Examples:

Characteristic equation	General solution
$(x-2)^3 = 0$	$(A + Bn + Cn^2)2^n$
$(x-1)(x-3)^2 = 0$	$A.1^n + (B + Cn)3^n$
$(x-4)^3(x+3)^4 = 0$	$(A + Bn + Cn^2)4^n + (D + En + Fn^2 + Gn^3)(-3)^n$

Example: (Fibonacci's Rabbits) Assume:

– every pair of rabbits produces its first pair of young when it is two months old, and one pair of young every month after that;

– rabbits never die.

If we begin with one pair of newborn rabbits, how many pairs are there after n months? History: posed by Fibonacci, 1220. Solved by de Moivre c.1720 using combinatorial generating functions; first use of combinatorial generating functions [Tucker].

Solution: Let number of pairs be F_n. We get rec. reln

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2$$

since there are F_{n-1} pairs still alive from last month, and F_{n-2} new pairs born to pairs alive two months ago. Initial conditions are

$$F_0 = F_1 = 1.$$

Now:

We have:	$F_n - F_{n-1} - F_{n-2} = 0$
Characteristic eqn:	$x^2 - x - 1 = 0$
Roots (quadratic formula):	$x = (1 \pm \sqrt{5})/2$
General solution:	$F_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$

So, apply initial conditions:

$$1 = F_0 = A + B \qquad (1)$$

$$1 = F_1 = \left(\tfrac{1+\sqrt{5}}{2}\right)A + \left(\tfrac{1-\sqrt{5}}{2}\right)B \qquad (2)$$

$$\tfrac{1-\sqrt{5}}{2} = \left(\tfrac{1-\sqrt{5}}{2}\right)A + \left(\tfrac{1-\sqrt{5}}{2}\right)B \qquad (3) = \left(\tfrac{1-\sqrt{5}}{2}\right) \times (1)$$

$$1 - \tfrac{1-\sqrt{5}}{2} = \tfrac{1+\sqrt{5}}{2} = \sqrt{5}A \qquad (2) - (3)$$

and therefore

$$A = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right), \qquad B = 1 - A = \frac{1}{\sqrt{5}}\left(\frac{\sqrt{5}-1}{2}\right) = -\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)$$

and so we have

$$\begin{aligned} F_n &= \frac{1}{\sqrt{5}}\left(\tfrac{1+\sqrt{5}}{2}\right)\left(\tfrac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\tfrac{1-\sqrt{5}}{2}\right)\left(\tfrac{1-\sqrt{5}}{2}\right)^n \\ &= \frac{1}{\sqrt{5}}\left(\left(\tfrac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\tfrac{1-\sqrt{5}}{2}\right)^{n+1}\right) \end{aligned}$$

– the formula for the nth Fibonacci number.

Note: $(1+\sqrt{5})/2 \simeq 1.618$ and $(1-\sqrt{5})/2 \simeq -0.618$, so as $n \to \infty$ we get

$$F_n \sim \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}.$$

Now, what about inhomogeneous ones? Situation very similar to solving DEs.

Theorem: Suppose we have a recurrence relation

$$a_n + c_1 a_{n-1} + \ldots + c_k a_{n-k} = f(n), \quad n \ge k \qquad \text{(I)}$$

with corresponding homogeneous equation

$$a_n + c_1 a_{n-1} + \ldots + c_k a_{n-k} = 0, \quad n \ge k \qquad \text{(H)}.$$

Let $a_n = p(n)$ be some solution of (I). Then $a_n = i(n)$ is a solution of (I) if and only if $i(n) = p(n) + h(n)$ for some solution $a_n = h(n)$ of (H).

Consequence: To solve a rec. reln (I), find:
(a) *complementary solution*: general solution of corresponding homogeneous eqn (H); and
(b) *particular solution*: any solution of (I).
Then
(c) add them to get general solution of (I), and
(d) if required, use initial conditions to determine constants.

Can do (a): what about (b)? Look at an example.

Example: $w_n = n + 3 + 6w_{n-1} - 9w_{n-2}$, $n \ge 2$, with $w_0 = 2$, $w_1 = 4$.

Solution: We have

$$w_n - 6w_{n-1} + 9w_{n-2} = n + 3 \qquad \text{(I)}.$$

(a) Now:

Homogeneous eqn:	$w_n - 6w_{n-1} + 9w_{n-2} = 0$	(H)
Characteristic eqn:	$x^2 - 6x + 9 = 0$	
	$(x-3)^2 = 0$	
Roots:	$x = 3$ twice	
Complementary solution:	$w_n = (A + Bn)3^n$	

(b) Need to find a particular solution of (I). Try something similar to RHS: $w_n = C + Dn$, say: substitute in (I) to see if can find values of C, D which will make this work. Like solving DEs by method of undetermined coefficients.

$$\begin{aligned} n + 3 &= w_n - 6w_{n-1} + 9w_{n-2} \\ &= (C + Dn) - 6(C + D(n-1)) + 9(C + D(n-2)) \\ &= 4C - 12D + 4Dn \end{aligned}$$

Now
$[n^1]:\quad 4D = 1$ so $D = 1/4$.
$[n^0]:\quad 4C - 12D = 4C - 3 = 3$ so $4C = 6$ so $C = 3/2$.
Since $C = 3/2$, $D = 1/4$ make all coefficients agree, $w_n = 3/2 + n/4$ is a particular solution of (I).
Note: can also find eqns for C and D by inserting particular values of n; disadvantage of this is that it doesn't show that $C + Dn$ actually satisfies the rec. reln for those values of C and D.

(c) So general solution of (I) is $w_n = 3/2 + n/4 + (A + Bn)3^n$.

(d) Initial conditions:

$2 = w_0 = 3/2 + 1/4.0 + (A + 0B)3^0 = 3/2 + A$ so $A = 1/2$,
$4 = w_1 = 3/2 + 1/4.1 + (A + B.1)3^1 = 7/4 + 3A + 3B = 13/4 + 3B$ so $3B = 3/4$ and $B = 1/4$.

Thus, solution is $w_n = 3/2 + n/4 + (1/2 + n/4)3^n$.

Finding a particular solution: (Method of undetermined coefficients) Guess something that looks like the RHS of (I). May have problem if RHS is like solution of (H) (this is why we solve (H) *first*). In this case, multiply guess by 'large enough' power of n to make it work.

Specifically, if RHS includes $p(n)x^n$, where $p(n)$ is a polynomial, guess $q(n)x^n$ where q is a polynomial of the same degree as p with undetermined coefficients. If x is a root of the characteristic equation of multiplicity m, guess $n^m q(n)x^n$ instead.

Examples:

Solution of (H)	RHS of (I)	Guess
$A.2^n + Bn + C$	3^n	$D3^n$
"	16	Dn^2
"	$17n + 16, 17n$	$Dn^2 + En^3$
"	4.2^n	$Dn2^n$
"	$5n2^n$	$(Dn + En^2)2^n$
$A(-1)^n + B4^n$	12	D
"	$12n - 11$	$Dn + E$
"	$12 + 3.5^n$	$D + E5^n$
"	$4.(-1)^n + n^2 4^n$	$Dn(-1)^n + (En + Fn^2 + Gn^3)4^n$

Recurrence relations and generating functions

Have alternative method for solving linear rec. relns with constant coefficients, and also some other types of rec. reln, using g.f.s

Idea: Use rec. reln to get equation for g.f. of sequence $(a_0, a_1, \ldots)$. Solve eqn, extract g.f. coefficient to get formula for a_n.

Example: $a_n = a_{n-1} + 6a_{n-2} + 5.3^n$, $n \geq 2$, with $a_0 = 4$, $a_1 = 6$.

Solution: Define g.f.

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots.$$

We have

$$a_n - a_{n-1} - 6a_{n-2} = 5.3^n, \quad n \geq 2 \quad \text{(important)}$$

$$\text{so} \quad \sum_{n=2}^{\infty} (a_n - a_{n-1} - 6a_{n-2} x^n = 5 \sum_{n=2}^{\infty} 3^n$$

$$\sum_{n=2}^{\infty} a_n x^n - \sum_{n=2}^{\infty} a_{n-1} x^n - 6 \sum_{n=2}^{\infty} a_{n-2} x^n = 5 \sum_{n=2}^{\infty} 3^n x^n$$

$$(m = n) \quad (m = n-1) \quad (m = n-2) \quad (m = n-2)$$

$$\sum_{m=2}^{\infty} a_m x^m - \sum_{m=1}^{\infty} a_m x^{m+1} - 6 \sum_{m=0}^{\infty} a_m x^{m+2} = 5 \sum_{m=0}^{\infty} 3^{m+2} x^{m+2}$$

$$\sum_{m=2}^{\infty} a_m x^m - x \sum_{m=1}^{\infty} a_m x^m - 6x^2 \sum_{m=0}^{\infty} a_m x^m = 5.3^2 x^2 \sum_{m=0}^{\infty} (3x)^m$$

$$(A(x) - a_0 - a_1 x) - x(A(x) - a_0) - 6x^2 A(x) = 45x^2(1 + (3x) + (3x)^2 + \ldots)$$

$$A(x) - 4 - 6x - x(A(x) - 4) - 6x^2 A(x) = 45x^2/(1 - 3x)$$

$$(1 - x - 6x^2)A(x) - 4 - 6x + 4x = 45x^2/(1 - 3x)$$

$$A(x)(1 - x - 6x^2) = \frac{45x^2}{1 - 3x} + 4 + 2x$$