Subset problems

Examples: (1) Find the number of (a) 5-subsets, (b) k-subsets, of $\mathbb{N}_n$ with no two consecutive integers.

Solution: To count subsets, we are going to count *difference vectors*.

(a) Consider $S = \{a_1, a_2, \ldots, a_5\} \subseteq \mathbb{N}_n$. Assume that $a_1 < a_2 < \ldots < a_5$. Construct the difference vector for this subset:

$$\vec{d} = (a_1, a_2 - a_1, a_3 - a_2, a_4 - a_3, a_5 - a_4, n - a_5) = (a_1, d_1, d_2, d_3, d_4, d_5)$$

Notice that the weight of this vector is $a_1 + d_1 + d_2 + d_3 + d_4 + d_5 = n$. We can set up a 1-1 correspondence between 5-subsets of $\mathbb{N}_n$ and certain 6-difference vectors of weight n. So, want $[x^n]$ in g.f. for appropriate d.v.s:

$a_1 \geq 1$	$x + x^2 + x^3 + \ldots = x(1-x)^{-1}$
$d_1 \geq 2$ (no consecutive numbers)	$x^2 + x^3 + x^4 + \ldots = x^2(1-x)^{-1}$
$d_2 \geq 2$	”
$d_3 \geq 2$	”
$d_4 \geq 2$	”
$d_5 \geq 0$	$1 + x + x^2 + x^3 + \ldots = (1-x)^{-1}$

So, what we want is

$$\begin{aligned} &[x^n]\; x(1-x)^{-1}[x^2(1-x)^{-1}]^4(1-x)^{-1} \\ =&[x^n]\; x^9(1-x)^{-6} = [x^{n-9}]\; (1-x)^{-6} \\ =&\left(\!\!\binom{6}{n-9}\!\!\right) = \binom{n-4}{5} \end{aligned}$$

provided $n \geq 9$; otherwise we get 0.

(b) More generally, for k-subsets set up difference vector

$$\vec{d} = (a_1, a_2 - a_1, a_3 - a_2, \ldots, a_k - a_{k-1}, n - a_k) = (a_1, d_1, d_2, \ldots, d_{k-1}, d_k)$$

and get g.f. coefficient

$$\begin{aligned} &[x^n] \quad \underbrace{x(1-x)^{-1}}_{\text{g.f. for } a_1} \quad \underbrace{[x^2(1-x)^{-1}]^{k-1}}_{\text{g.f. for } d_1, \ldots, d_{k-1}} \quad \underbrace{(1-x)^{-1}}_{\text{g.f. for } d_k} \\ =&[x^n]\; x^{2k-1}(1-x)^{-(k+1)} = [x^{n-2k+1}]\; (1-x)^{-(k+1)} \\ =&\left(\!\!\binom{k+1}{n-2k+1}\!\!\right) = \binom{n-k+1}{k} \end{aligned}$$

provided $n \geq 2k - 1$; otherwise we get 0.

(2) Find the number of 4-subsets of $\mathbb{N}_{100}$ in which the largest element is even.

Solution: Look at more general problem: find all 4-subsets of $\mathbb{N}_n$ in which the largest element is odd when n is odd, even when n is even. Set up d.v. $(a_1, d_1, d_2, d_3, d_4)$:

$a_1 \geq 1$	$x + x^2 + x^3 + \ldots = x(1-x)^{-1}$
$d_1, d_2, d_3 \geq 1$	$(x + x^2 + x^3 + \ldots)^3 = (x(1-x)^{-1})^3$
$d_4 = 0, 2, 4, 6, \ldots$	$1 + x^2 + x^4 + x^6 + \ldots = (1-x^2)^{-1}$

So g.f. is

$$\begin{aligned} &x(1-x)^{-1}(x(1-x)^{-1})^3(1-x^2)^{-1} \\ =&x^4(1-x)^{-4}(1-x^2)^{-1} = x^4\left(\frac{1+x}{1-x^2}\right)^4(1-x^2)^{-1} \\ =&x^4(1+x)^4(1-x^2)^{-5} = x^4(1 + 4x + 6x^2 + 4x^3 + x^4)(1-x^2)^{-5} \end{aligned}$$

So, our answer is

$$\begin{aligned}
&[x^{100}]\ x^4(1+4x+6x^2+4x^3+x^4)(1-x^2)^{-5} \\
=&[x^{96}]\ (1+4x+6x^2+4x^3+x^4)(1-x^2)^{-5} \\
=&[x^{96}]\ (1-x^2)^{-5}+6[x^{94}]\ (1-x^2)^{-5}+[x^{92}]\ (1-x^2)^{-5} \\
&\qquad \text{noting that coeffs of odd powers of } x \text{ in } (1-x^2)^{-5} \text{ are } 0 \\
=&[y^{48}]\ (1-y)^{-5}+6[y^{47}]\ (1-y)^{-5}+[y^{46}]\ (1-y)^{-5} \qquad \text{where } y=x^2 \\
=&\left(\!\!\binom{5}{48}\!\!\right)+6\left(\!\!\binom{5}{47}\!\!\right)+\left(\!\!\binom{5}{46}\!\!\right) \\
=&\binom{52}{4}+6\binom{51}{4}+\binom{50}{4}=270\,725+6(249\,900)+230\,300=2\,000\,425.
\end{aligned}$$

Partitions

Partitions: A *partition* of n is an unordered collection (multiset) of positive integers with sum n.

Example: Partitions of 5: $1+1+1+1+1$, $1+1+1+2$, $1+1+3$, $1+2+2$, $1+4$, $2+3$, 5.

We can think of partitions as the finite-weight elements of the set

$$\{0,1,1+1,1+1+1,\ldots\}\times\{0,2,2+2,2+2+2,\ldots\}\times\{0,3,\ldots\}\times\ldots$$

– an infinite Cartesian product: means set of sequences $(a_1, a_2, a_3, \ldots)$ with a_1 from first set, a_2 from second set, etc. - e.g. $1+2+2 \leftrightarrow (1, 2+2, 0, 0, \ldots)$. Weight of a partition is its sum, e.g. weight of $(1, 2+2, 0, 0, \ldots)$ is $1+2+2=5$. So, get g.f.

for 1's	$1+x+x^2+x^3+\ldots=(1-x)^{-1}$
for 2's	$1+x^2+x^4+x^6+\ldots=(1-x^2)^{-1}$
for 3's	$1+x^3+x^6+x^9+\ldots=(1-x^3)^{-1}$
$\vdots$	$\vdots$

So g.f. for partitions is *infinite* product

$$(1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}(1-x^4)^{-1}\ldots=\prod_{k=1}^{\infty}(1-x^k)^{-1}=\lim_{m\to\infty}\prod_{k=1}^{m}(1-x^k)^{-1}$$

Limit works in $\mathbb{C}[[x]]$ because coefficient of x^n only comes from first n terms. The number of partitions of n is just:

$$\begin{aligned}
&[x^n]\ (1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}(1-x^4)^{-1}\ldots \\
=&[x^n]\ (1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}\ldots(1-x^n)^{-1}
\end{aligned}$$

Examples: (1) (Polya's change-making example) How many ways are there to change \$1 into pennies, nickels, dimes and quarters?

Solution: Think of handful of coins as element of the set

$$\underset{\text{pennies}}{\{0,1,1+1,1+1+1,\ldots\}}\times\underset{\text{nickels}}{\{0,5,5+5,\ldots\}}\times\underset{\text{dimes}}{\{0,10,10+10,\ldots\}}\times\underset{\text{quarters}}{\{0,25,25+25,\ldots\}}$$

giving g.f.

$$\begin{aligned}
&(1+x+x^2+x^3+\ldots)(1+x^5+x^{10}+\ldots)(1+x^{10}+x^{20}+\ldots)(1+x^{25}+x^{50}+\ldots) \\
=&(1-x)^{-1}(1-x^5)^{-1}(1-x^{10})^{-1}(1-x^{25})^{-1}
\end{aligned}$$

Thus, our answer can be expressed as

$$[x^{100}]\ (1-x)^{-1}(1-x^5)^{-1}(1-x^{10})^{-1}(1-x^{25})^{-1}$$

If wanted to, could calculate, but lengthy: need common denominator:

$$\begin{aligned}
&\frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})}\\
=&\frac{1+x+x^2+\ldots+x^{49}}{1-x^{50}}\ \frac{1+x^5+x^{10}+\ldots+x^{45}}{1-x^{50}}\ \frac{1+x^{10}+x^{20}+x^{30}+x^{40}}{1-x^{50}}\ \frac{1+x^{25}}{1-x^{50}}\\
=&(1+x+\ldots+x^{49})(1+x^5+\ldots+x^{45})(1+x^{10}+\ldots+x^{40})(1+x^{25})(1-x^{50})^{-4}
\end{aligned}$$

For \$1 just as easy to enumerate cases as to multiply this out - but for \$1000 this is much better!

(2) In Australia there used to be 1c, 2c, 5c, 10c, 20c, 50c, \$1 and \$2 coins, and \$1 and \$2 notes. In how many ways could an Australian make change for a \$5 note?

Solution:

$$[x^{500}]\ \frac{1}{(1-x)(1-x^2)(1-x^5)(1-x^{10})(1-x^{20})(1-x^{50})(1-x^{100})^2(1-x^{200})^2}$$

— last two terms in the denominator are squared because there are two types of \$1's and \$2's.

Can also look at partitions with a given number of parts. First need to look at graphical representation of partition.

Ferrers diagrams: The *Ferrers diagram* for partition $a_1+a_2+\ldots+a_n$ ($a_1 \geq a_2 \geq \ldots \geq a_n$) has a_i dots in row i. (Can also use open squares, then Young diagram, basis for Young tableau.) E.g., $7+6+4$:

	3	3	3	3	2	2	1
7	•	•	•	•	•	•	•
6	•	•	•	•	•	•	
4	•	•	•	•			

Conjugate partition: Transpose Ferrers diagram, get Ferrers diagram of new partition, called *conjugate* of original. E.g., conjugate of $7+6+4$ is $3+3+3+3+2+2+1$.

Now let $p_n(r)$ and $p_{\leq n}(r)$ denote the number of partitions of r with exactly n and at most n parts, respectively. Using $1-1$ correspondences given by conjugacy, we get

$$\begin{aligned}
p_{\leq n}(r) &= \text{no. of partitions of } r \text{ with parts of size} \leq n\\
&= [x^r]\frac{1}{(1-x)}\frac{1}{(1-x^2)}\frac{1}{(1-x^3)}\cdots\frac{1}{(1-x^n)}
\end{aligned}$$

and

$$\begin{aligned}
p_n(r) &= \text{no. of partitions of } r \text{ with largest part of size exactly } n\\
&= [x^r]\frac{1}{(1-x)}\frac{1}{(1-x^2)}\frac{1}{(1-x^3)}\cdots\frac{x^n}{(1-x^n)}\\
&= [x^{r-n}]\frac{1}{(1-x)}\frac{1}{(1-x^2)}\frac{1}{(1-x^3)}\cdots\frac{1}{(1-x^n)} = p_{\leq n}(r-n)
\end{aligned}$$

Rota/Stanley's "12-fold way" (revisited): distributing balls into boxes. Two new entries.

r balls $\to$	n boxes	any way	≤ 1 ball per box	≥ 1 ball per box
distinct	distinct	n^r	$P(n,r)$	(from Stirling number)
identical	distinct	$\left(\!\binom{n}{r}\!\right) = \binom{n+r-1}{r}$	$\binom{n}{r}$	$\left(\!\binom{n}{r-n}\!\right) = \binom{r-1}{r-n}$
distinct	identical	(from Stirling number)	1 if $r \le n$ 0 if $r > n$	(Stirling number)
identical	identical	** $p_{\le n}(r)$ **	1 if $r \le n$ 0 if $r > n$	** $p_n(r)$ **