Recurrence relations and generating functions (ctd)

Example: $a_n = a_{n-1} + 6a_{n-2} + 5.3^n$, $n \geq 2$, with $a_0 = 4$, $a_1 = 6$.

Solution: Define g.f.

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots.$$

We have

$$\sum_{n=2}^{\infty}(a_n - a_{n-1} - 6a_{n-2})x^n = 5\sum_{n=2}^{\infty} 3^n$$

$$\vdots$$

$$\begin{aligned}
A(x)(1 - x - 6x^2) &= \frac{45x^2}{1-3x} + 4 + 2x \\
&= \frac{45x^2 + 4 - 10x - 6x^2}{1-3x} \\
&= \frac{39x^2 - 10x + 4}{1-3x}
\end{aligned}$$

$$\text{so} \quad A(x) = \frac{39x^2 - 10x + 4}{(1-3x)(1-x+6x^2)} = \frac{39x^2 - 10x + 4}{(1-3x)(1-3x)(1+2x)}$$

To extract coefficients, expand using partial fractions:

$$A(x) = \frac{39x^2 - 10x + 4}{(1-3x)^2(1+2x)} = \frac{\alpha}{1-3x} + \frac{\beta}{(1-3x)^2} + \frac{\gamma}{(1+2x)}$$

$$39x^2 - 10x + 4 = \alpha(1-3x)(1+2x) + \beta(1+2x) + \gamma(1-3x)^2$$

$x = 1/3$: $\quad 5 = 39/9 - 10/3 + 4 = 5\beta/3 \quad$ so $\quad \beta = 3$

$x = -1/2$: $\quad 75/4 = 39/4 + 10/2 + 4 = 25\gamma/4 \quad$ so $\quad \gamma = 3$

$x = 0$: $\quad 4 = \alpha + \beta + \gamma = \alpha + 6 \quad$ so $\quad \alpha = -2$

Thus

$$A(x) = \frac{-2}{1-3x} + \frac{3}{(1-3x)^2} + \frac{3}{1+2x}$$

$$\begin{aligned}
\text{so} \quad a_n = [x^n]\, A(x) &= -2[x^n]\,(1-3x)^{-1} + 3[x^n]\,(1-3x)^{-2} + 3[x^n]\,(1+2x)^{-1} \\
&= -2.3^n + 3.3^n\binom{n+1}{1} + 3(-2)^n \\
&= -2.3^n + 3(n+1)3^n + 3(-2)^n \\
&= (3n+1)3^n + 3(-2)^n
\end{aligned}$$

which agrees with $a_0 = 4$ and $a_1 = 6$.

Notes: (1) Long!
(2) G.f. method solves particular cases, doesn't give general solution.
(3) Be very careful with summation indices.

Recurrence relations and multiplication of generating functions

Sometimes to solve rec. reln need to look at product of g.f. with itself.

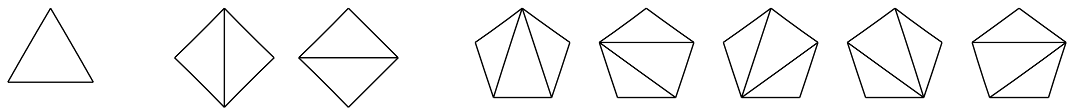
Recall: For two g.f.s,

$$\begin{aligned} A(x)B(x) &= (a_0 + a_1x + a_2x^2 + a_3x^3 + \dots)(b_0 + b_1x + b_2x^2 + b_3x^3 + \dots) \\ &= a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + \dots \\ &= \sum_{n=0}^{\infty} (a_0b_n + a_1b_{n-1} + \dots + a_nb_0)x^n \end{aligned}$$

Example: Find the number of triangulations t_n of a convex n-gon (with labelled vertices 1, 2, …, n).

$t_0 = 0$

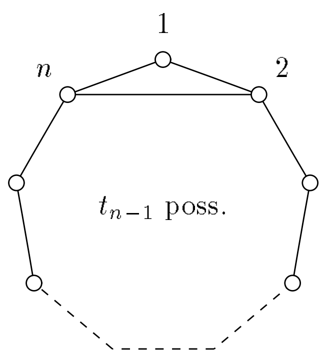
$t_1 = 0$



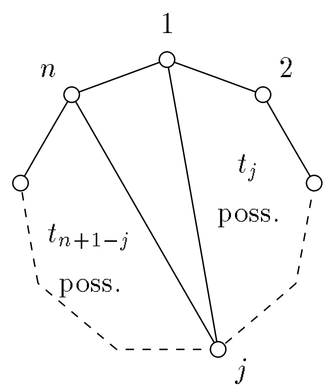
$t_2 = 0$ $t_3 = 1$ $t_4 = 2$ $t_5 = 5$

Could also argue that $t_2 = 1$ and work with that, but we will use $t_2 = 0$.

Solution: Edge $(n, 1)$ must be in some triangle with third vertex j, $2 \le j \le n-1$.



$j = 2$ or $n-1$: t_{n-1} possibilities



$3 \le j \le n-2$: t_jt_{n+1-j} possibilities

Therefore,

$$t_n = 2t_{n-1} + (t_3t_{n-2} + t_4t_{n-3} + \dots + t_{n-2}t_3), \qquad n \ge 4$$

(not valid for $n = 3$). Let $T(x) = \sum_{n=0}^{\infty} t_nx^n$. Notice that $3 + (n-2) = 4 + (n-3) = \dots = n+1$, so we want this to be coefficient of x^{n+1} in our g.f. (so multiply both sides of rec. reln by x^{n+1} and sum from 4 to ∞):

$$\sum_{n=4}^{\infty} t_nx^{n+1} = 2\sum_{n=4}^{\infty} t_{n-1}x^{n+1} + \sum_{n=4}^{\infty} (t_3t_{n-2} + t_4t_{n-3} + \dots + t_{n-2}t_3)x^{n+1}$$

$$(m = n-1) \qquad\qquad (m = n+1)$$

$$x\sum_{n=4}^{\infty} t_nx^n = 2x^2\sum_{m=3}^{\infty} t_mx^m + \sum_{m=5}^{\infty} (t_3t_{m-3} + t_4t_{m-4} + \dots + t_{m-3}t_3)x^m$$

$$x(T(x) - t_3x^3) = 2x^2T(x) + \sum_{m=0}^{\infty} (t_0t_m + t_1t_{m-1} + t_2t_{m-2} + t_3t_{m-3} + \dots + t_mt_0)x^m$$

(Note: all extra terms in last sum, including those with $m = 0$ to 4, involve t_0, t_1 or t_2 and so are all 0). But now, from our formula above, the last term here is just $T(x)T(x) = T(x)^2$. So

$$\begin{aligned} x(T(x) - x^3) &= 2x^2T(x) + T(x)^2 \\ T(x)^2 + (2x^2 - x)T(x) + x^4 &= 0 \end{aligned}$$

Solve by quadratic formula:

$$\begin{aligned} T(x) &= \frac{x-2x^2 \pm \sqrt{(2x-x)^2-4x^4}}{2} \\ &= \frac{x-2x^2 \pm \sqrt{-4x^3+x^2}}{2} \\ &= \frac{x-2x^2 \pm x\sqrt{1-4x}}{2} \end{aligned}$$

Now we know that $[x^1]\,T(x) = t_1 = 0$ so choose minus sign here (plus sign would give $[x^1]\,T(x) = 1$). Therefore

$$T(x) = \frac{x-2x^2 - x\sqrt{1-4x}}{2}.$$

Now, for $n \geq 3$, we have

$$\begin{aligned} t_n = [x^n]\,T(x) = [x^n]\,(-\tfrac{1}{2}x\sqrt{1-4x}) \\ &= -\tfrac{1}{2}[x^{n-1}]\,\sqrt{1-4x} \\ &= -\tfrac{1}{2}[x^{n-1}]\,\sum_{k=0}^{\infty}\binom{\frac{1}{2}}{k}(-4x)^k \quad \text{by the Generalised Binomial Theorem} \\ &= -\tfrac{1}{2}(-4)^{n-1}\binom{\frac{1}{2}}{n-1} \end{aligned}$$

Now, in general we have

$$\begin{aligned} \binom{\frac{1}{2}}{k} &= \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})\dots(\frac{1}{2}+1-k)}{1.2.3\dots k} = (-1)^{k-1}\frac{1.3.5\dots(2k-3)}{2^k k!} \\ &= \frac{(-1)^{k-1}}{2^k k!}\frac{1.2.3.4\dots(2k-2)}{2.4\dots(2k-2)} = \frac{(-1)^{k-1}}{2^k k!}\frac{1.2.3.4\dots(2k-2)}{2^{k-1}1.2\dots(k-1)} \\ &= \frac{(-1)^{k-1}}{2^k k!}\frac{(2k-2)!}{2^{k-1}(k-1)!} = \frac{(-1)^{k-1}}{k2^{2k-1}}\frac{(2k-2)!}{(k-1)!(k-1)!} = \frac{(-1)^{k-1}}{k2^{2k-1}}\binom{2k-2}{k-1} \end{aligned}$$

and so, for $n \geq 3$,

$$\begin{aligned} t_n &= -\tfrac{1}{2}(-4)^{n-1}\binom{\frac{1}{2}}{n-1} = -\tfrac{1}{2}(-1)^{n-1}2^{2n-2}\left(\frac{(-1)^{n-2}}{(n-1)2^{2n-3}}\binom{2n-4}{n-2}\right) \\ &= \frac{1}{n-1}\binom{2n-4}{n-2} \end{aligned}$$

which is an example of a *Catalan number*. We can check t_3, t_4, t_5:

$$t_3 = \frac{1}{2}\binom{2}{1} = 1 \qquad t_4 = \frac{1}{3}\binom{4}{2} = 2 \qquad t_5 = \frac{1}{4}\binom{6}{3} = 5$$

which all agree with what we know.

Recurrence relations and differential equations for g.f.s

Sometimes rec. relns contain expressions which can give us derivatives of g.f. for sequence.

Note:

$$A'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\sum_{n=0}^{\infty}a_n x^n = \sum_{n=0}^{\infty}na_n x^{n-1} = \sum_{n=1}^{\infty}na_n x^{n-1} = \sum_{m=0}^{\infty}(m+1)a_{m+1}x^m.$$

So expect to get derivatives when we have things like na_n in rec. reln.

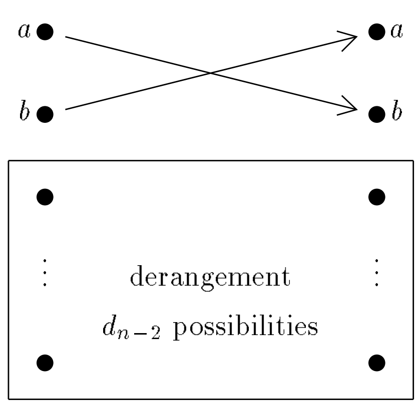
Derangements: A *derangement* of a set S is a bijection (one-to-one and onto function) $f : S \to S$ such that $f(s) \neq s$ for all $s \in S$.
A derangement of $\mathbb{N}_n = \{1, 2, \ldots, n\}$ can be thought of as a permutation $x_1x_2 \ldots x_n$ of 1, 2, …, n where $x_i = f(i) \neq i$ for all i. For example:

$4\underline{2}31$	not a derangement
4321	is a derangement

Hat check problem: If n people check their hats at a restaurant, in how many ways can they be returned so that nobody gets his own hat?

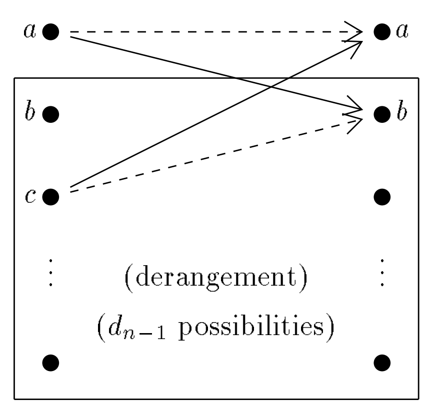
Solution: Want d_n, number of derangements of the n-set S of people ($f(s)$ = person who gets person s's hat). First, find rec. reln. Consider fixed $a \in S$, and let $b = f(a) \neq a$. There are $n-1$ choices for b. Given b, two possible cases:

(1) $f(b) = a$



f

(2) $f(c) = a$, $c \neq a, b$



f (f^*, dashed)

(In (2) there is a $1-1$ correspondence between f and f^*, and d_{n-1} choices for f^*.) So we get

$$d_n = (n-1)(d_{n-1} + d_{n-2}), \qquad n \geq 2$$
$$d_0 = 1, d_1 = 0.$$

We can set up DE for $D(x) = d_0 + d_1x + d_2x^2 + \ldots$. But we expect d_n to be something like $n!$, and this is an arrangement-type problem, so we can't expect a nice answer unless we use *exponential* g.f.:

$$\overline{D}(x) = d_0 + d_1x + d_2\frac{x^2}{2!} + d_3\frac{x^3}{3!} + \ldots$$
$$= \sum_{n=0}^{\infty} d_n\frac{x^n}{n!} = \sum_{n=0}^{\infty} \overline{d}_n x^n, \qquad \overline{d}_n = \frac{d_n}{n!}$$

To get eqn for $\overline{D}(x)$, rewrite rec. reln in terms of $\overline{d}_n$ (since $n-2$ lowest subscript, divide all terms by $(n-2)!$):

$$\frac{d_n}{(n-2)!} = (n-1)\frac{d_{n-1}}{(n-2)!} + (n-1)\frac{d_{n-2}}{(n-2)!}$$
$$n(n-1)\frac{d_n}{n!} = (n-1)(n-1)\frac{d_{n-1}}{(n-1)!} + (n-1)\frac{d_{n-2}}{(n-2)!}$$
$$n\overline{d}_n = (n-1)\overline{d}_{n-1} + \overline{d}_{n-2}, \qquad n \geq 2.$$

where $\overline{d}_0 = d_0/0! = 1$ and $\overline{d}_1 = d_1/1! = 0$. Therefore,

$$\sum_{n=2}^{\infty} n\overline{d}_n x^n = \sum_{n=2}^{\infty}(n-1)\overline{d}_{n-1}x^n + \sum_{n=2}^{\infty}\overline{d}_{n-2}x^n$$

$$(m = n-1) \qquad (m = n-2)$$

$$x\sum_{n=2}^{\infty} n\overline{d}_n x^{n-1} = x^2\sum_{m=1}^{\infty} m\overline{d}_m x^{m-1} + x^2\sum_{m=0}^{\infty}\overline{d}_m x^m$$

$$x\overline{D}(x) = x(\overline{D}(x) - 1d_1) = x^2\overline{D}'(x) + x^2\overline{D}(x)$$

$$(x - x^2)\overline{D}'(x) = x^2 D(x)$$

$$\frac{\overline{D}'(x)}{\overline{D}(x)} = \frac{x^2}{x-x^2} = \frac{x}{1-x} = \frac{1}{1-x} - 1$$

Now, integrate:

$$\ln \overline{D}(x) = -\ln(1-x) - x + C.$$

But $\overline{D}(0) = d_0 = 1$, so

$$0 = \ln 1 = \ln \overline{D}(0) = -\ln(1-0) - 0 + C = C.$$

Therefore,

$$\ln \overline{D}(x) = -\ln(1-x) - x$$

$$\overline{D}(x) = \frac{e^{-x}}{1-x} = e^{-x}(1-x)^{-1}$$

Now

$$d_n = \left[\frac{x^n}{n!}\right]\overline{D}(x) = n![x^n]\left(1 - \frac{x^1}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots\right)(1 + x + x^2 + x^3 + \dots)$$

$$= n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^n}{n!}\right)$$

Check: $d_3 = 6(1 - 1 + \frac{1}{2} - \frac{1}{6}) = 2$: 231, 312.
$d_4 = 24(1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24}) = 9$: 2143, 2341, 2413, 3142, 3412, 3421, 4321, 4312, 4123.

Note: As $n \to \infty$,

$$d_n \sim n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots\right) = n!\left(\frac{1}{e}\right) = n!/e.$$

So proportion of permutations which are derangements approaches a constant, $1/e$, as $n \to \infty$.