{m} s_{m+2} - \ldots + (-1)^{n-m} \binom{n}{m} s_n$.

Proof: (a) Substitute $x-1$ for x in Theorem above.
(b)

$$\begin{aligned} e_m &= [x^m]\ E(x) = [x^m]\ S(x-1) \\ &= [x^m]\ \sum_{k=0}^{n} s_k (x-1)^k \\ &= [x^m]\ \sum_{k=0}^{n} s_k \sum_{m=0}^{k} \binom{k}{m} (-1)^{k-m} x^m && \text{by Binomial Theorem} \\ &= [x^m] \sum_{0 \le m \le k \le n} (-1)^{k-m} \binom{k}{m} s_k x^m \\ &= \sum_{k=m}^{n} (-1)^{k-m} \binom{k}{m} s_k. \end{aligned}$$

Note: Principle of inclusion-exclusion is a special case of this: $e_0 = s_0 - s_1 + s_2 - \ldots + (-1)^n s_n$ – just put $m = 0$.

Example: In how many ways can 9 people be put into 5 distinct rooms so that exactly two are empty?

Can do this using Stirling numbers: later.

Solution: Let P_i = 'room i is empty'. We want exactly two empty rooms so number we want is e_2: need s_2, s_3, s_4 and s_5:

$$\begin{aligned} N(P_1P_2) &= \ldots = N(P_4P_5) = 3^9 && \text{so } s_2 = \binom{5}{2} 3^9 \\ N(P_1P_2P_3) &= \ldots = N(P_3P_4P_5) = 2^9 && \text{so } s_3 = \binom{5}{3} 2^9 \\ N(P_1P_2P_3P_4) &= \ldots = N(P_2P_3P_4P_5) = 1^9 = 1 && \text{so } s_4 = \binom{5}{4} \\ N(P_1P_2P_3P_4P_5) &= 0 && \text{so } s_5 = 0 \end{aligned}$$

Therefore

$$\begin{aligned} e_2 &= s_2 - \binom{3}{2}s_3 + \binom{4}{2}s_4 - \binom{5}{2}s_5 \\ &= \binom{5}{2}3^9 - \binom{3}{2}\binom{5}{3}2^9 + \binom{4}{2}\binom{5}{4} - 0 \\ &= 10.19683 - 3.10.512 + 6.5 = 181\,500 \end{aligned}$$

Alternative solution: Choose 3 nonempty rooms: $\binom{5}{3} = 10$ ways. Then distribute 9 people into 3 distinct nonempty rooms: $T(9,3) = 3!S(9,3) = 6S(9,3)$ ways. So answer is $10.6S(9,3) = 60S(9,3)$ and looking up $S(9,3)$ in book, $S(9,3) = 3025$ so answer is $60.3025 = 181\,500$.

Incidentally, inc-exc can be used for nonempty box problems: see 4.6.

Also, often want to know how many things there are with at least or at most m of the properties we are interested in, without working out the e_ks and adding them up.

Notation: c_m = number of objects with *at least* m of properties $P_1, \ldots, P_n$; a_m = number with *at most* m.

Theorem: For $m \geq 1$, $c_m = \sum_{k=m}^{n} (-1)^{k-m}\binom{k-1}{m-1}s_k$, and $c_0 = N = s_0$. Then $a_m = N - c_{m+1} = s_0 - c_{m+1}$.

Proof: Problem 4.13, using g.f.s.

Example: How many ways are there to arrange the letters of COMMITTEE so that there are at least two consecutive pairs of identical letters?

Solution: Let properties be M, T, E – consecutive Ms, Ts, Es. We want c_2 so we need s_2 and s_3:

$$N(MT) = \frac{7!}{2!} \quad \begin{array}{l} \leftarrow \text{arrange C, O, MM, I, TT, E, E} \\ \qquad\qquad\qquad \leftarrow \text{two Es} \end{array}$$

and so $N(MT) = N(ME) = N(TE) = 7!/2!$ and $s_2 = 3.7!/2! = 3.5040/2 = 7560$; also

$$N(MTE) = 6! = 720 \qquad \leftarrow \text{arrange C, O, MM, I, TT, EE}$$

so $s_3 = 720$. Now

$$c_2 = s_2 - \binom{2}{1}s_3 = 7560 - 2.720 = 6120.$$

Example: How many permutations of $\mathbb{N}_{10}$ are there in which at least 6 integers are deranged?

Solution: As before let P_i mean that we have an i in position i, i.e. i is not deranged. We want

$$\begin{aligned} M &= \text{ number with at least 6 deranged} \\ &= \text{ number with at most 4 fixed } (a_4) \\ &= 10! - \text{ number with at least 5 fixed} = 10! - c_5 \end{aligned}$$

To get c_5, need $s_5, s_6, \ldots, s_{10}$:

$$N(P_1 P_2 \ldots P_k) = \ldots = (10-k)!$$

$$\text{so} \quad s_k = \binom{10}{k}(10-k)! = \frac{10!}{k!(10-k)!}(10-k)! = \frac{10!}{k!} = P(10, 10-k)$$

Therefore

$$\begin{aligned} c_5 &= s_5 - \binom{5}{4}s_6 + \binom{6}{4}s_7 - \binom{7}{4}s_8 + \binom{8}{4}s_9 - \binom{9}{4}s_{10} \\ &= P(10,5) - \binom{5}{4}P(10,4) + \binom{6}{4}P(10,3) - \binom{7}{4}P(10,2) + \binom{8}{4}P(10,1) - \binom{9}{4}P(10,0) \\ &= 30\,240 - 5.5040 + 15.720 - 35.90 + 70.10 - 126.1 = 13\,264 \end{aligned}$$

and hence our answer is $M = 10! - c_5 = 3\,628\,800 - 13\,264 = 3\,615\,536$.

Alternative solution: Pick k things to derange, $6 \le k \le 10$: $\binom{10}{k}$ ways. Then derange them: d_k ways. So we have

$$\begin{aligned} M &= \binom{10}{6}d_6 + \binom{10}{7}d_7 + \binom{10}{8}d_8 + \binom{10}{9}d_9 + \binom{10}{10}d_{10} \\ &= 210.265 + 120.1854 + 45.14\,833 + 10.133\,496 + 1.1\,334\,961 \\ &= 3\,615\,536. \end{aligned}$$

(Values of d_n obtained from a book.)

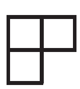
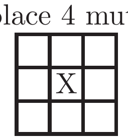
Rook polynomials

Motivation: Consider: how many ways to assign 4 people (A, B, C, D) to jobs (1, 2, 3, 4) if A cannot do 1 or 2, B and C cannot do 3, and D cannot do 2?

	1	2	3	4
A	X	X		
B			X	
C			X	
D		X		

Want to choose one non-X in each row and column

Think of table as chessboard: want to place 4 mutually nonthreatening rooks on non-X squares.

Notation: B = board, e.g.  

etc.

$r_k(B)$ = number of ways to put k nonthreatening rooks on B

$R(x, B) = r_0(B) + r_1(B)x + r_2(B)x^2 + \ldots = \sum_{k=0}^{\infty} r_k(B)x^k$

– the *rook polynomial* of B (o.g.f. for sequence of $r_k(B)$'s)

Notes: $r_0(B) = 1$ for any board.

$R(x, -) = 1$ where $-$ is the empty board.

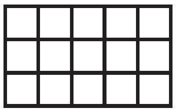
$R(x, B)$ is invariant under permutations of the rows and columns, and under transposition.

Boards and rooks just a way of representing matchings in bipartite graphs, rook polynomial closely related to matching polynomial.

Examples:

n squares

$$R(x, B) = 1 + nx$$



$m \times n$ board

$$r_0 = 1$$

$$r_1 = mn$$

$$r_2 = \binom{m}{2} n(n-1)$$

$$\vdots$$

$$r_k = \binom{m}{k} n(n-1)\dots(n-k+1) = \binom{m}{k} P(n,k) = \binom{m}{k}\binom{n}{k} k!$$

$$\vdots$$

$$R(x, B) = \sum_{k=0}^{m} \binom{m}{k}\binom{n}{k} k! x^k = \sum_{k=0}^{m} \frac{m! n! x^k}{k!(m-k)!(n-k)!}$$